# Physics-based Performance Enhancement in Computational Electromagnetics: A Review

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***Abstract*—Maxwell's electrodynamic differential equations in general bi-anisotropic media have been split into an independent $4 \times 4$ diagonalized and a dependent $2 \times 4$ supplementary system of equations, referred to as the D- and S- forms, respectively. The forms have been utilized to construct standard singular Dyadic Green's functions (DGFs). Problem-tailored expressions for the Dirac's $\delta-$function have been obtained using Fourier integral representations for the DGFs. The resulting expressions for the $\delta-$ function have been used to regularize the originating DGFs exponentially. On the other hand, employing standard finite-support basis- and testing functions, the DGFs have been regularized algebraically. Since the geneses of the exponential and algebraic regularization techniques are conceptually different they can be employed independently or in unison. Finally, frequency-, material- and geometry independent universal functions have been constructed for accelerated and highly performance-enhanced computation of the self- and mutual interactions in the method of moments applications.**

## I. Introduction

In this contribution several physics-inspired measures for enhancing the performance of computations in electromagnetic applications have been reviewed. In spite of the fact that the scope of the developed theory extends beyond electrodynamics the emphasis in this paper has been Maxwell's equations ([1] and references therein). The theory is based on six concepts which are briefly touched upon in six sections. (i) Diagonalization of Maxwell's equations leading to the D- and S-forms, [2], [3]; (ii) construction of standard singular dyadic Green's functions (DGFs); (iii) construction of problem-tailored integral representations for the Dirac's $\delta-$function; (iv) exponential regularization of DGFs, [4]; (v) algebraic regularization of DGFs, [5]; (vi) construction of frequency- and geometry independent universal functions, [1].

It is worth pointing out the following distinguishing features of the theory: The procedures (i)-(vi) can be implemented analytically or numerically, whichever the case may be. Thereby, the steps leading to the D- and S-forms can algorithmically be automated using symbolic languages. The D-form in spectral domain is an eigenvalue problem, depending only on the slowness variable $s = k/\omega$, rather than on $k$ and $\omega$ individually. Here, $k$ and $\omega$ refer to the wavenumber and the frequency, respectively. The dependence of the eigen equation on $s$ is the origin for the $s-$dependence of the eigenpairs and thus the $s-$dependence of the Green's functions and consequently the $s-$dependence of the universal function.

## II. Diagonalization

### A. Diagonalization of Maxwell's equations with respect to the variable $x$ in bi-anisotropic media: The $D_x-$form

$$
\left\{ \begin{bmatrix} \zeta_{32} & \zeta_{33} & \mu_{32} & \mu_{33} \\ -\zeta_{22} & -\zeta_{23} & -\mu_{22} & -\mu_{23} \\ -\varepsilon_{32} & -\varepsilon_{33} & -\xi_{32} & -\xi_{33} \\ \varepsilon_{22} & \varepsilon_{23} & \xi_{22} & \xi_{23} \end{bmatrix} + \begin{bmatrix} -\zeta_{31} - \frac{\partial_y}{j\omega} & -\mu_{31} \\ \zeta_{21} - \frac{\partial_z}{j\omega} & \mu_{21} \\ \varepsilon_{31} & \xi_{31} - \frac{\partial_y}{j\omega} \\ -\varepsilon_{21} & -\xi_{21} - \frac{\partial_z}{j\omega} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \xi_{11} \\ \zeta_{11} & \mu_{11} \end{bmatrix}^{-1} \times \begin{bmatrix} \varepsilon_{12} & \varepsilon_{13} & \zeta_{12} - \frac{\partial_z}{j\omega} & \zeta_{13} + \frac{\partial_y}{j\omega} \\ \zeta_{12} + \frac{\partial_z}{j\omega} & \zeta_{13} - \frac{\partial_y}{j\omega} & \mu_{12} & \mu_{13} \end{bmatrix} \right\}
\times \begin{bmatrix} E_2 \\ E_3 \\ H_2 \\ H_3 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \zeta_{31} + \frac{\partial_y}{j\omega} & \mu_{31} \\ -\zeta_{21} + \frac{\partial_z}{j\omega} & -\mu_{21} \\ -\varepsilon_{31} & -\xi_{31} + \frac{\partial_y}{j\omega} \\ \varepsilon_{21} & \xi_{21} + \frac{\partial_z}{j\omega} \end{bmatrix}
\times \begin{bmatrix} \varepsilon_{11} & \xi_{11} \\ \zeta_{11} & \mu_{11} \end{bmatrix}^{-1} \begin{bmatrix} J_1 \\ 0 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} 0 \\ 0 \\ J_3 \\ -J_2 \end{bmatrix}
= \frac{\partial_x}{j\omega} \begin{bmatrix} E_2 \\ E_3 \\ H_2 \\ H_3 \end{bmatrix} \quad (1)
$$

### B. The equation supplementary to the $x-$diagonalized form: The $S_x-$form

$$
\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = - \begin{bmatrix} \varepsilon_{11} & \xi_{11} \\ \zeta_{11} & \mu_{11} \end{bmatrix}^{-1} \times \begin{bmatrix} \varepsilon_{12} & \varepsilon_{13} & \zeta_{12} - \frac{\partial_z}{j\omega} & \zeta_{13} + \frac{\partial_y}{j\omega} \\ \zeta_{12} + \frac{\partial_z}{j\omega} & \zeta_{13} - \frac{\partial_y}{j\omega} & \mu_{12} & \mu_{13} \end{bmatrix} \times \begin{bmatrix} E_2 \\ E_3 \\ H_2 \\ H_3 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \varepsilon_{11} & \xi_{11} \\ \zeta_{11} & \mu_{11} \end{bmatrix}^{-1} \begin{bmatrix} J_1 \\ 0 \end{bmatrix} \quad (2)
$$

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### *C. Diagonalization of Maxwell's equations with respect to the variable $y$ in bi-anisotropic media: The $D_y-$form*

$$
\left\{ \begin{bmatrix} \zeta_{13} & \zeta_{11} & \mu_{13} & \mu_{11} \\ -\zeta_{33} & -\zeta_{31} & -\mu_{33} & -\mu_{31} \\ -\varepsilon_{13} & -\varepsilon_{11} & -\xi_{13} & -\xi_{11} \\ \varepsilon_{33} & \varepsilon_{31} & \xi_{33} & \xi_{31} \end{bmatrix} \right.
+ \begin{bmatrix} -\zeta_{12} - \frac{\partial_z}{j\omega} & -\mu_{12} \\ \zeta_{32} - \frac{\partial_x}{j\omega} & \mu_{32} \\ \varepsilon_{12} & \xi_{12} - \frac{\partial_z}{j\omega} \\ -\varepsilon_{32} & -\xi_{32} - \frac{\partial_x}{j\omega} \end{bmatrix} \begin{bmatrix} \varepsilon_{22} & \xi_{22} \\ \zeta_{22} & \mu_{22} \end{bmatrix}^{-1}
$$
$$
\left. \times \begin{bmatrix} \varepsilon_{23} & \varepsilon_{21} & \zeta_{23} - \frac{\partial_x}{j\omega} & \zeta_{21} + \frac{\partial_z}{j\omega} \\ \zeta_{23} + \frac{\partial_x}{j\omega} & \zeta_{21} - \frac{\partial_z}{j\omega} & \mu_{23} & \mu_{21} \end{bmatrix} \right\}
\times \begin{bmatrix} E_3 \\ E_1 \\ H_3 \\ H_1 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \zeta_{12} + \frac{\partial_z}{j\omega} & \mu_{12} \\ -\zeta_{32} + \frac{\partial_x}{j\omega} & -\mu_{32} \\ -\varepsilon_{12} & -\xi_{12} + \frac{\partial_z}{j\omega} \\ \varepsilon_{32} & \xi_{32} + \frac{\partial_x}{j\omega} \end{bmatrix}
$$
$$
\times \begin{bmatrix} \varepsilon_{22} & \xi_{22} \\ \zeta_{22} & \mu_{22} \end{bmatrix}^{-1} \begin{bmatrix} J_2 \\ 0 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} 0 \\ 0 \\ J_1 \\ -J_3 \end{bmatrix}
= \frac{\partial_y}{j\omega} \begin{bmatrix} E_3 \\ E_1 \\ H_3 \\ H_1 \end{bmatrix} \quad (3)
$$

### *D. The equation supplementary to the $y-$diagonalized form: The $S_y-$form*

$$
\begin{bmatrix} E_2 \\ H_2 \end{bmatrix} = - \begin{bmatrix} \varepsilon_{22} & \xi_{22} \\ \zeta_{22} & \mu_{22} \end{bmatrix}^{-1}
\times \begin{bmatrix} \varepsilon_{23} & \varepsilon_{21} & \zeta_{23} - \frac{\partial_x}{j\omega} & \zeta_{21} + \frac{\partial_z}{j\omega} \\ \zeta_{23} + \frac{\partial_x}{j\omega} & \zeta_{21} - \frac{\partial_z}{j\omega} & \mu_{23} & \mu_{21} \end{bmatrix}
\times \begin{bmatrix} E_3 \\ E_1 \\ H_3 \\ H_1 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \varepsilon_{22} & \xi_{22} \\ \zeta_{22} & \mu_{22} \end{bmatrix}^{-1} \begin{bmatrix} J_2 \\ 0 \end{bmatrix} \quad (4)
$$

### *E. Diagonalization of Maxwell's equations with respect to the variable $z$ in bi-anisotropic media: The $D_z-$form*

$$
\left\{ \begin{bmatrix} \zeta_{21} & \zeta_{22} & \mu_{21} & \mu_{22} \\ -\zeta_{11} & -\zeta_{12} & -\mu_{11} & -\mu_{12} \\ -\varepsilon_{21} & -\varepsilon_{22} & -\xi_{21} & -\xi_{22} \\ \varepsilon_{11} & \varepsilon_{12} & \xi_{11} & \xi_{12} \end{bmatrix} \right.
+ \begin{bmatrix} -\zeta_{23} - \frac{\partial_x}{j\omega} & -\mu_{23} \\ \zeta_{13} - \frac{\partial_y}{j\omega} & \mu_{13} \\ \varepsilon_{23} & \xi_{23} - \frac{\partial_x}{j\omega} \\ -\varepsilon_{13} & -\xi_{13} - \frac{\partial_y}{j\omega} \end{bmatrix} \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1}
$$
$$
\left. \times \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \zeta_{31} - \frac{\partial_y}{j\omega} & \zeta_{32} + \frac{\partial_x}{j\omega} \\ \zeta_{31} + \frac{\partial_y}{j\omega} & \zeta_{32} - \frac{\partial_x}{j\omega} & \mu_{31} & \mu_{32} \end{bmatrix} \right\}
\times \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \zeta_{23} + \frac{\partial_x}{j\omega} & \mu_{23} \\ -\zeta_{13} + \frac{\partial_y}{j\omega} & -\mu_{13} \\ -\varepsilon_{23} & -\xi_{23} + \frac{\partial_x}{j\omega} \\ \varepsilon_{13} & \xi_{13} + \frac{\partial_y}{j\omega} \end{bmatrix}
$$
$$
\times \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1} \begin{bmatrix} J_3 \\ 0 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} 0 \\ 0 \\ J_2 \\ -J_1 \end{bmatrix}
= \frac{\partial_z}{j\omega} \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \quad (5)
$$

### *F. The equation supplementary to the $z-$diagonalized form: The $S_z-$form*

$$
\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = - \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1}
\times \begin{bmatrix} \varepsilon_{31} & \varepsilon_{32} & \zeta_{31} - \frac{\partial_y}{j\omega} & \zeta_{32} + \frac{\partial_x}{j\omega} \\ \zeta_{31} + \frac{\partial_y}{j\omega} & \zeta_{32} - \frac{\partial_x}{j\omega} & \mu_{31} & \mu_{32} \end{bmatrix}
\times \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} + \frac{1}{j\omega} \begin{bmatrix} \varepsilon_{33} & \xi_{33} \\ \zeta_{33} & \mu_{33} \end{bmatrix}^{-1} \begin{bmatrix} J_3 \\ 0 \end{bmatrix} \quad (6)
$$

*Comments:* Each pair of the above D-forms and their corresponding S-forms; i.e., $\{D_x - \text{form}, S_x - \text{form}\}$, $\{D_y - \text{form}, S_y - \text{form}\}$ or $\{D_z - \text{form}, S_z - \text{form}\}$ is equivalent with the Maxwell's equations in bi-anisotropic media characterized by material matrices $\underline{\underline{\varepsilon}}$, $\underline{\underline{\xi}}$, $\underline{\underline{\zeta}}$ and $\underline{\underline{\mu}}$, and thus constitutive equations $\mathbf{D} = \underline{\underline{\varepsilon}}\mathbf{E} + \underline{\underline{\xi}}\mathbf{H}$ and $\mathbf{B} = \underline{\underline{\zeta}}\mathbf{E} + \underline{\underline{\mu}}\mathbf{H}$. Thereby, $J_i$ $(i = 1, 2, 3)$ is the component of the electric current in the $i^{\text{th}}$ direction. Assuming, e.g., $J_1(x, y, z) = J_1\delta(x - x', y - y', z - z')$, $J_2(x, y, z) = 0$, and $J_3(x, y, z) = 0$ the resulting field components $E_i$ and $H_i$, $(i = 1, 2, 3)$ correspond to the Green's functions $G_{E_i,J_1}(x, y, z|x', y', z')$ and $G_{H_i,J_1}(x, y, z|x', y', z')$.

It is worth mentioning that while manipulations and computations can be carried out in most general cases, the discussion in the remaining sections will focus on free-space elucidating details of the ideas involved rather than on the manipulatory complexities. To this end consider free-space with $J_1(x,y,z) = J_1\delta(x-x', y-y', z-z')$, $J_2(x,y,z) = J_2\delta(x-x', y-y', z-z')$, and $J_3(x,y,z)=0$, to obtain:

*G. Diagonalization of Maxwell's equations with respect to the variable $z$ in free space with $J_3 = 0$: The $D_z$−form*

$$\left\{\begin{bmatrix} 0 & 0 & 0 & \mu \\ 0 & 0 & -\mu & 0 \\ 0 & -\varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -\frac{\partial_x}{j\omega} & 0 \\ -\frac{\partial_y}{j\omega} & 0 \\ 0 & -\frac{\partial_x}{j\omega} \\ 0 & -\frac{\partial_y}{j\omega} \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 0 & -\frac{\partial_y}{j\omega} & \frac{\partial_x}{j\omega} \\ \frac{\partial_y}{j\omega} & -\frac{\partial_x}{j\omega} & 0 & 0 \end{bmatrix}\right\} \times \begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} + \frac{1}{j\omega}\begin{bmatrix} \frac{\partial_x}{j\omega} & 0 \\ \frac{\partial_y}{j\omega} & 0 \\ 0 & \frac{\partial_x}{j\omega} \\ 0 & \frac{\partial_y}{j\omega} \end{bmatrix} + \frac{1}{j\omega}\begin{bmatrix} 0 \\ 0 \\ J_2 \\ -J_1 \end{bmatrix} = \frac{\partial_z}{j\omega}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \quad (7)$$

*H. The equation supplementary to the $z$−diagonalized form in free space with $J_3 = 0$: The $S_z$−form*

$$\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = -\begin{bmatrix} \varepsilon & 0 \\ 0 & \mu \end{bmatrix}^{-1} \times \begin{bmatrix} 0 & 0 & -\frac{\partial_y}{j\omega} & \frac{\partial_x}{j\omega} \\ \frac{\partial_y}{j\omega} & -\frac{\partial_x}{j\omega} & 0 & 0 \end{bmatrix}\begin{bmatrix} E_1 \\ E_2 \\ H_1 \\ H_2 \end{bmatrix} \quad (8)$$

## III. Conventional Singular Dyadic Green's Functions in 3D Electrodynamics

*Statement of the problem:* Consider the dipoles $J_1\mathbf{e}_1\ \delta(x-x', y-y', z-z')$ and $J_2\mathbf{e}_2\ \delta(x-x',\ y-y',\ z-z')$ located at $(x', y', z')$ in a medium characterized by constant scalar permittivity $\varepsilon$ and permeability $\mu$. The unit vectors in the $x-$ and $y-$ directions are denoted by $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Consider (7) and (8) and calculate the electric field vector $\mathbf{E}(x,y,z)$ and the magnetic field vector $\mathbf{H}(x,y,z)$ in entire $(x,y,z)$−space as the response of the medium to the assumed unit dipoles.

*Solution procedure:* Partition $(x,y,z)$−space into subspaces $z > z'$ and $z < z'$ by introducing the fictitious plane $z = z'$. Find solution ansatzes for $\mathbf{E}(x,y,z)$ and $\mathbf{H}(x,y,z)$, involving *a priori* unknown coefficients, in regions $z > z'$ and $z < z'$. Satisfy Sommerfeld radiation conditions at infinity along with ''interface'' conditions at $z = z'$ to determine the unknown coefficients.

It is advantageous to introduce $W(k_1,k_2)$:

$$W = \begin{cases} \sqrt{k_1^2 + k_2^2 - \varepsilon\mu\omega^2}; & k_1^2 + k_2^2 - \varepsilon\mu\omega^2 > 0 \\ -j\sqrt{\varepsilon\mu\omega^2 - (k_1^2 + k_2^2)}; & \varepsilon\mu\omega^2 - (k_1^2 + k_2^2) > 0 \end{cases} \quad (9)$$

Since there is no danger of ambiguity the same symbols are used to denote fields in spatial- and spectral domain, e.g., $E_1(x,y)$ and $E_1(k_1,k_2)$, rather than $E_1(x,y)$ and $\overline{E}_1(k_1,k_2)$.

Interface conditions (suppressing $e^{jk_1(x-x')}e^{jk_2(y-y')}$) read:

$$E_1^{z>z'}(k_1,k_2) - E_1^{z<z'}(k_1,k_2) = 0 \quad (10a)$$
$$E_2^{z>z'}(k_1,k_2) - E_2^{z<z'}(k_1,k_2) = 0 \quad (10b)$$
$$H_1^{z>z'}(k_1,k_2) - H_1^{z<z'}(k_1,k_2) = J_2 \quad (10c)$$
$$H_2^{z>z'}(k_1,k_2) - H_2^{z<z'}(k_1,k_2) = -J_1 \quad (10d)$$

Explicit expressions for the dyadic Green's functions in spectral domain read:

$$G_{E_1,J_1}(k_1,k_2) = \frac{j}{2\varepsilon\omega}\frac{k_2^2 - W^2}{W} \quad (11a)$$
$$G_{E_1,J_2}(k_1,k_2) = \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \quad (11b)$$
$$G_{E_2,J_1}(k_1,k_2) = \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \quad (11c)$$
$$G_{E_2,J_2}(k_1,k_2) = \frac{j}{2\varepsilon\omega}\frac{k_1^2 - W^2}{W} \quad (11d)$$
$$G_{H_1,J_1}(k_1,k_2) = 0 \quad (11e)$$
$$G_{H_1,J_2}(k_1,k_2) = \text{sgn}(z-z')\frac{1}{2} \quad (11f)$$
$$G_{H_2,J_1}(k_1,k_2) = -\text{sgn}(z-z')\frac{1}{2} \quad (11g)$$
$$G_{H_2,J_2}(k_1,k_2) = 0 \quad (11h)$$
$$G_{E_3,J_1}(k_1,k_2) = \text{sgn}(z-z')\frac{k_1}{2\omega\varepsilon} \quad (11i)$$
$$G_{E_3,J_2}(k_1,k_2) = \text{sgn}(z-z')\frac{k_2}{2\omega\varepsilon} \quad (11j)$$
$$G_{H_3,J_1}(k_1,k_2) = -\frac{jk_2}{2W} \quad (11k)$$
$$G_{H_3,J_2}(k_1,k_2) = \frac{jk_1}{2W} \quad (11l)$$

## IV. Integral Representations for Problem-tailored Field-induced Dirac's $\delta-$Functions

Consider $G_{H_2,J_1}(k_1,k_2)$. In real space we have:

$$G_{H_2,J_1}(\mathbf{x}|\mathbf{x}') = -\mathrm{sgn}(z-z')\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \times e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W|z-z'|} \quad (12)$$

In view of (9) $W=\sqrt{k^2-k_0^2}$ if $k^2-k_0^2>0$, and $W=-j\sqrt{k_0^2-k^2}$ if $k_0^2-k^2>0$. Here, $k^2=k_1^2+k_2^2$ and $k_0=\omega/c_0$ with $c_0$ being speed of light in free space, and $\omega$ the angular frequency.

Consistent with Maxwell's equations the following ''interface'' relationship holds true:

$$-\lim_{z\to z'_+} G^{z>z'}_{H_2,J_1}(\mathbf{x}|\mathbf{x}') + \lim_{z\to z'_-} G^{z<z'}_{H_2,J_1}(\mathbf{x}|\mathbf{x}') = \delta(x-x',y-y') \quad (13)$$

Substituting (12) into ( (13)) and replacing $|z-z'|$ by $\eta(>0)$ lead to:

$$\delta(x-x',y-y') = \lim_{\eta\to 0}\delta_\eta(x-x',y-y') = \lim_{\eta\to 0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\eta} \quad (14)$$

*Remarks:* It is claimed that the procedure captured in Eqs. (12)-(14) for constructing ''physics-inspired'' integral representations for the Dirac's delta function has been overlooked in literature. For $\eta$ finite, however small, (14) defines the distributed smeared-out source function $\delta_\eta(x-x',y-y')$, which smoothly approaches the symbolic generalized function $\delta(x-x',y-y')$. Exciting the medium (here free-space) with $\delta_\eta(x-x',y-y')$, and following the procedure for the construction of standard DGFs in the preceding section, lead to regularized dyadic Green's functions. This fact will be demonstrated in the next section. In view of the fact that $W$ can assume complex values, the validity of the relationships in (14) is far from trivial. This non-obviousness is particularly obscuring when the expressions are not available in closed form. The following constructive proof has been devised to clarify matters.

### A. $D-$**Theorem**

The relationships in (14) are valid, [2].

Proof: Symmetry considerations lead to

$$\delta(x-x',y-y') = \lim_{\eta\to 0}\frac{1}{\pi^2}\int_0^{\infty}\int_0^{\infty}dk_1dk_2 \times\cos[k_1(x-x')]\cos[k_2(y-y')]e^{-W\eta}. \quad (15)$$

Denote the double integral in (15) by $I$, change to polar coordinates, partition the integration range over $k$ into $[0,k_0]$ and $[k_0,\infty]$, and use the definition of $W$ to obtain:

$$\begin{aligned} I &= \lim_{\eta\to 0}\frac{1}{\pi^2}\int_0^{k_0}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right)e^{j\sqrt{k_0^2-k^2}\eta} \\ &+ \lim_{\eta\to 0}\frac{1}{\pi^2}\int_{k_0}^{\infty}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right)e^{-\sqrt{k^2-k_0^2}\eta} (= I_1+I_2) \end{aligned} \quad (16)$$

Denote the 1st term at the RHS of (16) by $I_1$, and use the limit $e^{j\sqrt{k_0^2-k^2}\eta}\to 1$ for $\eta\to 0$ $(0\le k\le k_0)$, to obtain:

$$I_1 = \frac{1}{\pi^2}\int_0^{k_0}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \times \cos\left(\sin\theta|y-y'|\right) \quad (17)$$

Denote the 2nd term at the RHS of 16) by $I_2$, and use the limit $e^{-\sqrt{k^2-k_0^2}\eta}\to e^{-k\eta}$ for $\eta\to 0$ $(k_0\le k<\infty)$ to obtain

$$I_2 = \lim_{\eta\to 0}\frac{1}{\pi^2}\int_{k_0}^{\infty}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \times \cos\left(\sin\theta|y-y'|\right)e^{-k\eta} \quad (18)$$

Rearrangement of the integration range $\int_{k_0}^{\infty}dk=\int_0^{\infty}dk-\int_0^{k_0}dk$ transforms (18) to

$$\begin{aligned} I_2 &= \lim_{\eta\to 0}\frac{1}{\pi^2}\int_0^{\infty}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right)e^{-k\eta} \\ &- \lim_{\eta\to 0}\frac{1}{\pi^2}\int_0^{k_0}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right)e^{-k\eta} \end{aligned} \quad (19)$$

Consider the limit $e^{-k\eta}\to 1$ for $\eta\to 0$ $(0\le k\le k_0)$ in the 2nd integral at the RHS of (19), to obtain:

$$\begin{aligned} I_2 &= \lim_{\eta\to 0}\frac{1}{\pi^2}\int_0^{\infty}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right)e^{-k\eta} \\ &- \frac{1}{\pi^2}\int_0^{k_0}\int_0^{\pi/2}dkd\theta k\cos\left(\cos\theta|x-x'|\right) \\ &\times \cos\left(\sin\theta|y-y'|\right) \end{aligned} \quad (20)$$

The $2^{\text{nd}}$ integral in (20) equals $-I_1$. Rearrangement yields:

$$I_1 + I_2 = \lim_{\eta\to 0} \frac{1}{\pi^2} \int_0^{\infty} \int_0^{\pi/2} dk d\theta k \cos\left(\cos\theta |x-x'|\right) \times \cos\left(\sin\theta |y-y'|\right) e^{-k\eta} = I \tag{21}$$

In earlier work it was established that the double integral in (21) is a valid integral representation for the Dirac's $\delta-$function. △

To demonstrate the power of the method for the construction of integral representations for the Dirac's $\delta-$function it is instructive to consider the Green's function $G_{H_3,J_1}(k_1,k_2)$. To fully appreciate the underlying intricate relationships it is also instructive to visualize the relative orientation of the $H_3$ field component to the orientation of the dipole $J_1$: Moving along the $y-$axis and crossing the line $y=y'$, the field component $H_3$ has a jump discontinuity equal to $\delta(x-x', z-z')$. This can be shown as follows. Consider the Green's function $G_{H_3,J_1}(k_1,k_2)$ in real space:

$$G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \left\{-\frac{1}{2}\frac{jk_2}{W}\right\} \times e^{jk_1(x-x')} e^{jk_2(y-y')} e^{-W|z-z'|} \tag{22}$$

Denote $G_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$ in region $y-y'>0$ ($y-y'=|y-y'|$) by $G^{y>y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$. Likewise, denote $G_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$ in region $y-y'<0$ ($y-y'=-|y-y'|$) by $G^{y<y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$. Then, consistency with Maxwell's equations requires that:

$$\lim_{|y-y'|\to 0} G^{y>y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}') - \lim_{|y-y'|\to 0} G^{y<y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \delta(x-x', z-z') \tag{23}$$

It should be noted that the limiting process acts on the variable in the oscillating rather than the decaying exponential, thus making this formula exceptionally interesting and important.

Replacing $|y-y'|$ by $\eta$ and substituting the resulting expressions for $G^{y>y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$ and $G^{y<y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}')$ into (23) yields the desired distributed source function, a relationship the validity of which will be established in virtue of the S-Theorem:

$$\begin{aligned} \delta(x-x', z-z') &= \lim_{\eta\to 0} \delta_\eta(x-x', z-z') \\ &= \lim_{\eta\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \left\{-\frac{1}{2}\frac{jk_2}{W}\right\} \\ &\times e^{jk_1(x-x')} e^{jk_2\eta} e^{-W|z-z'|} \\ &- \lim_{\eta\to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \left\{-\frac{1}{2}\frac{jk_2}{W}\right\} \\ &\times e^{jk_1(x-x')} e^{-jk_2\eta} e^{-W|z-z'|} \end{aligned} \tag{24}$$

*Remarks:* It is claimed that the above procedure for constructing ''physics-inspired'' delta functions has been overlooked in literature. For $\eta$ finite, however small, (24) defines the distributed source function $\delta_\eta(x-x', z-z')$ which smoothly approaches the symbolic Dirac's $\delta(x-x', z-z')$.

### *B. S−*Theorem

The relationships in (24) are valid, [3].

*Proof:* Symmetry considerations in (22) followed by taking the limit $\lim_{|y-y'|\to 0}$ gives

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \frac{1}{2\pi^2} \lim_{|y-y'|\to 0} \int_0^{\infty}\int_0^{\infty} dk_1 dk_2 \left\{\frac{k_2}{W}\right\} \times \cos[k_1(x-x')] \sin[k_2(y-y')] e^{-W|z-z'|} \tag{25}$$

Observe that the term $\lim_{|y-y'|\to 0} \sin[k_2(y-y')]$ is non-zero only for $k_2 \to \infty$ (and thus for $k=\sqrt{k_1^2+k_2^2} \to \infty$). Consequently, with $\lim_{k\to\infty} W \propto k$, (25) is equivalent with

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \frac{1}{2\pi^2} \lim_{|y-y'|\to 0} \int_0^{\infty}\int_0^{\infty} dk_1 dk_2 \times \cos[k_1(x-x')] \left\{k_2 \sin[k_2(y-y')]\right\} \left\{\frac{e^{-k|z-z'|}}{k}\right\} \tag{26}$$

With $k_2 \sin[k_2(y-y')] = -\partial/\partial y \cos[k_2(y-y')]$ and $e^{-k|z-z'|}/k = -\text{sgn}(z-z') \int dz e^{-k|z-z'|}$ (26) takes the form

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \frac{1}{2\pi^2} \lim_{|y-y'|\to 0} \int_0^{\infty}\int_0^{\infty} dk_1 dk_2 \times \cos[k_1(x-x')] \left\{\frac{\partial}{\partial y}\cos[k_2(y-y')]\right\} \times \left\{\text{sgn}(z-z') \int dz e^{-k|z-z'|}\right\} \tag{27}$$

Exchanging the order of integral- and differential operators $\int_0^{\infty}\int_0^{\infty} dk_1 dk_2 \frac{\partial}{\partial y} \int dz \Longrightarrow \frac{\partial}{\partial y}\int dz \int_0^{\infty}\int_0^{\infty} dk_1 dk_2$ yields:

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \lim_{|y-y'|\to 0} \frac{\partial}{\partial y} \text{sgn}(z-z') \int dz \times \frac{1}{2\pi^2} \int_0^{\infty}\int_0^{\infty} dk_1 dk_2 \cos[k_1(x-x')] \cos[k_2(y-y')] \times e^{-k|z-z'|} \tag{28}$$

The double integral can be calculated in closed form. Thus,

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \lim_{|y-y'|\to 0} \frac{\partial}{\partial y} \text{sgn}(z-z') \int dz \times \frac{1}{4\pi} \frac{|z-z'|}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}. \tag{29}$$

Absorbing $\mathrm{sgn}(z-z')$ into $|z-z'|$ yields:

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \lim_{|y-y'|\to 0} \frac{\partial}{\partial y} \times \frac{1}{4\pi}\int dz \frac{z-z'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \quad (30)$$

The following calculation is a delicate interplay of terms, essentially replacing $z-z'$ in the numerator in (30) by $|y-y'|$, which is crucially important for further arguments.

The integral in (30) can also be calculated in closed form:

$$\lim_{|y-y'|\to 0} G_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \lim_{|y-y'|\to 0} \frac{\partial}{\partial y} \times \frac{1}{4\pi}\left\{-\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}\right\} \quad (31)$$

Differentiating with respect to $y$ and using $y-y'=|y-y'|$ and $y-y'=-|y-y'|$ for $y>y'$ and $y<y'$, respectively, in the numerators of the resulting expressions, result in:

$$\lim_{|y-y'|\to 0} G^{y>y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}') - \lim_{|y-y'|\to 0} G^{y<y'}_{H_3,J_1}(\mathbf{x}|\mathbf{x}') = \lim_{|y-y'|\to 0} \frac{1}{2\pi}\frac{|y-y'|}{[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}} \quad (32)$$

Identifying the limits at the RHS as $\delta(x-x', z-z')$ the claim in the S-Theorem is immediate. △

## V. Exponential Regularization of Singular Dyadic Green's Functions

The representation in (14) enables the regularization of singular integral expressions arising in the Method of Moments (MoM) applications. The integral in (14) is well-defined for any finite value $\eta > 0$, and can be utilized for defining the ''smeared out'' $\delta-$function source, $\rho_\eta(x-x', y-y')$:

$$\begin{aligned} \delta_\eta(x-x', y-y') &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \\ &\times\ e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\eta} \\ &\Longleftrightarrow\ e^{-W\eta} \ \ \text{(spectral domain)} \end{aligned} \quad (33)$$

Thus, rather than exciting the medium (free space in the present case) with sources $\mathbf{J}_1 = J_1\mathbf{e}_1\delta(x-x', y-y')$ and $\mathbf{J}_2 = J_2\mathbf{e}_2\delta(x-x', y-y')$, consider their corresponding field-theoretically constructed distributed ''smeared out'' counterparts $\mathbf{J}_{1,\eta} = J_1\mathbf{e}_1\delta_\eta(x-x', y-y')$ and $\mathbf{J}_{2,\eta} = J_2\mathbf{e}_2\delta_\eta(x-x', y-y')$, respectively. Note that the support of $\mathbf{J}_{1,\eta}$ and $\mathbf{J}_{1,\eta}$ are confined to the interface. Using these problem-tailored distributed sources in the interface conditions we obtain the following exponentially $\eta-$regularized Green's functions:

$$\left\{\begin{array}{c} G_{E_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_2,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_3,J_1,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_3,J_1,\eta}(\mathbf{x}|\mathbf{x}') \end{array}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \times \left\{\begin{array}{c} \frac{j}{2\varepsilon\omega}\frac{k_2^2-W^2}{W} \\ \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \\ 0 \\ -\mathrm{sgn}(z-z')\frac{1}{2} \\ \mathrm{sgn}(z-z')\frac{k_1}{2\varepsilon\omega} \\ -\frac{jk_2}{2W} \end{array}\right\} \times\ e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\left(|z-z'|+\eta\right)} \quad (34)$$

Similarly,

$$\left\{\begin{array}{c} G_{E_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_2,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_1,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{E_3,J_2,\eta}(\mathbf{x}|\mathbf{x}') \\ G_{H_3,J_2,\eta}(\mathbf{x}|\mathbf{x}') \end{array}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \times \left\{\begin{array}{c} \frac{j}{2\varepsilon\omega}\frac{-k_1k_2}{W} \\ \frac{j}{2\varepsilon\omega}\frac{k_1^2-W^2}{W} \\ \mathrm{sgn}(z-z')\frac{1}{2} \\ 0 \\ \mathrm{sgn}(z-z')\frac{k_2}{2\varepsilon\omega} \\ \frac{jk_1}{2W} \end{array}\right\} \times\ e^{jk_1(x-x')}e^{jk_2(y-y')}e^{-W\left(|z-z'|+\eta\right)} \quad (35)$$

Note that the original slowly-convergent or even divergent integrals have become regularized with the appearance of a problem-specific exponential damping term. The subindex $\eta$ is a reminder of this property!

## VI. Algebraic Regularization of Dyadic Green's Functions

### A. Preparatory considerations

Consider the following generic convolution type integral in spectral domain, [1]:

$$\varphi(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\overline{G}(k_1,k_2)\overline{\rho}(k_1,k_2)e^{jk_1x}e^{jk_2y} \tag{36}$$

A bar indicates quantities in $(k_1,k_2)-$spectral domain. Furthermore, $\overline{G}(k_1,k_2)$ stands for a generic scalar-valued Green's function. It has been assumed that $G(x,y)$ is translationally invariant; i.e., $G(x-x',y-y')$, where $(x,y)$ and $(x',y')$, respectively, refer to the observation- and source point. In addition we shall assume the following source distribution with *a priori* unknown coefficients $\rho_n$:

$$\rho(x,y)=\sum_{n=1}^{N}\rho_n b_n(x,y) \tag{37}$$

with

$$b_n(x,y)=\begin{cases}1 & x_n^b<x<x_n^e \text{ and } y_n^b<y<y_n^e\\ 0 & \text{elsewhere}\end{cases} \tag{38}$$

The Fourier transform of $\rho(x,y)$ denoted by $\overline{\rho}(k_1,k_2)$ is:

$$\begin{aligned}\overline{\rho}(k_1,k_2) &= \sum_{n=1}^{N}\frac{Q_n}{-4\triangle x_n\triangle y_n k_1k_2}\left(e^{-jk_1x_n^e}-e^{-jk_1x_n^b}\right)\\ &\times\left(e^{-jk_2y_n^e}-e^{-jk_2y_n^b}\right)\end{aligned} \tag{39}$$

where $2\triangle x_n(=x_n^e-x_n^b)$ and $2\triangle y_n(=y_n^e-y_n^b)$, respectively. Furthermore, denoting the integral of the source on the $n^{\text{th}}$ sub-square by $Q_n$ we have $Q_n=\rho_n(2\triangle x_n)(2\triangle y_n)$.

Testing $\varphi(x,y)$ by the weighting functions $b_m(x,y)$, $m=1,\cdots,M$, and denoting the average of $\varphi(x,y)$ on the $m^{\text{th}}$ sub-sqaure by $\varphi_m$ we have:

$$\varphi_m=\frac{1}{(x_m^e-x_m^b)(y_m^e-y_m^b)}\int_{x_m^b}^{x_m^e}\int_{y_m^b}^{y_m^e}dxdy\varphi(x,y) \tag{40}$$

Substituting (36) into (40) and rearranging the order of integrals we obtain

$$\varphi_m=\int_{\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\overline{G}(k_1,k_2)\underbrace{\overline{w}_m(k_1,k_2)\overline{\rho}(k_1,k_2)}_{\overline{R}_m(k_1,k_2)} \tag{41}$$

with

$$\begin{aligned}\overline{w}_m(k_1,k_2) &= \frac{1}{-4\triangle x_m\triangle y_m k_1k_2}\left(e^{jk_1x_m^e}-e^{jk_1x_n^b}\right)\\ &\times\left(e^{jk_2y_m^e}-e^{jk_1y_n^b}\right);\ m=1,\cdots,M\end{aligned} \tag{42}$$

Note the introduction of $\overline{R}_m(k_1,k_2)$ in (41). Also note that the solvability condition requires $M=N$.

### B. Automatic emergence of Hadamard Finite Parts

Consider $\overline{R}_m(k_1,k_2)$ as introduced in (41). It is evident that $\overline{R}_m(k_1,k_2)$ only depends on the selected basis- and testing functions, implying that the Green's function does not play any role in the structure of $\overline{R}_m(k_1,k_2)$. On the other hand $\overline{R}_m(k_1,k_2)$ shall play a significant role in the algebraic regularization of the field integrals, to any degree desirable. Furthermore, $\overline{R}_m(k_1,k_2)$ will automatically give rise to the emergence of Hadamard Finite Parts. In order to establish these results we substitute for $\overline{\rho}(k_1,k_2)$ and $\overline{w}_m(k_1,k_2)$, respectively, from (39) and (42), and rearrange to obtain:

$$\begin{aligned}\overline{R}_m(k_1,k_2) &= \sum_{n=1}^{N}Q_n\frac{1}{16\triangle x_m\triangle y_m\triangle x_n\triangle y_n k_1^2k_2^2}\\ &\times\Big[e^{j(k_1x_m^e+k_2y_m^e)}-e^{j(k_1x_m^e+k_2y_m^b)}\\ &\quad -e^{j(k_1x_m^b+k_2y_m^e)}+e^{j(k_1x_m^b+k_2y_m^b)}\Big]\\ &\times\Big[e^{-j(k_1x_n^e+k_2y_n^e)}-e^{-j(k_1x_n^e+k_2y_n^b)}\\ &\quad -e^{-j(k_1x_n^b+k_2y_n^e)}+e^{-j(k_1x_n^b+k_2y_n^b)}\Big]\end{aligned} \tag{43}$$

*Lemma:* The following relationship holds true:

$$\begin{aligned}\overline{\mathcal{R}}_m(k_1,k_2) &= \sum_{n=1}^{N}Q_n\frac{1}{16\triangle x_m\triangle y_m\triangle x_n\triangle y_n k_1^2k_2^2}\\ &\times\Big[+\Big\{1+[j(k_1x_m^e+k_2y_m^e)]+\frac{1}{2}[j(k_1x_m^e+k_2y_m^e)]^2\Big\}\\ &\quad-\Big\{1+[j(k_1x_m^e+k_2y_m^b)]+\frac{1}{2}[j(k_1x_m^e+k_2y_m^b)]^2\Big\}\\ &\quad-\Big\{1+[j(k_1x_m^b+k_2y_m^e)]+\frac{1}{2}[j(k_1x_m^b+k_2y_m^e)]^2\Big\}\\ &\quad+\Big\{1+[j(k_1x_m^b+k_2y_m^b)]+\frac{1}{2}[j(k_1x_m^b+k_2y_m^b)]^2\Big\}\Big]\\ &\times\Big[+\Big\{1+[-j(k_1x_n^e+k_2y_n^e)]+\frac{1}{2}[-j(k_1x_n^e+k_2y_n^e)]^2\Big\}\\ &\quad-\Big\{1+[-j(k_1x_n^e+k_2y_n^b)]+\frac{1}{2}[-j(k_1x_n^e+k_2y_n^b)]^2\Big\}\\ &\quad-\Big\{1+[-j(k_1x_n^b+k_2y_n^e)]+\frac{1}{2}[-j(k_1x_n^b+k_2y_n^e)]^2\Big\}\\ &\quad+\Big\{1+[-j(k_1x_n^b+k_2y_n^b)]+\frac{1}{2}[-j(k_1x_n^b+k_2y_n^b)]^2\Big\}\Big]\\ &=\sum_{n=1}^{N}Q_n\end{aligned} \tag{44}$$

## VII. UNIVERSAL FUNCTIONS

Inspired by the aforementioned properties we define the geometry-independent ''Universal function,'' which can be pre-computed and stored for future numerical calculation, [1]:

$$\begin{aligned} U(X,Y) &= \int_0^{2\pi} d\theta \frac{1}{\sin^2\theta\cos^2\theta} \\ &\times \Big\{ \int_0^{k_c} dk k\overline{\mathcal{G}}(k,\theta)\frac{1}{k^4}\Big(e^{jk(\sin\theta X+\cos\theta Y)} \\ &- \sum_{l=0}^{3}\frac{1}{l!}\Big[jk(\sin\theta X+\cos\theta Y)\Big]^l\Big) \\ &+ \int_{k_c}^{\infty} dk k\overline{\mathcal{G}}(k,\theta)\frac{1}{k^4}e^{jk(\sin\theta X+\cos\theta Y)}\Big\} \quad (45)\end{aligned}$$

The $\int_0^{k_c} dk-$integral in (45) resembles Hadamard Finite Part. Note that for $k << k_c$ the first dominant term in the numerator of the expression in this integral is $o(k^4)$, which cancels the $1/k^4-$term perfectly. An important question is how to proceed if $\overline{\mathcal{G}}(k,\theta) \propto 1/|k|$ for $k << k_c$. In such a case, as it turns out, the total source integral in the ''universe'' must add up to zero ($\sum_{n=1}^{N} Q_n = 0$) (regularizing condition in infrared region). In such a case we can include in (45) the $4^{\text{th}}-$order terms as well by letting $l$ run from 0 to 4. On the other hand, it must be pointed out, that the $\int_{k_c}^{\infty} dk-$integral in (45) decays sufficiently strongly for $k \to \infty$ to ensure convergence of the integral. Consequently, $U(X,Y)$ is regular in the infrared- as well as ultraviolet region. In electrodynamics certain dyadic Green's functions are proportional to $k$ for $k \to \infty$, implying that $k\overline{\mathcal{G}}(k,\theta)/k^4$ behaves according to $1/k^2$. Considering the $\int_{k_c}^{\infty} dk-$integral it can be concluded that the integrability in ultraviolet region is safely guaranteed even in such cases.

Upon construction it is immediate that the interaction elements in the Method of Moments applications, $A_{mn}$, can be written in terms of the Universal Function $U(X,Y)$ as follows:

$$\begin{aligned} A_{mn} &= \frac{1}{64\pi^2\triangle x_m\triangle y_m\triangle x_n\triangle y_n}\Big\{ \\ &+ U(x_{mn}^{ee},y_{mn}^{ee}) - U(x_{mn}^{ee},y_{mn}^{eb}) \\ &- U(x_{mn}^{eb},y_{mn}^{ee}) + U(x_{mn}^{eb},y_{mn}^{eb}) \\ &- U(x_{mn}^{ee},y_{mn}^{be}) + U(x_{mn}^{ee},y_{mn}^{bb}) \\ &+ U(x_{mn}^{eb},y_{mn}^{be}) - U(x_{mn}^{eb},y_{mn}^{bb}) \\ &- U(x_{mn}^{be},y_{mn}^{ee}) + U(x_{mn}^{be},y_{mn}^{eb}) \\ &+ U(x_{mn}^{bb},y_{mn}^{ee}) - U(x_{mn}^{bb},y_{mn}^{eb}) \\ &+ U(x_{mn}^{be},y_{mn}^{be}) - U(x_{mn}^{be},y_{mn}^{bb}) \\ &- U(x_{mn}^{bb},y_{mn}^{be}) + U(x_{mn}^{bb},y_{mn}^{bb})\Big\} \quad (46)\end{aligned}$$

It is worth noting that replacing $e^{jk(\sin\theta X+\cos\theta Y)}$ with $e^{jk(\sin\theta X+\cos\theta)Y} - 1$ in the $\int_{k_c}^{\infty} dk-$integral, (45), does not alter the value of $A_{mn}$. Therefore, denoting the corresponding Universal Function by $V(X,Y)$, it can been shown that $V(0,0) = 0$, a result which is of theoretical and computational significance. For completeness it should be mentioned that the application of MoM leads to $\sum_{n=1}^{N} A_{mn}Q_n = \varphi_m$.

## VIII. CONCLUSION

A sequence of six constructive measures was reviewed for the performance enhancement of computations in the Method of Moments (MoMs) applications in computational electromagnetics. Given a particular direction in space, it was shown that Maxwell's electrodymic equations can be split into a diagonal form and a corresponding supplementary form, the D- and S-forms, respectively. The D- and S- forms were subsequently utilized to construct standard singular dyadic Green's functions (DGFs) in spectral domain. The DGFs were employed to construct novel problem-tailored integral representations for the Dirac's $\delta-$function. The resulting distributed ''smeared out'' Dirac's $\delta-$functions were employed to regularize DGFs exponentially. Furthermore, standard finite-support basis- and testing functions were used to additionally regularize DGFs algebraically. The algebraic regularization scheme enabled the construction of frequency-, and geometry independent Universal functions for the calculation of self- and mutual interaction elements arising in the MoMs applications.

Future work shall focus on the application of the method to the investigation of small-scale phenomena in material science, material engineering, and device modeling and simulation. Furthermore, boundary value problems with fairly arbitrary geometries should be investigated. It is also desirable to construct problem-characteristic integral representations for the Dirac's $\delta-$function in complex media. Finally, the method will be applied to realistic problems to gauge its performance under realistic conditions.


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## REFERENCES


[1] Baghai-Wadji, A. R., ''3D Electrostatic Charge Distribution on Finitely-thick Bus-bars in Micro-acoustic Devices: Combined Regularization in the Near- and Far-field,'' *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, Special Issue, 2015 (invited).

[2] Baghai-Wadji, A. R., ''D-Theorem (On Regularization): Green's Function-induced Distributed Elementary Sources − First Kind,'' in *Proceedings of the IEEE Antennas and Propagation Symposium*, Memphis, Mississippi, USA, 2014.

[3] Baghai-Wadji, A. R., ''S-Theorem (On Regularization): Green's Function-induced Distributed Elementary Sources − Second Kind,'' in *Proceedings of the IEEE Antennas and Propagation Symposium*, Memphis, Mississippi, USA, 2014.

[4] Baghai-Wadji, A. R., ''Exponential Regularization of EM Dyadic Green's Functions via Green's Functions-induced Dirac $\delta-$Functions,'' in *Proceedings of the Progress in Electromagnetic Research Symposium*, Prague, Czech Republic, July, 2015.

[5] Baghai-Wadji, A. R., ''Algebraic Regularization of Universal Functions in EM via Self-induced Hadamard Finite Parts,'' in *Proceedings of the Progress in Electromagnetic Research Symposium*, Prague, Czech Republic, July, 2015.