Discrete-time Quadratic-Optimal Hedging Strategies for European Contingent Claims

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***Abstract*—We revisit the problem of optimally hedging a European contingent claim (ECC) using a hedging portfolio consisting of a risky asset that can be traded at pre-specified discrete times. The objective function to be minimized is either the second-moment or the variance of the hedging error calculated in the market probability measure. The main outcome of our work is to show that unique solutions exist in a larger class of admissible strategies under integrability and non-degeneracy conditions on the hedging asset price process that are weaker than popular descriptions provided previously. Specifically, we do not require the hedging asset price process to be square-integrable, and do not use the bounded mean-variance trade off assumption. Our criterion for admissible strategies only requires the cumulative trading gain, and not the incremental trading gains, to be square integrable. We derive explicit expressions for the second-moment and the variance of the hedging error to arrive at the respective optimal hedging strategies. Further, we explain the connections between our work and those of the previous formulations.**

I. Introduction

We revisit the problem of discrete-time hedging of a European contingent claim (ECC) with a portfolio consisting of a risky security and a risk-less money market asset that grows continuously at the risk-free rate. The risky security, known as the hedging asset, may or may not be the underlying on which the ECC is written and it is to be traded at pre-specified discrete-times. The objective is to seek a hedging strategy to minimize a quadratic-cost criterion on the profit or loss to the seller of the ECC. Problems of such kind have been considered previously by several authors, for example, [1]–[5] in continuous time and [6]–[9] in discrete-time. The pre-cursor to the current work lies in the discrete-time hedging literature. In particular, we build on the work of [7] by weakening the assumptions on the hedging asset price process and widening the set of admissible trading strategies. Our approach to finding optimal hedging strategies mainly involve manipulations of conditional expectations.

A. Background and Motivation

Much of the work on hedging have their origin in the seminal works of Black, Scholes and Merton [10], [11] wherein the authors considered the problem of replicating an ECC and eliminating all risk under ideal market conditions such as continuous trading, absence of transaction costs and the assumption that the underlying asset price follows a geometric Brownian motion (GBM). However, in reality, trading occurs at discrete time and in a discrete-time setting it is not possible to hedge seller's risk completely. Our first motivation is to consider the problem of hedging in a discrete-time setting. To this end, a quadratic criterion on the profit or loss is usually considered as a criterion function to be optimized [1]–[3]. Our second motivation is consider the problem of quadratic hedging with weak assumptions on the hedging asset price process than previously considered before. In here, assumptions on the hedging asset price process are centered on integrablity or measurability properties.

B. Scope of work

The current work seeks self financing trading strategies in a discrete-time setting with a non-degeneracy condition that is more general than [6], [7]. Our approach to finding optimal hedging strategies involves computing analytical expressions for mean, variance and second-moment of profit or loss resulting from an arbitrary trading strategy. These expressions are then used to arrive at the candidate strategies for the respective moment minimizing problems. The candidate strategies are shown to belong to a set of admissible trading strategies.

C. Outline

The outline of the current work is as follows. In Section II we provide a mathematical formulation to the problem. We introduce our non-degeneracy condition on the hedging asset process and the admissibility criterion on the set of trading strategies. Relationship of the problem formulation with previous works are discussed in Section III. In Section IV several preliminary notions are introduced to provide a platform for arguments in subsequent sections. Section V contains the derivation of explicit expressions for the mean, second-moment and variance of the profit or loss to the seller. These analytical expressions are later used in Section VI to arrive at the optimal hedging strategies for various optimization problems considered. We summarize our results in Section VII. The appendix contain proofs of certain intermediate Lemmas stated in various sections.

II. Problem setup

A. Notations

Consider a probability space $(\Omega, \mathcal{F}, Q)$. Denote by $L^0(\Omega, \mathcal{F}, Q)$, $L^1(\Omega, \mathcal{F}, Q)$ and $L^2(\Omega, \mathcal{F}, Q)$ the set of measurable, integrable, and square integrable random variables on $(\Omega, \mathcal{F}, Q)$, respectively. Given $X \in L^1(\Omega, \mathcal{F}, Q)$, $Y, Z \in$

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$L^2(\Omega, \mathcal{F}, Q)$, with $\mathcal{G} \subseteq \mathcal{F}$ being a σ-algebra, $\mathbb{E}^Q(X)$, $\mathbb{E}^Q(X|\mathcal{G})$, $\mathrm{var}^Q(Y|\mathcal{G})$ and $\mathrm{covar}^Q(Y, Z|\mathcal{G})$ denote, respectively, the expectation of X, the conditional expectation of X given $\mathcal{G}$, the conditional variance of Y given $\mathcal{G}$, and the conditional covariance between Y and Z given $\mathcal{G}$, with all moments taken under the measure Q. We emphasize that all inequalities and equalities involving random variables on $(\Omega, \mathcal{F}, Q)$ are to be understood to hold Q-almost surely.

The ECC to be hedged matures at time T. Trading on the risky asset occurs at pre-specified discrete-times $\{T_0, T_1, \cdots, T_n\}, T_{k-1} < T_k, k \in \{1, \ldots, n\}$, with $T_n = T$. Denote S_k as the discounted price of the hedging asset at time T_k, for each $k \in \{0, \ldots, n\}$ and $\{S_k\}_{k=0}^n$ as the discounted hedging asset price process on the probability space $(\Omega, \mathcal{F}, Q)$. We note that all discounting happens to time T_0. Let $\beta \in \mathbb{R}$ be the initial amount invested at time T_0 in the money market asset which includes the premium received for the ECC and let $V \in L^2(\Omega, \mathcal{F}, Q)$ be the discounted payoff of the ECC. Finally, denote Δ_k as the amount of the risky asset to be held in the hedging portfolio during the time interval $(T_{k-1}, T_k]$, $k \in \{1, \ldots, n\}$.

The probability space $(\Omega, \mathcal{F}, Q)$ is augmented with a filtration $\{\mathcal{F}_k\}_{k=0}^n$ with $\mathcal{F}_k$ being the σ-algebra generated by the observations up to and including the hedging time T_k, $k \in \{0, \ldots, n\}$. We assume $\mathcal{F}_0$ to be the trivial σ-algebra augmented with Q-null sets and set $\mathcal{F}_n = \mathcal{F}$. Lastly, we denote $\mathbb{E}_i^Q(\cdot)$ and $\mathrm{var}_i^Q(\cdot)$ to be conditional expectation and conditional variance with respect to σ-algebra $\mathcal{F}_i$ in measure Q.

B. Problem statement

We restrict ourselves to self-financing hedging strategies in the sense that, at hedging times, T_k $1 \le k < n$, changes to positions in the money market asset are funded only by cash flows resulting from the trading of the risky asset. At maturity time T_n, we liquidate the risky asset to deliver the payoff of the ECC to the buyer. The hedging error, which is the discounted final money-market position of the seller of the ECC is then given by,

$$H(\Delta, \beta, V) = \sum_{i=1}^{n} \Delta_i (S_i - S_{i-1}) + \beta - V. \quad (2.1)$$

The hedging error (2.1) represents the profit or loss to the seller of the ECC, with positive values of H indicating profit. We seek hedging strategies $\Delta = (\Delta_1, \ldots, \Delta_k)$ such that the seller's risk represented by the second-moment or the variance of the hedging error is minimized. To this end, we assume a certain non-degeneracy condition on the hedging asset process and define a admissibility criterion on the set of trading strategies.

1) Non-degeneracy condition on the underlying asset price process: We assume the discounted hedging asset price process S_k to be $\mathcal{F}_k$-measurable for $k \in \{0, \ldots, n\}$ with the following non-degeneracy condition:

Assumption 1: (**Non-degeneracy condition**) (**c.f.** [7]) For each $k \in \{1, \ldots, n\}$, $S_k - S_{k-1} = M_{k-1} I_k$, where $M_{k-1} \in L^0(\Omega, \mathcal{F}_{k-1}, Q)$ and $I_k \in L^2(\Omega, \mathcal{F}, Q)$ are such that M_{k-1} is positive Q-almost surely and $\mathrm{var}_{k-1}^Q(I_k) > 0$.

Assumption 1 states that the increment in the discounted hedging asset price, across a hedging interval, has two factors. The measurability condition on the first factor implies that it is determined by observations available up to the current hedging instant. The positivity of the conditional variance of the second factor is a non-degeneracy assumption which represents the uncertainty in the hedging asset price increment not captured through observations up to and until the current hedging instant. The motivation for the factorization provided in Assumption 1 is that it subsumes two special cases that are listed below. The first of these is in terms of arithmetic returns on the discounted hedging asset prices defined by $R_k = \frac{S_k - S_{k-1}}{S_{k-1}}$ for each $k \in \{1, \ldots, n\}$ which is defined whenever S_{k-1} is non-zero Q-almost surely.

Assumption 1A: For each $k \in \{1, \ldots, n\}$, S_{k-1} is $\mathcal{F}_{k-1}$-measurable and non-zero Q-almost surely. Further, $R_k \in L^2(\Omega, \mathcal{F}, Q)$ with $\mathrm{var}_{k-1}^Q(R_k) > 0$, Q-almost surely.

Assumption 1B: For each $k \in \{1, \ldots, n\}$, $S_k - S_{k-1} \in L^2(\Omega, \mathcal{F}, Q)$ and $\mathrm{var}_{k-1}^Q(S_k - S_{k-1}) > 0$, Q-almost surely.

Assumption 1A implies that Assumption 1 holds with $M_{k-1} = S_{k-1}$ and $I_k = R_k$ for $k \in \{1, \ldots, n\}$, while Assumption 1B implies Assumption 1 holds with $M_{k-1} = 1$ and $I_k = S_k - S_{k-1}$ for $k \in \{1, \ldots, n\}$.

2) Admissibility of trading strategies: Represent a hedging strategy by an n-tuple $\Delta = (\Delta_1, \ldots \Delta_n)$ of random numbers on $(\Omega, \mathcal{F}, Q)$. In practice, any hedging decision taken at time T_k, $k \in \{0, \ldots, n-1\}$ can only be based on observations made up to time T_k. Hence it is natural to restrict ourselves to hedging strategies that are predictable in the sense that Δ_k is $\mathcal{F}_{k-1}$-measurable for each $k \in \{1, \ldots, n\}$. We will denote the set of all predictable hedging strategies by $\mathcal{P}$.

The discounted gain from trading accumulated up to time T_k, $k \in \{1, \ldots, n\}$, that accrues when the hedging strategy $\Delta \in \mathcal{P}$ is used, can be expressed as,

$$G_k(\Delta) \triangleq \sum_{i=1}^{k} \Delta_i (S_i - S_{i-1}) = \sum_{i=1}^{k} \Delta_i M_{i-1} I_i. \quad (2.2)$$

We adopt the convention that $G_0(\Delta) = 0$. The sequence $\{G_i(\Delta)\}_{i=0}^n$ is referred to as the gains process resulting from the hedging strategy Δ. We now define our criterion for admissibility as follows (**c.f.** [9]),

$$\Theta \triangleq \{\Delta \in \mathcal{P} : G_n(\Delta) \in L^2(\Omega, \mathcal{F}, Q)\}. \quad (2.3)$$

Observe that the set Θ is a non-trivial real vector space. The non-triviality follows because the strategy $(M_0^{-1}, \ldots, M_{n-1}^{-1})$ is contained in Θ under Assumption 1.

III. Comparison of problem formulation

The solution to a quadratic hedging problem depends on the integrability and measurability assumptions made on the underlying asset price process referred to as non-degeneracy conditions. While the assumption of square integrability on the hedging asset price process increments were used as the non-degeneracy (ND) condition in [6], [7], [12]–[14], references [9], [15] considered a certain no-arbitrage condition as the ND on the underlying asset price process. Additional assumptions on the asset price process like deterministic mean-variance trade-off [6], [13], bounded mean-variance trade off [7], positive variance, stationary and independent returns [12] were also made. The work of Melnikov and Nechaev in [8], has solved the minimum-variance hedging problem without imposing any non-degeneracy condition. However, we believe that some non-degeneracy condition is required to ensure uniqueness of optimal trading strategies. We further conjecture that in a set up that is as general as in [8], an optimization problem can have multiple trading strategies as solutions with all of these solutions having same trading gain. Although, having no ND condition should not be viewed as a disadvantage to a model description, a non-degeneracy assumption helps to restrict the class of asset price processes to meaningful ones [9]. Finally, we note that the current exposition is for hedging with single asset, whereas some previous works [9], [16]–[18] have considered the case of multiple hedging assets.

Previous literature also differ in the choice of the admissibility criteria for trading strategies and there are two popular choices. While, works in [8], [9], [15], like us, considered trading strategies whose final cumulative trading gain is square integrable as being admissible, references [6], [7], [12]–[14] considered trading strategies which yield square-integrable incremental discounted gain across each trading interval as being admissible. A rigorous definition of the latter can be stated as below (see [7]).

$$\hat{\Theta} \stackrel{\triangle}{=} \{\Delta \in \mathcal{P} : \Delta_k(S_k - S_{k-1}) \in L^2(\Omega, \mathcal{F}, Q)\}, \qquad (3.4)$$

for $k \in \{1, \ldots, n\}$. Note that a strategy $\Delta \in \mathcal{P}$ is contained in $\hat{\Theta}$ if and only if the incremental discounted gain that results at each trading time from applying the strategy is square integrable on $(\Omega, \mathcal{F}, Q)$. The criterion for admissibility given in the current work, which defines Θ in (2.3) requires only that the accumulated discounted gain up to T_n to be square integrable, and is thus weaker than the admissibility criterion in (3.4). In other words, $\hat{\Theta} \subseteq \Theta$. It is also worth while to note that $\hat{\Theta} = \Theta$ under the stronger assumption of bounded mean-variance trade off provided in [7]. One can therefore argue that the stronger bounded mean-variance trade off assumption is only needed to ensure that the solutions of the mean-variance hedging problems, which always exist in the bigger set Θ are actually contained in $\hat{\Theta}$.

IV. Mathematical Preliminaries

We begin this section with two useful results whose proofs are given in [19]. Thereafter, we introduce certain variables that will play a key role in the subsequent sections. We end this section with a result that provides a necessary and sufficient condition for a trading strategy to be admissible according to (2.3).

Lemma 4.1: Suppose $W \in L^2(\Omega, \mathcal{F}, Q)$, $Y \in L^0(\Omega, \mathcal{F}, Q)$ and $\mathcal{G} \subseteq \mathcal{F}$, a σ-algebra, are such that $\mathrm{var}^Q(W|\mathcal{G}) > 0$ and $0 < Y^2 \leq 1$. Then $\mathbb{E}^Q(W^2Y^2|\mathcal{G}) > 0$ and $0 < \mathbb{E}^Q(Y^2|\mathcal{G}) - [\mathbb{E}^Q(WY^2|\mathcal{G})]^2/\mathbb{E}^Q(W^2Y^2|\mathcal{G}) \leq 1$.

Let Z be a random variable of the form $Z = X_1Y_1 + \cdots + X_lY_l$. Clearly, if each of the products X_iY_i is integrable, then Z is integrable. Lemma 4.2 provides a weaker sufficiency condition for integrability of Z which will be used subsequently in this exposition.

Lemma 4.2: Suppose $W_i \in L^1(\Omega, \mathcal{F}, Q)$ and $Y_i \in L^0(\Omega, \mathcal{G}, Q)$ for every $i \in \{1, \ldots, l\}$, where $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra. Let $Z = W_1Y_1 + \cdots + W_lY_l$, and $Z' = \mathbb{E}^Q(W_1|\mathcal{G})Y_1 + \cdots + \mathbb{E}^Q(W_l|\mathcal{G})Y_l$. Then the following statements hold.

1) If $Z \in L^1(\Omega, \mathcal{F}, Q)$, then $Z' \in L^1(\Omega, \mathcal{F}, Q)$ and $\mathbb{E}^Q(Z|\mathcal{G}) = Z'$.
2) If Z is nonnegative and $Z' \in L^1(\Omega, \mathcal{F}, Q)$, then $Z \in L^1(\Omega, \mathcal{F}, Q)$, and $\mathbb{E}^Q(Z|\mathcal{G}) = Z'$.

Next, we define a sequence of random variables $\{U_i\}_{i=0}^n$ in the following way. Assuming that, for some $i \in \{1, \ldots, n\}$, we have defined U_i satisfying $0 < U_i \leq 1$, define U_{i-1} by

$$U_{i-1} = \sqrt{\mathbb{E}^Q_{i-1}(U_i^2) - \frac{[\mathbb{E}^Q_{i-1}(U_i^2 I_i)]^2}{\mathbb{E}^Q_{i-1}(U_i^2)I_i^2}}. \qquad (4.5)$$

Applying Lemma 4.1 under Assumption 1 with $Y = U_i$ and $W = I_i$ implies that the right hand side of (4.5) is well defined and real-valued, and takes values in the interval $(0, 1]$. Our definition of the random variables $U_0, \ldots, U_n$ is now completed by letting $U_n = 1$. Note that $U_i \in L^2(\Omega, \mathcal{F}_i, Q)$ for each $i \in \{0, \ldots, n\}$.

Next, for each $i \in \{1, \ldots, n\}$, define a measure Q_i on $(\Omega, \mathcal{F})$ by letting

$$\frac{\mathrm{d}Q_i}{\mathrm{d}Q} = \frac{U_i^2}{\mathbb{E}^Q(U_i^2)}. \qquad (4.6)$$

It is easy to see that $Q_n = Q$. Since $U_i > 0$, it follows that each Q_i is equivalent to Q, for $i \in \{1, \cdots, n\}$. Since $U_i \leq 1$, it also follows that $L^j(\Omega, \mathcal{F}, Q) \subseteq L^j(\Omega, \mathcal{F}, Q_i)$ for each $j \in \{1, 2\}$ and $i \in \{1, \ldots, n\}$. In particular, by Assumption 1, $I_i \in L^2(\Omega, \mathcal{F}, Q_i)$ for each $i \in \{1, \ldots, n\}$. Finally, we note that, given $i \in \{1, \ldots, n\}$, $W \in L^2(\Omega, \mathcal{F}, Q_i)$ if and only if $U_iW \in L^2(\Omega, \mathcal{F}, Q)$.

Next, for each $i \in \{1, \ldots, n\}$, we introduce

$$\mu_i = M_{i-1}\mathbb{E}^Q_{i-1}(U_i^2 I_i), \;\; \nu_i = M_{i-1}\sqrt{\mathbb{E}^Q_{i-1}(U_i^2 I_i^2)}. \qquad (4.7)$$

Note that $\nu_i > 0$ for each $i \in \{1, \ldots, n\}$ by Lemma 4.1 and our assumption on M_{i-1}. Change of measure in conditional expectations allows us to rewrite (4.7) as

$$\begin{aligned} \mu_i &= M_{i-1}\mathbb{E}^{Q_i}_{i-1}(I_i)\mathbb{E}^Q_{i-1}(U_i^2), \\ \nu_i &= M_{i-1}\sqrt{\mathbb{E}^{Q_i}_{i-1}(I_i^2)\mathbb{E}^Q_{i-1}(U_i^2)}. \end{aligned} \qquad (4.8)$$

Finally, (4.7) and (4.8) enable us to rewrite (4.5) as

$$U_{i-1}^2 = \mathbb{E}^Q_{i-1}(U_i^2) - \frac{\mu_i^2}{\nu_i^2} \qquad (4.9)$$

For later use, we state a useful identity, which follow from (4.9). For each $i \in \{1, \dots, n-1\}$, the following equality holds,

$$U_i^2 = 1 - \sum_{j=i+1}^{n} \mathbb{E}_i^Q \left(\frac{\mu_j^2}{\nu_j^2} \right). \quad (4.10)$$

We next give a useful characterization of admissible strategies in the form of the following lemma. The proof of this Lemma is given in the appendix.

Lemma 4.3: Suppose $\Delta \in \mathcal{P}$ and $i \in \{1, \dots, n\}$. If $\Delta \in \Theta$, then we have $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$, and

$$\begin{aligned} \mathbb{E}_{i-1}^Q[U_i^2 G_i^2(\Delta)] &= U_{i-1}^2 G_{i-1}^2(\Delta) \\ &+ \nu_i^2 \left(\Delta_i + \frac{\mu_i}{\nu_i^2} G_{i-1}(\Delta) \right)^2, \quad (4.11) \\ \mathbb{E}_{i-1}^Q[U_i^2 G_i(\Delta)] &= U_{i-1}^2 G_{i-1}(\Delta) \\ &+ \mu_i \left(\Delta_i + \frac{\mu_i}{\nu_i^2} G_{i-1}(\Delta) \right). \quad (4.12) \end{aligned}$$

Furthermore, $\Delta \in \Theta$ if and only if $\nu_i \Delta_i + \frac{\mu_i}{\nu_i} G_{i-1}(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$ for each $i \in \{1, \dots, n\}$.

V. Moments of the hedging error

We now turn our attention to finding expressions for the mean, variance and second-moment of the hedging error of an arbitrary trading strategy that is admissible in the sense of (2.3). Our approach relies on algebraic manipulations of conditional expectations which are rigorously justified using Lemma 4.2. In the next section, we will use these expressions to obtain solutions of the hedging problems.

We begin by introducing the function $F_i : L^2(\Omega, \mathcal{F}, Q_i) \to L^0(\Omega, \mathcal{F}_{i-1}, Q)$, for each $i \in \{1, \dots, n\}$, by

$$F_i(X) = \mathbb{E}_{i-1}^{Q_i}(X) - \mathbb{E}_{i-1}^{Q_i}(I_i) \frac{\text{covar}_{i-1}^{Q_i}(I_i, X)}{\text{var}_{i-1}^{Q_i}(I_i)}. \quad (5.13)$$

Note that the right hand side of (5.13) is defined Q-almost everywhere since, by Assumption 1, the conditional variance appearing on the right hand side is positive Q-almost surely. One can show that the function F_i is a map from $L^2(\Omega, \mathcal{F}, Q_i)$ to $L^2(\Omega, \mathcal{F}_{i-1}, Q_{i-1})$ for every $i \in \{1, \dots, n\}$ [19].

Define a collection of functions X_i, $i \in \{0, \dots, n\}$ on $L^2(\Omega, \mathcal{F}, Q)$ by first setting $X_n(Z) = Z$ for all $Z \in L^2(\Omega, \mathcal{F}, Q)$, and then recursively define $X_{i-1} = F_i \circ X_i$. Each X_i is a map from $L^2(\Omega, \mathcal{F}, Q)$ to $L^2(\Omega, \mathcal{F}_i, Q_i)$. Next, given $i \in \{0, \dots, n\}$, $\Delta \in \mathcal{P}$, $\beta \in \mathbb{R}$, and $V \in L^2(\Omega, \mathcal{F}, Q)$, define

$$H_i(\Delta, \beta, V) = G_i(\Delta) + \beta - X_i(V). \quad (5.14)$$

Observe that H_n is same as the hedging error H of (2.1). Given $\Delta \in \mathcal{P}$, $i \in \{1, \cdots, n\}$, $\beta \in \mathbb{R}$, and $V \in L^2(\Omega, \mathcal{F}, Q)$, define $\eta_i(\Delta, \beta, V) \in L^0(\Omega, \mathcal{F}_{i-1}, Q)$ by

$$\begin{aligned} \eta_i(\Delta, \beta, V) &= -\frac{\mu_i}{\nu_i^2}[G_{i-1}(\Delta) + \beta] \\ &+ \frac{M_{i-1}}{\nu_i^2} \mathbb{E}_{i-1}^Q(U_i^2 I_i X_i(V)). \quad (5.15) \end{aligned}$$

Note that the conditional expectations in (5.15) exist because both $U_i X_i(V)$ and $U_i I_i$ are square integrable on $(\Omega, \mathcal{F}, Q)$, and the Cauchy-Schwartz inequality implies that $U_i^2 I_i X_i(V) \in L^1(\Omega, \mathcal{F}, Q)$. For later use, we observe that, if $i \in \{1, \dots, n\}$ and $\beta_1, \beta_2 \in \mathbb{R}$, then

$$\eta_i(\Delta, \beta_1, V) + \frac{\mu_i}{\nu_i^2}(\beta_1 - \beta_2) = \eta_i(\Delta, \beta_2, V). \quad (5.16)$$

Our next result gives a convenient alternative expression to (5.15) in addition to the integrability properties that we will need soon. The proof is given in appendix.

Lemma 5.1: Suppose $V \in L^2(\Omega, \mathcal{F}, Q)$, $i \in \{1, \dots, n\}$, and $\Delta \in \Theta$. Then we have $\nu_i(\Delta_i - \eta_i(\Delta, \beta, V)) \in L^2(\Omega, \mathcal{F}, Q)$ and $\mu_i(\Delta_i - \eta_i(\Delta, \beta, V)) \in L^2(\Omega, \mathcal{F}, Q)$. Moreover,

$$\eta_i(\Delta, \beta, V) = -\frac{\mu_i}{\nu_i^2} H_{i-1}(\Delta, \beta, V) + \frac{\text{covar}_{i-1}^{Q_i}(I_i, X_i(V))}{M_{i-1} \text{var}_{i-1}^{Q_i}(I_i)}. \quad (5.17)$$

We now state two key results in the form a Lemma and Theorem that will enable us to obtain expressions for the mean, variance and second moment of the hedging error resulting from a given admissible hedging strategy. While the proof of the Lemma, provided in the appendix, makes use of Lemma 4.2, the proof of the Theorem 5.3, provided below, makes use of Lemma 5.2 in a recursive manner.

Proof of Theorem 5.3 Recall that $H(\Delta, \beta, V) = H_n(\Delta, \beta, V)$, while $U_n = 1$. Hence applying (5.18) with $i = n$ gives

$$\begin{aligned} \mathbb{E}_{n-1}^Q[H(\Delta, \beta, V)] &= U_{n-1}^2 H_{n-1}(\Delta, \beta, V) \\ &+ \mu_n[\Delta_n - \eta_n(\Delta, \beta, V)]. \end{aligned}$$

Now suppose that, for some $j \in \{1, \dots, n-1\}$, the following holds.

$$\begin{aligned} \mathbb{E}_{n-j}^Q[H(\Delta, \beta, V)] &= U_{n-j}^2 H_{n-j}(\Delta, \beta, V) \\ &+ \sum_{k=n-j+1}^{n} \mathbb{E}_{n-j}^Q \left[\mu_k \{ \Delta_k - \eta_k(\Delta, \beta, V) \} \right]. \quad (5.24) \end{aligned}$$

Lemmas 5.1 and 5.2 imply that all terms on the right hand side of the above equation are integrable on $(\Omega, \mathcal{F}, Q)$. Taking the conditional expectation $\mathbb{E}_{n-j-1}^Q(\cdot)$ on both the sides of the above equation and using (5.18) with $i = n - j$ yields

$$\begin{aligned} \mathbb{E}_{n-j-1}^Q[H(\Delta, \beta, V)] &= U_{n-j-1}^2 H_{n-j-1}(\Delta, \beta, V) \\ &+ \sum_{k=n-j}^{n} \mathbb{E}_{n-j-1}^Q \left[\mu_k \{ \Delta_k - \eta_k(\Delta, \beta, V) \} \right] \quad (5.25) \end{aligned}$$

that is, (5.24) holds with j replaced by $j - 1$. Noting from (5.14) that $H_0(\Delta, \beta, V) = \beta - X_0(V)$ and applying induction yields (5.19). A similar induction argument based on (5.18) yields (5.21).

To prove (5.20), we use (4.10) with $i = 0$ to write (5.19) as

$$\sum_{i=1}^{n} \mathbb{E}^Q \left[\mu_i \left\{ \Delta_i - \eta_i(\Delta, \beta, V) - \frac{\mu_i}{\nu_i^2}(\beta - X_0(V)) \right\} \right] + (\beta - X_0(V)).$$

Lemma 5.2: Suppose $\Delta \in \Theta$, $V \in L^2(\Omega, \mathcal{F}, Q)$, $\beta \in \mathbb{R}$, and $i \in \{1, \ldots, n\}$. Then, $U_i H_i(\Delta, \beta, V) \in L^2(\Omega, \mathcal{F}, Q)$, and

$$\begin{aligned}
\mathbb{E}^Q_{i-1}[U_i^2 H_i^2(\Delta,\beta,V)] &= U_{i-1}^2 H_{i-1}^2(\Delta,\beta,V) + \nu_i^2[\Delta_i - \eta_i(\Delta,\beta,V)]^2 \\
&+ \mathbb{E}^Q_{i-1}(U_i^2)\left[\mathrm{var}^{Q_i}_{i-1}(X_i(V)) - \frac{\mathrm{covar}^{Q_i}_{i-1}(I_i, X_i(V))^2}{\mathrm{var}^{Q_i}_{i-1}(I_i)}\right] \\
\mathbb{E}^Q_{i-1}[U_i^2 H_i(\Delta,\beta,V)] &= U_{i-1}^2 H_{i-1}(\Delta,\beta,V) + \mu_i[\Delta_i - \eta_i(\Delta,\beta,V)]. \qquad (5.18)
\end{aligned}$$

Equation (5.20) now follows on using (5.16) with $\beta_1 = \beta$ and $\beta_2 = X_0(V)$.

Next, on using (5.16) with $\beta_1 = \beta$ and $\beta_2 = X_0(V)$ to substitute for $\eta_i(\Delta, \beta, V)$ in terms of $\eta_i(\Delta, X_0(V), V)$, we have $\sum_{i=1}^n \mathbb{E}^Q[\nu_i^2\{\Delta_i - \eta_i(\Delta,\beta,V)\}^2]$ equals

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}^Q[\nu_i^2\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}^2] \\
&\quad + [\beta - X_0(V)]^2 \sum_{i=1}^n \mathbb{E}^Q(\mu_i^2/\nu_i^2) \\
&\quad + 2[\beta - X_0(V)] \sum_{i=1}^n \mathbb{E}^Q\left[\mu_i\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}\right].
\end{aligned}$$

Using 4.10 with $i = 0$ to replace the last summation above and using the resulting expression in (5.21) yields $\mathbb{E}^Q[H^2(\Delta,\beta,V)]$ as

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E}^Q[\nu_i^2\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}^2] + [\beta - X_0(V)]^2 + \sigma_0^2 \\
&+ \quad 2[\beta - X_0(V)] \sum_{i=1}^n \mathbb{E}^Q[\mu_i\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}].
\end{aligned}$$

Squaring (5.20) and subtracting from the last equation gives (5.22). □

VI. Optimal Hedging Strategies

In this section, we use the expressions derived in the previous section to obtain solutions of various optimal hedging problems. We will soon see that all optimal strategies involve the function given by $\Delta^* : \mathbb{R} \times L^2(\Omega, \mathcal{F}, Q) \to \mathcal{P}$ defined by

$$\begin{aligned}
\Delta_i^*(\beta, V) &= -\frac{\mu_i}{\nu_i^2}[G_{i-1}(\Delta^*(\beta,V)) + \beta] \\
&+ \frac{M_{i-1}}{\nu_i^2}\mathbb{E}^Q_{i-1}(U_i^2 I_i X_i(V)), \qquad (6.26)
\end{aligned}$$

for each $i \in \{1, \ldots, n\}$.

Remark One can show that the function $\Delta_i^*(\beta, V)$ given by equation (6.26) does not depend on the factorization in Assumption 1. To this end, we first argue that the variables $\{U_i\}_{i=1}^n, \{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ do not depend on the said factorization. Let the asset price increments have an alternate factorization given by, $S_i - S_{i-1} = \hat{M}_{i-1}\hat{I}_i$ for each i, with $\hat{M}_{i-1}$ and $\hat{I}_i$ satisfying the conditions in Assumption 1. Define random variables $\{\hat{U}_i\}_{i=1}^n$ for the new factorization using (4.5). If

Theorem 5.3: Suppose $\Delta \in \theta$, $\beta \in \mathbb{R}$ and $V \in L^2(\Omega, \mathcal{F}, Q)$. Then

$$\begin{aligned}
\mathbb{E}^Q[H(\Delta,\beta,V)] &= \sum_{i=1}^n \mathbb{E}^Q[\mu_i\{\Delta_i - \eta_i(\Delta,\beta,V)\}] + U_0^2[\beta - X_0(V)] \qquad (5.19) \\
&= \sum_{i=1}^n \mathbb{E}^Q[\mu_i\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}] + (\beta - X_0(V)) \qquad (5.20) \\
\mathbb{E}^Q[H^2(\Delta,\beta,V)] &= \sum_{i=1}^n \mathbb{E}^Q[\nu_i^2\{\Delta_i - \eta_i(\Delta,\beta,V)\}^2] + U_0^2[\beta - X_0(V)]^2 + \sigma_0^2 \qquad (5.21) \\
\mathrm{var}^Q(H(\Delta,\beta,V)) &= \sum_{i=1}^n \mathbb{E}^Q[\nu_i^2\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}^2] \\
&- \left[\sum_{i=1}^n \mathbb{E}^Q[\mu_i\{\Delta_i - \eta_i(\Delta, X_0(V), V)\}]\right]^2 + \sigma_0^2 \qquad (5.22)
\end{aligned}$$

where

$$\sigma_0^2 = \sum_{i=1}^n \mathbb{E}^Q\left[\mathbb{E}^Q_{i-1}(U_i^2)\left\{\mathrm{var}^{Q_i}_{i-1}(X_i(V)) - \frac{\{\mathrm{covar}^{Q_i}_{i-1}(I_i, X_i(V))\}^2}{\mathrm{var}^{Q_i}_{i-1}(I_i)}\right\}\right]. \qquad (5.23)$$

TABLE I

SOLUTION FOR VARIOUS QUADRATIC HEDGING PROBLEMS

Quantities	Problems			
	1	1a	2	2a
Δ	$\Delta^*(X_0(V),V)$	$\Delta^*(\beta_0,V)$	$\Delta^*(X_0(V),V)$	$\Delta^*(X_0(V),V)$
β	$X_0(V)$	β_0	undetermined	β_0
$\mathbb{E}^Q[H(\Delta,\beta,V)]$	0	$U_0^2[\beta_0-X_0(V)]$	undetermined	$[\beta_0-X_0(V)]$
$\mathbb{E}^Q[H^2(\Delta,\beta,V)]-\sigma_0^2$	0	$U_0^2[\beta_0-X_0(V)]^2$	undetermined	$[\beta_0-X_0(V)]^2$
$\text{var}^Q(H(\Delta,\beta,V))-\sigma_0^2$	0	$U_0^2(1-U_0^2)[\beta_0-X_0(V)]^2$	0	0

$U_i=\hat{U}_i$ for some i, then $\hat{I}_i=M_{i-1}I_i/\hat{M}_{i-1}$. From Lemma 4.2, we have, $\mathbb{E}^Q_{i-1}(U_i^2\hat{I}_i^2)=(M_{i-1}/\hat{M}_{i-1})^2\mathbb{E}^Q_{i-1}(U_i^2I_i^2)$ and $\mathbb{E}^Q_{i-1}(U_i^2\hat{I}_i)=(M_{i-1}/\hat{M}_{i-1})\mathbb{E}^Q_{i-1}(U_i^2I_i)$, so that (4.5) yields $\hat{U}_{i-1}=U_{i-1}$. This implies $\hat{U}_i=U_i$ for all i, since $\hat{U}_n=U_n=1$. A similar reasoning can be put forth to show that the variables $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ are independent of the factorization. Next, observe that the term $G_{i-1}(\Delta^*(\beta,V))$ is a sum of asset price increments across trading intervals which is independent of the factorization. Lastly, the second term on the right hand side of equation (6.26) involves the product $M_{i-1}I_i$ which again is independent of the factorization.

One can show that the function Δ^* is a linear function from $\mathbb{R}\times L^2(\Omega,\mathcal{F},Q)$ to Θ and actually yields an admissible strategy for each initial investment $\beta\in\mathbb{R}$ and a payoff $V\in L^2(\Omega,\mathcal{F},Q)$. Moreover, for every $\beta\in\mathbb{R}$ and $V\in L^2(\Omega,\mathcal{F},Q)$, $\Delta^*(\beta,V)$ is the unique hedging strategy in $\mathcal{P}$ satisfying $\Delta_i=\eta_i(\Delta,\beta,V)$ for all $i\in\{1,\ldots,n\}$ [19].

The fact that $\Delta^*(\beta,V)$ satisfies $\Delta_i=\eta_i(\Delta,\beta,V)$ for all $i\in\{1,\ldots,n\}$ along with (5.17) yields the alternative expression

$$\Delta_i^*(\beta,V) = -\frac{\mu_i}{\nu_i^2}H_{i-1}(\Delta_i^*(\beta,V),\beta,V) + \frac{\text{covar}^{Q_i}_{i-1}(I_i,X_i(V))}{M_{i-1}\text{var}^{Q_i}_{i-1}(I_i)}, \quad (6.27)$$

for each $i\in\{1,\ldots,n\}$. In the rest of this section, we present solutions to problems which do not involve any constraints on the hedging strategy. For convenience we first list out the problems we solve.

1 Find the admissible trading strategy Δ and the initial investment β that minimizes the second-moment of the hedging error

1a Find admissible trading strategy Δ that minimizes the second-moment of the hedging error for a given initial investment β_0

2 Find admissible trading strategy Δ and initial investment β that minimizes the variance of the hedging error

2a Find admissible trading strategy Δ that minimizes the variance of the hedging error for a given initial investment β_0

Proposition 6.1: Suppose $V\in L^2(\Omega,\mathcal{F},Q)$, and $\beta_0\in\mathbb{R}$. Then, under Assumption 1, the unique solutions to the problems listed above as well as the corresponding values for the second moment, mean, and the variance of the hedging error are as given in Table I.

Proof. First, note that the constraint 1a fixes β to be β_0. The solution to problem 1a becomes evident on inspecting (5.21), and recalling that $\Delta^*(\beta,V)$ is the unique hedging strategy in Θ satisfying $\Delta_i=\eta_i(\Delta,\beta,V)$ for all $i\in\{1,\ldots,n\}$. The expressions for the mean and second moment of the hedging error follow from (5.19) and (5.21), respectively, while the variance is directly computed as $\mathbb{E}^Q[H^2(\Delta,\beta,V)]-[\mathbb{E}^Q[H(\Delta,\beta,V)]^2$.

The solution to problem 1, its uniqueness and the resulting second moment of the hedging error readily follow from observing (5.21) and fixing $\beta=X_0(V)$. The expression for the resulting hedging error variance follows from substituting $\Delta_i=\eta_i(\Delta,\beta,V)$ for all $i\in\{1,\ldots,n\}$ into (5.22), while $\mathbb{E}^Q[H(\Delta,\beta,V)]=0$ follows by letting $\beta=X_0(V)$ in the expression (5.19).

Recall that from equation (5.22) and that $\Delta^*(\cdot)$ is a unique hedging strategy in $\mathcal{P}$ satisfying $\Delta_i=\eta_i(\Delta,\beta,V)$ for all $i\in\{1,\ldots,n\}$ imply that $\text{var}^Q(H(\Delta,\beta,V))=\sigma_0^2$ for $\Delta=\Delta^*(X_0(V),V)$, while it can be shown that $\text{var}^Q(H(\Delta,\beta,V))\geq\sigma_0^2$ for all $\Delta\in\Theta$. It follows that the minimum in problem 2 is achieved by $\Delta=\Delta^*(X_0(V),V)$. Since $\text{var}^Q(H(\Delta,\beta,V))$ does not depend on β, the optimization in problem 2 does not determine β. Consequently, the mean and second moment of the hedging error resulting from the optimal strategy remains undetermined.

Since the variance of the hedging error does not depend on β, the solution to problem 2a is the same as the solution to problem 2. The expression for the mean of the hedging error follows by substituting $\Delta=\Delta^*(X_0(V),V)$ and $\beta=\beta_0$ in (5.20), while the expression for the second moment of the hedging error follows by using $\mathbb{E}^Q[H^2(\Delta,\beta,V)]=\text{var}^Q(H(\Delta,\beta,V))+[\mathbb{E}^Q[H(\Delta,\beta,V)]]^2$. □

VII. SUMMARY AND OUTLOOK

In this paper, we considered the problem of discrete-time quadratic optimal hedging of an ECC with a portfolio consisting of a risky asset. Specifically, optimal hedging strategies were obtained that minimize the second-moment and the variance of the profit and loss to the seller of the ECC. The problem setup differs from previous formulations by having weaker non-degeneracy assumptions on the hedging asset price process and a broader admissibility criterion for trading strategies. The solution approach relies mainly on manipulations of conditional expectations and sums of squares. We derived analytical expressions for second-order moments of the hedging error and used them to arrive at optimal trading strategies. A longer version of the work containing derivation of variance optimal martingale measures, mean-variance frontier determination and connections to previous literature is available at [19]

As noted in the introduction, the solution to the discrete-time hedging problems depend on the non-degeneracy assumptions on the underlying asset price process. In this context, it is worthwhile to identify the weakest non-degeneracy condition required for solving such hedging problems and eventually examine the need for any sort of non-degeneracy assumption. Such analysis will more shed light into the problem setting such as the one in [8]. It may also be useful to characterize equivalent non-degeneracy conditions for the multi-asset case involving covariances of different underlying asset prices and examine their role in solving the hedging problem. We end by noting that the solutions of various quadratic hedging problems presented in Section VI involve computationally intensive Monte-Carlo simulations. However, when underlying asset price process follows a definite model like GBM, it may be possible to arrive at closed form expressions for hedging problems. For example, closed form solutions to hedging problems for simple ECCs [20], [21] and path-dependent ECCs [22] are available for the GBM case.

VIII. Appendix

Proof of Lemma 4.3 Note that the right hand side of (4.11) is well defined since $\nu_i > 0$. For the first assertion, suppose $\Delta \in \Theta$. Then $U_n G_n(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$. To set up an induction argument, pick $i \in \{2, \ldots, n\}$, and suppose we have shown that $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$.

We may use (2.2) to expand $U_i^2 G_i^2(\Delta)$ into a sum $W_1 Y_1 + W_2 Y_2 + W_3 Y_3$, with $W_1 = U_i^2$, $Y_1 = G_{i-1}^2(\Delta)$, $W_2 = U_i^2 I_i$, $Y_2 = 2\Delta_i M_{i-1} G_{i-1}(\Delta)$, $W_3 = U_i^2 I_i^2$, and $Y_3 = \Delta_i^2 M_{i-1}^2$ Note that each $W_j, j \in \{1,2,3\}$ is integrable on $(\Omega, \mathcal{F}, Q)$, while each $Y_j, j \in \{1,2,3\}$ is $\mathcal{F}_{i-1}$-measurable. Assumption 1 implies that part 1) of Lemma 4.2 applies with $Z = U_i^2 G_i^2(\Delta)$. Hence, we have $\mathbb{E}_{i-1}^Q(U_i^2 G_i^2(\Delta)) = \sum_{j=1}^3 \mathbb{E}^Q(W_j|\mathcal{F}_{i-1}) Y_j = \mathbb{E}_{i-1}^Q(U_i^2) G_{i-1}^2(\Delta) + 2\mu_i \Delta_i G_{i-1}(\Delta) + \nu_i^2 \Delta_i^2$, where we have used (4.7). Using (4.9) to express $\mathbb{E}_{i-1}^Q(U_i^2)$ in terms of U_{i-1}^2, μ_i and ν_i, and rearranging terms yields (4.11). Equation (4.11) implies that $U_{i-1}^2 G_{i-1}^2(\Delta)$ is dominated by the integrable random variable $\mathbb{E}_{i-1}^Q[U_i^2 G_i^2(\Delta)]$, and hence is integrable on $(\Omega, \mathcal{F}, Q)$. Induction now completes the proof of the assertion on integrability, as well as that of (4.11).

To prove (4.12), pick $i \in \{1, \ldots, n\}$, and observe that $U_i^2 G_i(\Delta) = U_i^2 G_{i-1}(\Delta) + \Delta_i M_{i-1} U_i^2 I_i$. $U_i^2 G_i(\Delta)$ is integrable by Lemma 4.3. Therefore, by Lemma 4.2, $\mathbb{E}_{i-1}^Q(U_i^2 G_i(\Delta)) = \mathbb{E}_{i-1}^Q(U_i^2) G_{i-1}(\Delta) + \mu_i \Delta_i$. Substituting for $\mathbb{E}_{i-1}^Q(U_i^2)$ from (4.9) yields (4.12).

To prove necessity in the second assertion, suppose $\Delta \in \Theta$, and choose $i \in \{1, \ldots, n\}$. Then, by the first assertion, (4.11) holds, and $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$. Equation (4.11) implies that $(\nu_i \Delta_i + \frac{\mu_i}{\nu_i} G_{i-1}(\Delta))^2$ is dominated by the integrable random variable $\mathbb{E}_{i-1}^Q[U_i^2 G_i^2(\Delta)]$, and hence is integrable on $(\Omega, \mathcal{F}, Q)$. To prove sufficiency, consider $\Delta \in \mathcal{P}$. For each $i \in \{1, \ldots, n\}$, denote $N_i = \left(\nu_i \, \Delta_i \, + \frac{\mu_i}{\nu_i} \, G_{i-1}(\Delta)\right)$ and suppose $N_i \in L^2(\Omega, \mathcal{F}, Q)$ for $i \in \{1, \cdots, n\}$. The proof proceeds by induction on i. Observe that $U_1 G_1(\Delta) = U_1 \Delta_1 M_0 I_1$, which is square integrable on $(\Omega, \mathcal{F}, Q)$ since Δ_1 and M_0 are deterministic and $I_1 \in L^2(\Omega, \mathcal{F}, Q)$. Now suppose that $U_{i-1} G_{i-1} \in L^2(\Omega, \mathcal{F}, Q)$ for some $i \in \{2, \ldots, n\}$. Recall that we may expand $U_i^2 G_i^2(\Delta) = W_1 Y_1 + W_2 Y_2 + W_3 Y_3$, where $W_j \in L^1(\Omega, \mathcal{F}, Q)$ and $Y_j \in L^0(\Omega, \mathcal{F}_{i-1})$, $j \in \{1,2,3\}$, are as chosen earlier in the proof. An easy calculation yields $\sum_{j=1}^3 \mathbb{E}^Q(W_j|\mathcal{F}_{i-1}) Y_j = U_{i-1}^2 G_{i-1}^2(\Delta) + N_i^2$, which is integrable by the induction hypothesis and our assumption on N_i. Since $U_i^2 G_i^2(\Delta)$ is nonnegative, part 2) of Lemma 4.2 implies that $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$. It follows by induction that $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$ for all $i \in \{1, \ldots, n\}$. In particular, we have, $U_n G_n(\Delta) = G_n(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$. Thus $\Delta \in \Theta$, and sufficiency follows. □

Proof of Lemma 5.1 Substituting for $\eta_i(\Delta, \beta, V)$ from (5.15) yields

$$\begin{aligned}\nu_i(\Delta_i - \eta_i(\Delta, \beta, V)) &= [\nu_i \Delta_i + \frac{\mu_i}{\nu_i} G_{i-1}(\Delta)] - \frac{\mu_i}{\nu_i}\beta \\ &+ \frac{M_{i-1}}{\nu_i} \mathbb{E}_{i-1}^Q(U_i^2 I_i X_i(V)). \quad (8.28)\end{aligned}$$

The first term on the right hand side of (8.28) is in $L^2(\Omega, \mathcal{F}, Q)$ by Lemma 4.3, since $\Delta \in \Theta$. We can see from (4.8) that $\mu_i^2/\nu_i^2 \le 1$, and hence the second term $(\mu_i/\nu_i)\beta \in L^2(\Omega, \mathcal{F}, Q)$. On using the definition of ν_i from (4.7) and the Cauchy-Schwartz inequality, we find that the square of the third term on the right hand side of (8.28) is bounded above by $\mathbb{E}_{i-1}^Q[U_i^2 X_i^2(V)]$. Since $X_i(V) \in L^2(\Omega, \mathcal{F}, Q_i)$, we have $U_i^2 X_i^2(V) \in L^1(\Omega, \mathcal{F}, Q)$, and hence $\mathbb{E}_{i-1}^Q[U_i^2 X_i^2(V)] \in L^1(\Omega, \mathcal{F}, Q)$ and $\frac{M_{i-1}}{\nu_i}\mathbb{E}_{i-1}^Q(U_i^2 I_i X_i(V)) \in L^2(\Omega, \mathcal{F}, Q)$. This proves the first inclusion. The second inclusion follows from the first by noting from (4.8) that $\mu_i^2 \le \nu_i^2$ Q-almost surely.

Next, let Z denote the right hand side of (5.17). On substituting for H_{i-1} from (5.14) into the right hand side of (5.17), we get

$$\begin{aligned}Z + \frac{\mu_i}{\nu_i^2}[G_{i-1}(\Delta) + \beta] &= \frac{\mu_i}{\nu_i^2} X_{i-1}(V) + \frac{\mathrm{covar}_{i-1}^{Q_i}(I_i, X_i(V))}{M_{i-1}\mathrm{var}_{i-1}^{Q_i}(I_i)} \\ &= \frac{1}{M_{i-1}}\left[\frac{\mathbb{E}_{i-1}^{Q_i}(I_i)\mathbb{E}_{i-1}^{Q_i}[X_i(V)] + \mathrm{covar}_{i-1}^{Q_i}(I_i, X_i(V))}{\mathbb{E}_{i-1}^{Q_i}(I_i^2)}\right] \\ &= \frac{1}{M_{i-1}} \frac{\mathbb{E}_{i-1}^{Q_i}(I_i X_i(V))}{\mathbb{E}_{i-1}^{Q_i}(I_i^2)} \\ &= \frac{1}{M_{i-1}} \frac{\mathbb{E}_{i-1}^{Q}(U_i^2 I_i X_i(V))}{\mathbb{E}_{i-1}^{Q}(U_i^2 I_i^2)} = \frac{M_{i-1}}{\nu_i^2}\mathbb{E}_{i-1}^Q(U_i^2 I_i X_i(V)),\end{aligned}$$

where the second equality uses (5.13), and the last equality use (4.7). A comparison of (8.29) with (5.15) shows that $Z = \eta_i(\Delta, \beta, V)$, that is, (5.17) holds. □

Proof of Lemma 5.2 Note that $U_i G_i(\Delta) \in L^2(\Omega, \mathcal{F}, Q)$ by Lemma 4.3, $\beta U_i \in L^2(\Omega, \mathcal{F}, Q)$ by boundedness, and

$U_iX_i(V) \in L^2(\Omega, \mathcal{F}, Q)$ by construction of X_i. Hence, by (5.14), $U_iH_i(\Delta, \beta, V) = U_iG_i(\Delta) + \beta U_i - U_iX_i(V) \in L^2(\Omega, \mathcal{F}, Q)$. We note that $U_iH_i(\Delta, \beta, V)$ can be written as,

$$\begin{aligned} U_iH_i(\Delta, \beta, V) &= \underbrace{U_iI_i}_{W_1}\underbrace{[\Delta_i - \eta_i(\Delta, \beta, V)]M_{i-1}}_{Y_1} \\ &+ \underbrace{U_i\left(1 - \frac{\mu_i}{\nu_i^2}M_{i-1}I_i\right)}_{W_2}\underbrace{H_{i-1}(\Delta, \beta, V)}_{Y_2} \\ &+ \underbrace{U_i[\mathbb{E}^{Q_i}_{i-1}[X_i(V)] - X_i(V)]}_{W_3} \\ &+ \underbrace{U_i[I_i - \mathbb{E}^{Q_i}_{i-1}(I_i)]}_{W_4}\underbrace{\frac{\text{covar}^{Q_i}_{i-1}(I_i, X_i(V))}{\text{var}^{Q_i}_{i-1}(I_i)}}_{Y_4}. \end{aligned}$$

From the right hand side of equation (8.29), one can use algebraic manipulations involving expressions for $H_i(\Delta, \beta, V)$ and $H_{i-1}(\Delta, \beta, V)$ from (5.14), $\eta_i(\Delta, \beta, V)$ from (5.17) and the relationship between X_i and X_{i-1} from (5.13) to arrive at the left hand side of (8.29).

We observe that $U_iH_i(\Delta, \beta, V)$ is a sum $\sum_{j=1}^4 W_jY_j$, where $Y_3 = 1$, and Y_1, Y_2, Y_4 and W_j, $j \in \{1, \ldots, 4\}$ are as indicated in (8.29). The random variables Y_j, $j \in \{1, \ldots, 4\}$ are clearly $\mathcal{F}_{i-1}$-measurable, while $W_1, W_3, W_4 \in L^2(\Omega, \mathcal{F}, Q)$. Applying Lemma 4.2 to the expansion of W_2^2 and using (4.5) yields $W_2 \in L^2(\Omega, \mathcal{F}, Q)$ and $\mathbb{E}^Q_{i-1}(W_2^2) = U_{i-1}^2$. Our observation following Lemma 4.2 now shows that

$$\mathbb{E}^Q_{i-1}[U_i^2H_i^2(\Delta, \beta, V)] = \sum_{j,k\leq 4}\mathbb{E}^Q_{i-1}(W_jW_k)Y_jY_k. \quad (8.29)$$

We next compute the terms on the right hand side of (8.29).

Equation (4.7) readily yields $\mathbb{E}^Q_{i-1}(W_1^2)Y_1^2 = \nu_i^2[\Delta_i - \eta_i(\Delta, \beta, V)]^2$. The products W_1W_2, W_1W_3 and W_1W_4 are contained in $L^1(\Omega, \mathcal{F}, Q)$ by the Cauchy-Schwartz inequality. Hence, applying Lemma 4.2 to the expansions of W_1W_2, W_1W_3 and W_1W_4, and using (4.7) and (4.6) yields $\mathbb{E}^Q_{i-1}(W_1W_2) = 0$, $\mathbb{E}^Q_{i-1}(W_1W_3) = -\mathbb{E}^Q_{i-1}(U_i^2)\text{covar}^{Q_i}_{i-1}(I_i, X_i(V))$ and $\mathbb{E}^Q_{i-1}(W_1W_4) = \mathbb{E}^Q_{i-1}(U_i^2)\text{var}^{Q_i}_{i-1}(I_i)$, so that we have $\mathbb{E}^Q_{i-1}(W_1W_3)Y_1Y_3 + \mathbb{E}^Q_{i-1}(W_1W_4)Y_1Y_4 = 0$. Similarly, applying Lemma 4.2 to the expansions of W_2W_3 and W_2W_4, and using (4.8) and (4.6) yields $\mathbb{E}^Q_{i-1}(W_2W_3) = \mathbb{E}^Q_{i-1}(U_i^2)\mathbb{E}^{Q_i}_{i-1}(I_i)\text{covar}^{Q_i}_{i-1}(I_i, X_i(V))/\mathbb{E}^{Q_i}_{i-1}(I_i^2)$ and further $\mathbb{E}^Q_{i-1}(W_2W_4) = -\mathbb{E}^Q_{i-1}(U_i^2)\mathbb{E}^{Q_i}_{i-1}(I_i)\text{var}^{Q_i}_{i-1}(Ii)/\mathbb{E}^{Q_i}_{i-1}(I_i^2)$, so that $\mathbb{E}^Q_{i-1}(W_2W_3)Y_2Y_3 + \mathbb{E}^Q_{i-1}(W_2W_4)Y_2Y_4 = 0$.

A direct computation shows $\mathbb{E}^Q_{i-1}(W_3^2) = \mathbb{E}^Q_{i-1}(U_i^2)\text{var}^{Q_i}_{i-1}(X_i(V))$, $\mathbb{E}^Q_{i-1}(W_4^2) = \mathbb{E}^Q_{i-1}(U_i^2)\text{var}^{Q_i}_{i-1}(I_i)$, and $\mathbb{E}^Q_{i-1}(W_3W_4) = -\mathbb{E}^Q_{i-1}(U_i^2)\text{covar}^{Q_i}_{i-1}(I_i, X_i(V))$, so that $\sum_{j,k\in\{3,4\}}\mathbb{E}^Q_{i-1}(W_jW_k)Y_jY_k$ can be written as

$$\mathbb{E}^Q_{i-1}(U_i^2)\left[\text{var}^{Q_i}_{i-1}(X_i(V)) - \frac{\{\text{covar}^{Q_i}_{i-1}(I_i, X_i(V))\}^2}{\text{var}^{Q_i}_{i-1}(I_i)}\right].$$

Substituting into (8.29) now yields (5.18). The integrability of $U_iH_i(\Delta, \beta, V)$ implies that $U_i^2H_i(\Delta, \beta, V) \in L^1(\Omega, \mathcal{F}, Q)$. Multiplying (8.29) by U_i yields an expansion of $U_i^2H_i(\Delta, \beta, V)$, to which Lemma 4.2 is applied to obtain $\mathbb{E}^Q_{i-1}(U_i^2H_i(\Delta, \beta, V)) = \sum_{j\leq 4}\mathbb{E}^Q_{i-1}(W_j)Y_j$. Equations (4.7) and (4.9) readily yield $\mathbb{E}^Q_{i-1}(W_1) = \mu_i/M_{i-1}$, $\mathbb{E}^Q_{i-1}(W_2) = U_{i-1}^2$, and $\mathbb{E}^Q_{i-1}(W_3) = \mathbb{E}^Q_{i-1}(W_4) = 0$. Equation (5.18) now follows immediately. □

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