Dempster-Shafer Logical Model for Fuzzy Description Logics

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***Abstract*—Description Logics, defined as a family of knowledge representation languages, have gained a lot of popularity, due to their connection with the Semantic Web, and more precisely, with the *Web Ontology Language* - OWL (OWL-DL). Vague information cannot be considered negligible when dealing with Semantic Web tasks. In this context, the definition of *fuzzy DLs* has been emerged. The Semantics of any DL is defined by an *interpretation*, which can be considered as a state of a world, where a DL formula (crisp or fuzzy) holds. In our method, we consider an uncertainty extension in a fuzzy DL, in the sense that an axiom *holds* with a *belief degree*. In order to represent these axioms, we assume Dempster-Shafer basic probability assignments on states of world (interpretations). We define the concept of *Dempster-Shafer Fuzzy interpretation*, in order to define semantics for our DL.**

I. Introduction

The Semantic Web applications [1] are strongly related to knowledge representation mechanisms. The concept of *ontology*, defined as an *explicit and formal specification of a conceptualization*, as it is used in [1], is applied for the definition of any domain of interest. Ontologies are defined through the *Web Ontology Language - OWL*, with OWL2 being the current version [2].

Description Logics (DLs) [3], [4], [5], have been applied extensively in Semantic Web applications, as they are the logics behind the most widely used version of OWL, OWL-DL. DLs allow for the representation of a domain of Knowledge, by providing *Concepts* (unary predicates) along with *Roles* (binary predicates), e.g: $CheapHotel \equiv Hotel \sqcap (\leq 100\ hasCost)$. This DL formula defines the Concept *CheapHotel* as a *Hotel with cost value less than 100*. One main issue with this formula is that it only provides for true/false statements, i.e a *hotel* can be either *cheap* or *not*. *Fuzzy DLs* [6], [7], have been defined as a way to represent vague statements, i.e DL formulas with a truth value in $[0, 1]$. The fuzzy extension allows for the definition of fuzzy DL formulas, i.e $CheapHotel \equiv Hotel \sqcap (hasCost.Cheap)$. Fuzzy DLs apply membership functions to describe vague concepts. Thus, $Cheap$ can be described by a membership function, that assigns each *cost value* a *membership degree* in $[0, 1]$. This means that an individual belongs to the class $CheapHotel$ with a degree value in $[0, 1]$.

Another aspect that we have to consider in Semantic Web information is *information incompleteness*. Dempster-Shafer theory [8] is an effective framework for representing information incompleteness. In our example, let us suppose that we know a hotel's cost per night with a *belief degree*, i.e:

$$h_1 : Hotel \sqcap (hasCost.\{100\}) \textit{ with belief degree 0.8}$$

If we want to compute the *truthness* of the *DL formula*

$$h_1 : CheapHotel$$

then we have to consider the following:

- Fuzziness, as a result to the fuzzy Concept $CheapHotel$
- Information incompleteness, as the exact cost value is missing

This means that any DL statement should be described by two degrees, a *fuzzy degree* and an *uncertainty degree*.

Extending Semantic web concepts with fuzzy sets requires a method for defining these sets. Fuzzy clustering [9] is considered a method for deriving fuzzy sets. Fuzzy C-Means is the most well known fuzzy clustering algorithm. A method that applies fuzzy clustering in a semantic web ontology can be found in [10]. Genetic Algorithms (GAs) [11] are also considered a framework for fuzzy clustering procedures. In our method, we consider existing fuzzy sets, by predefined membership functions.

The rest of this paper is organized as follows: In Section II, the concepts of uncertainty and vagueness are outlined. In Section III, the basics of Description Logics, along with fuzzy and uncertainty extensions are presented. In Section IV, the Dempster-Shafer theory is presented and a fuzzy extension of it is also considered. In Section V, our Dempster-Shafer Fuzzy DL is defined, based on $\mathcal{ALC}$ crisp DL and a fuzzy framework of it. In Section VI, we present a case example that illustrates our method. Finally, in Conclusion, we consider some further work that extends our model.

II. Uncertainty and Vagueness Handling in Semantic Web

Uncertainty and *Vagueness* constitute *deficient* knowledge in Semantic Web community. These two concepts can be described in the following way:

- *Uncertainty*: It refers to situations of information incompleteness (epistemic uncertainty) or randomness behaviour of a system (aleatory uncertainty) [12]
- *Vagueness*: It refers to situations of imprecise information [13]

As a special case, we have to consider situations where both uncertainty and vagueness are present, like in our introductory example. Zadeh's Z-numbers [14] can be considered a framework for representing both degrees. More precisely, a Z-number is defined as $Z = (A, B)$, where A denotes the restriction on the values a variable X can take and B is a measure of reliability of A.

In order to deal with uncertainty and vagueness, the crisp Description Logics should be extended in order to represent these concepts. Following, we overview the basics of DLs and present some of these extensions.

III. Description Logics

Generally speaking, Description Logics (DLs) are a family of knowledge representation languages. In literature there exist a lot of papers that introduce the basics of DLs [3], [4], [5]. Description Logics define a Knowledge Base as a triple:

$$< \mathcal{T}, \mathcal{R}, \mathcal{A} >$$

of axioms, where:

- $\mathcal{T}$: It is the $TBox$ of the Knowledge Base, which contains axioms concerning *DL Concepts*
- $\mathcal{R}$: It is the $RBox$ of the Knowledge Base, which contains axioms concerning *DL Roles*
- $\mathcal{A}$: It is the $ABox$ of the Knowledge Base, which contains axioms concerning *DL individuals*

DLs employ naming convention which describes the characteristics of the language. $\mathcal{ALC}$, an acronym of $\mathcal{A}$ttribute $\mathcal{L}$anguage with $\mathcal{C}$omplement, is considered as the basic DL.

A DL *interpretation* $\mathcal{I}$ is defined as $< \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} >$, where $\Delta^{\mathcal{I}}$ is the interpretation domain and $\cdot^{\mathcal{I}}$ is the interpretation function. An interpretation actually assigns a true/false value to each DL axiom. In case an axiom τ is true wrt. $\mathcal{I}$, this is denoted as $\mathcal{I} \models \tau$. *Subsumption*, *instantiation* and *consistency checking* are the main forms of reasoning in DLs.

A. The DL $\mathcal{ALC}$

$\mathcal{ALC}$ is considered the basic DL language. Its syntax applies the following sets, N_C (the set of concept names), N_R (the set of role names) and N_I (the set of individuals). Also, a set of syntax rules is applied, in order to build complex concepts. More precisely, $\mathcal{ALC}$ considers the following as $\mathcal{ALC}$ *concepts*:

- $\top, \bot, A \in N_C$, where A is a primitive concept
- If C, D are $\mathcal{ALC}$ *concepts*, then $C \sqcap D$, $C \sqcup D$, $\neg C$, $\forall r.C$ and $\exists r.C$, where r is a *DL Role*, are $\mathcal{ALC}$ *concepts*.

An interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ performs the following mapping:

$$\top^{\mathcal{I}} = \Delta^{\mathcal{I}}, \bot^{\mathcal{I}} = \emptyset, C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}, r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$$
$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}, (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \backslash C^{\mathcal{I}}$$
$$(\forall r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \forall d', (d, d') \in r^{\mathcal{I}} \textit{ implies } d' \in C^{\mathcal{I}}\}$$
$$(\exists r.C)^{\mathcal{I}} = \{d \in \Delta^{\mathcal{I}} : \exists d', (d, d') \in r^{\mathcal{I}} \textit{ and } d' \in C^{\mathcal{I}}\}$$

Also $\mathcal{ALC}$ considers two kinds of assertions for an individual α:

- $C(\alpha)$, meaning α is an instance of C (*concept assertion*)
- $r(\alpha, \beta)$, meaning the is a relation r between α, β (*role assertion*)

A set of concept assertions $\{C(\alpha_1), \ldots, C(\alpha_n)\}$ is *satisfied* in an interpretation $\mathcal{I}$, iff $\alpha_i^{\mathcal{I}} \in C^{\mathcal{I}}, i = 1, \ldots, n$. A set of role assertions $\{r(\alpha_1, \beta_1), \ldots, r(\alpha_n, \beta_n)\}$ is *satisfied* in an interpretation $\mathcal{I}$, iff $(\alpha_i^{\mathcal{I}}, \beta_i^{\mathcal{I}}) \in r^{\mathcal{I}}, i = 1, \ldots, n$.

B. DL Fuzzy extensions

A Fuzzy extension of a DL [6], [7], [15], [16] considers axioms that have a *truthness* degree rather than a *true/false* value. Any fuzzy extension is based on Fuzzy Sets and Fuzzy Logic [17], [18], in order to define a *fuzzy interpretation*. This means that an axiom is true in an interpretation $\mathcal{I}$ with a degree value in $[0, 1]$. Semantics of a fuzzy DL is defined based on the family of Fuzzy Logics [6].

DL fuzzy extensions consider a particular crisp DL and define degrees of truth on DL axioms. In [15], the extension of DL $\mathcal{ALC}$ is defined. This approach defines fuzzy assertional axioms of the form $< \tau \quad \alpha >$, meaning that τ *is true with degree* α. Another approach is considered in [7] where the concept of the concrete fuzzy domain predicate is considered, as an extension to the concrete domain. A concrete domain D is defined as a pair $< \Delta_{\mathrm{D}}, \Phi_{\mathrm{D}} >$, where Δ_{D} is an interpretation domain and Φ_{D} is the set of concrete domain predicates d with arity n and interpretation $d^{\mathrm{D}} \subseteq \Delta_{\mathrm{D}}^n$. A concrete fuzzy domain predicate d, with arity n has an interpretation $d^{\mathrm{D}} : \Delta_{\mathrm{D}}^n \rightarrow [0, 1]$. The concrete fuzzy domain predicates allow for defining fuzzy concepts.

C. DL Uncertainty extensions

Uncertainty extensions approaches usually consider *interpretations* as *possible worlds*. This means that a DL axiom is true/false in an interpretation. Following, these approaches assign a distribution (probabilistic/possibilistic) on the set of possible worlds.

The *probabilistic knowledge base* defined in [13] is built upon the probabilistic logic. A probability distribution, $\mathcal{P}$, is defined on the set of possible worlds. A *probabilistic interpretation* defines a mapping, μ from the set of possible worlds $\mathcal{W}$ as: $\mu : \mathcal{W} \rightarrow [0, 1]$.

The probability degree of a formula ϕ is defined as:

$$\mathcal{P}(\phi) = \sum_{I \models \phi} \mathcal{P}(I)$$

where I is an interpretation (possible world) and $I \models \phi$ means that ϕ is true in I.

In addition, a *possibilistic knowledge base* defined in [13] annotates DL formulas with a possibility and necessity measures, based on theory of possibility [19].

IV. Dempster-Shafer Theory and Fuzzy Sets

Dempster-Shafer theory [8] is a framework for representing incomplete information. It applies the concept of Belief function in order to define a measure of evidence that supports an event.

A Dempster-Shafer framework considers the *Frame of Discernment*, $\mathcal{W}$, which is defined as the set of exhaustive and mutually exclusive events. Let $2^{\mathcal{W}}$ the powerset of $\mathcal{W}$ and $A \in 2^{\mathcal{W}}$. Then, the following functions are defined: Based on these sets, the following functions are defined:

- Basic probability assignment m: It is defined as a function $m : 2^{\mathcal{W}} \to [0,1]$
- Belief and Plausibility functions Bel and Pl: They are defined as:

$$Bel(A) = \sum_{B \subseteq A} m(B), \quad Pl(A) = \sum_{B \cap A \neq \emptyset} m(B)$$

where $A \in 2^{\mathcal{W}}$.

Dempster's rule of Combination [12] is defined on two basic probability assignments m_1, m_2, derived from independent sources:

$$m_1 \bigoplus \; m_2(B) = \frac{\sum_{A_i \bigcap \; A_j = B} m_1(A_i) \times m_2(A_j)}{1 - \sum_{A_i \bigcap \; A_j = \emptyset} m_1(A_i) \times m_2(A_j)}$$

A. Dempster-Shafer and Fuzzy Sets

The basic probability assignment m, is defined upon the concept of a compatibility relation C [20]. More precisely, if we consider two spaces, X and $\mathcal{W}$, along with a probability distribution p on X, then the following statement holds:

$x \in X$ is compatible to $y \in \mathcal{W}$, denoted as xCy, if it is possible x is an answer to X and y is an answer to $\mathcal{W}$ at the same time

Based on the compatibility relation, the *granule* of an element $x \in X$ is defined as: $G(x) = \{y \mid \; y \in \; \mathcal{W}, xCy\}$. Then, the basic probability assignment $m(A)$, $A \subseteq \mathcal{W}$ is defined as:

$$m(A) = \frac{\sum_{G(x_i)=A} p(x_i)}{1 - \sum_{G(x_i)=\emptyset} p(x_i)}$$

In [21] a generalization of Dempster-Shafer theory in order to account for fuzzy sets is defined. The theory is based on joint possibility distributions. More precisely, the compatibility relation is now defined as follows:

$$C(x,y) = \Pi_{\Psi,\Omega}(x,y), x \in \; X, y \in \; \mathcal{W}$$

where C is the compatibility relation, Π is a joint possibility distribution and Ψ,Ω are variables that take values from spaces X and $\mathcal{W}$ respectively.

The granule $G(x)$ is now defined as:

$$G(x) = \Pi_{\Omega|\Psi=x}$$

The basic probability assignment of a fuzzy subset $A \subset \; \mathcal{W}$ is induced by the equation

$$m(A) = \frac{\sum_{G(x_i)=A} p(x_i)}{1 - \sum_{G(x_i)=\emptyset} p(x_i)}$$

In this approach, a method for inducing a belief degree for a fuzzy set B from a set of non-fuzzy focals is outlined. More precisely, the Belief-Plausibility degrees are considered as lower-upper probabilities of the set. Then, the Belief degree of the set B induced by a set of non-fuzzy focals $\mathcal{A}$ is the optimal solution to the linear program $\sum_{x_i \in \; B} \sum_j m(x_i : A_j)$, where $A_j \in \mathcal{A}$.

B. Dempster-Shafer and Logic view

The logical view of Dempster-Shafer theory has been studied in various works in the literature [22], [23], [24], [25], [26]. In [24] a set of first-order formulas is annotated with belief and plausibility degrees. Formulas are of the form $\phi[a,b]$, where ϕ is first-order formula and a,b constitute the Belief-Plausibility degrees. A set of classical first-order interpretations, $\mathcal{I}$, is considered, and a *bf-interpretation* is defined as:

$$\mathcal{M} : 2^{\mathcal{I}} \to [0,1]$$

A *bf-interpretation* is constrained by the following:

$$\begin{aligned}
&Bel_{\mathcal{M}}(2^{\mathcal{I}}) = 1 \\
&if \quad X \cup Y \neq 2^{\mathcal{I}}, then \\
&Bel_{\mathcal{M}}(X \cap Y) \geq Bel_{\mathcal{M}} \times Bel_M(Y)
\end{aligned}$$

where Bel denotes the belief degree of elements in $2^{\mathcal{I}}$. Also, the concept of *hyper-interpretation* is employed, defined as an element of $2^{\mathcal{I}}$.

V. A Dempster-Shafer Fuzzy Description Logic

In this section we define a DL suitable for representing *vague incomplete information*. Our intention is to have DL formulas with semantics based on Fuzzy Logic and Dempster-Shafer framework. As opposed to current uncertainty and vagueness handling approaches, our method represents fuzzy statements in an uncertainty framework. In our approach, we consider the classical DL *interpretation* as a possible world. In each interpretation, a set of DL axioms holds. More precisely, we present a Dempster-Shafer extension of the *fuzzy* $\mathcal{ALC}$(D), which is defined in [15].

Following, we introduce the basic concepts of the *fuzzy* $\mathcal{ALC}$.

A. Fuzzy $\mathcal{ALC}$

The fuzzy extension of $\mathcal{ALC}$ is defined in [15]. In this approach, the fuzzy statements are only the assertions of the DL Knowledge Base, i.e the *ABox*.

The syntax of the fuzzy $\mathcal{ALC}$ considers fuzzy assertions of the form $< a \quad n >$, where a is a crisp DL assertion and $n \in [0,1]$.

The crisp DL interpretation is now extended into a *fuzzy interpretation*. Generally, concepts are considered as fuzzy subsets of $\Delta^{\mathcal{I}}$, and roles as fuzzy subsets of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. More precisely, a fuzzy interpretation $\mathcal{I}$ assigns to each fuzzy Concept C a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0,1]$ and each fuzzy Role R a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0,1]$.

This means that, in the semantics of the assertions, there is a truthness degree that is defined as follows:

- $C^{\mathcal{I}}(a^{\mathcal{I}})$ constitutes the *membership degree* of individual $a^{\mathcal{I}}$ in fuzzy subset $C^{\mathcal{I}}$
- $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$ constitutes the *membership degree* of individual $(a^{\mathcal{I}}, b^{\mathcal{I}})$ in fuzzy subset $R^{\mathcal{I}}$

B. Dempster Shafer Fuzzy $\mathcal{ALC}$

Our method extends the fuzzy DL described previously, with Dempster-Shafer modules. Our extension is based on the fuzzy extension defined in [21].

More precisely, we consider fuzzy subsets of a domain. In addition, we consider the set of all the interpretations $\mathcal{W}$ and define a basic probability assignment on subsets of it. As we consider fuzzy interpretations, $\mathcal{W}$ is an infinite set. In our approach, we assume that $\mathcal{W}$ is defined as $\{\mathcal{I}_1, \mathcal{I}_2, \dots\}$, where each $\mathcal{I}_i$ contains DL assertions along with their membership degree under a fuzzy DL interpretation. As an example of a $\mathcal{I}_i$ let us consider the following:

$$\mathcal{I}_i = \{\{0.5/CheapHotel(a), 0.8/CloseToMetroStation(a)\}, \{0.7/CheapHotel(a)\}\}$$

Following, we define the syntax and semantics of our DL.

C. Syntax

Definition 1. A *Dempster-Shafer Fuzzy assertion* is defined as

$$< a \quad : k >, \tag{1}$$

where a is a fuzzy $\mathcal{ALC}$ DL assertion and $k \in [0,1]$.

In this assertion, k represents the belief degree lower bound. For example

$$< CheapHotel(c) \quad 0.5 \quad : 0.6 >$$

denotes that c is a $CheapHotel$ with *fuzzy lower degree* 0.5 and *belief lower degree* 0.6.

Next, we define a *Dempster-Shafer Fuzzy Knowledge Base* $\mathcal{KB}_{DS}$ as a set of *Dempster-Shafer Fuzzy assertions*.

Definition 2. A *Dempster-Shafer Fuzzy ABox* is defined as a set of assertion axioms, as follows:

$$< C(i) \quad n \quad : k >, i \in N_I \tag{2}$$

$$< R(i_1, i_2) \quad n \quad : k >, i_1, i_2 \in N_I \tag{3}$$

where C is an $\mathcal{ALC}$ DL concept, R a $\mathcal{ALC}$ DL Role and $n, k \in [0,1]$.

D. Semantics

In our approach, we interpret Concepts as fuzzy subsets of a domain $\Delta^{\mathcal{I}_{DS}}$ and Roles as fuzzy subsets of a domain $\Delta^{\mathcal{I}_{DS}} \times \Delta^{\mathcal{I}_{DS}}$. Also, we consider Zadeh's semantics, in order to define interpretation of the $\mathcal{ALC}$ *concepts*, described in Section III. In that sense, our interpretation resembles the fuzzy interpretation described in Section V-A.

In the crisp $\mathcal{ALC}$, an assertion axiom is *satisfied* in an interpretation $\mathcal{I}$, iff the axiom is true in this interpretation, denoted that the axiom *holds* in $\mathcal{I}$. The fuzzy extension considers a truthness degree of satisfaction. Thus, a fuzzy axiom holds with a certain *degree*.

Our innovation considers the set of possible worlds, denoted as $\mathcal{W} = \{\mathcal{I}_1, \mathcal{I}_2, \dots\}$.

We introduce the following definitions:

Definition 3. An assertion $< C(a) \quad n \quad : k >$ *holds* in an interpretation or possible world $\mathcal{I}_j, j = 1, \dots, n$, denoted as $\mathcal{I}_j \models < C(a) \quad n >$, iff $C^{\mathcal{I}_j}(a^{\mathcal{I}_j}) \geq \; n$, where $C^{\mathcal{I}_j}(a^{\mathcal{I}_j})$ is the *membership degree* of $a^{\mathcal{I}_j}$ being $C^{\mathcal{I}_j}$.

Definition 4. An assertion $< R(a,b) \quad n \quad : k >$ *holds* in an interpretation or possible world $\mathcal{I}_j, j = 1, \dots, n$, denoted as $\mathcal{I}_j \models < R(a,b) \quad n >$, iff $R^{\mathcal{I}_j}(a^{\mathcal{I}_j}, b^{\mathcal{I}_j}) \geq \; n$, where $R^{\mathcal{I}_j}(a^{\mathcal{I}_j}, b^{\mathcal{I}_j})$ is the *membership degree* of $(a^{\mathcal{I}_j}, b^{\mathcal{I}_j})$ being $R^{\mathcal{I}_j}$.

From now on, if τ is a Dempster-Shafer Fuzzy assertion, we denote as $\mu_{\mathcal{I}_j}(\tau)$ the membership degree of the corresponding individual a (or (a,b)) that τ describes, under the interpretation $\mathcal{I}_j$.

Definition 5. A Dempster-Shafer Fuzzy interpretation $\mathcal{I}_{DS}$ is defined as:

$$\mathcal{I}_{DS} = (\Delta^{\mathcal{I}_{DS}}, \cdot^{\mathcal{I}_{DS}}, \mathcal{W}, m) \tag{4}$$

where $\Delta^{\mathcal{I}_{DS}}$ is the interpretation domain, $\cdot^{\mathcal{I}_{DS}}$ is an interpretation function [1], $\mathcal{W}$ is the set of possible worlds and m is a basic probability assignment on subsets of $\mathcal{W}$. Any $\mathcal{I}_j \in \mathcal{W}$, such that $m(\mathcal{I}_j) > 0$ is called a *focal possible world*.

Also, we define the fuzzy counterpart of a Dempster-Shafer Fuzzy assertion $\tau = < a \quad : k >$, as $\tau_{Fuzzy} \equiv a$.

The satisfaction of a Fuzzy counterpart assertion τ_{fuzzy} in a set of possible worlds $\{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$ is defined as:

Definition 6. A set of possible worlds $\mathcal{T} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$ *satisfies* a fuzzy assertion $< C(a) \quad n >$,
iff:

$$\forall \mathcal{I}_j \in \mathcal{T}, \mathcal{I}_j \models < C(a) \quad n >$$

Definition 7. A set of possible worlds $\mathcal{T} = \{\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_m\}$ *satisfies* a fuzzy assertion $< R(a,b) \quad n >$,
iff:

$$\forall \mathcal{I}_j \in \mathcal{T}, \mathcal{I}_j \models < R(a,b) \quad n >$$

[1] We consider that the interpretation function performs the same mappings as in the fuzzy $\mathcal{ALC}$ case

Definition 8. A set of possible worlds $\mathcal{T} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m\}$ *does not satisfy* a fuzzy assertion $< C(a) \quad n >$, iff:

$$\exists \mathcal{I}_j \in \mathcal{T}, \mathcal{I}_j \not\models < C(a) \quad n >$$

Definition 9. A set of possible worlds $\mathcal{T} = \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_m\}$ *does not satisfy* a fuzzy assertion $< R(a, b) \quad n >$, iff:

$$\exists \mathcal{I}_j \in \mathcal{T}, \mathcal{I}_j \not\models < R(a, b) \quad n >$$

If an assertion $< a \quad n >$ is satisfied under a set $\mathcal{T}$, then it is denoted as:

$$\mathcal{T} \models < a \quad n >$$

If an assertion $< a \quad n >$ is not satisfied under a set $\mathcal{T}$, then it is denoted as:

$$\mathcal{T} \not\models < a \quad n >$$

The insertion of the basic probability assignment m on $\mathcal{W}$ imposes the definition of a *belief degree* for the DL assertions. In order to derive such degrees, we consider the focal possible worlds of $\mathcal{W}$ with a basic probability assignment to play the role of *non-fuzzy focal elements*, described in Section IV.

In that sense, we define the *Belief degree* of a fuzzy assertion τ_{fuzzy}, under $\mathcal{W}$, as follows:

Definition 10.

$$Bel(\tau_{fuzzy}) = \sum_{\mathcal{T} \models \tau_{fuzzy}} m(\mathcal{T}) \times inf_{\mathcal{I}_j \in \mathcal{T}} \mu_{\mathcal{I}_j}(a) \quad (5)$$

where $\mu_{\mathcal{I}_j}(a)$ is the *membership degree* of $a^{\mathcal{I}_j}$ in $C^{\mathcal{I}_j}$ (Resp. $(a^{\mathcal{I}_j}, b^{\mathcal{I}_j}) \in R^{\mathcal{I}_j}$) in possible world $\mathcal{I}_j$.

In the same way, we define the Plausibility degree as follows:

Definition 11.

$$Pl(\tau_{fuzzy}) = \sum_{\mathcal{T} \models \tau_{fuzzy}} m(\mathcal{T}) \times sup_{\mathcal{I}_j \in \mathcal{T}} \mu_{\mathcal{I}_j}(a) \quad (6)$$

where $\mu_{\mathcal{I}_j}(a)$ is the *membership degree* of $a^{\mathcal{I}_j}$ in $C^{\mathcal{I}_j}$ (Resp. $(a^{\mathcal{I}_j}, b^{\mathcal{I}_j}) \in R^{\mathcal{I}_j}$) in possible world $\mathcal{I}_j$.

The intuition behind the definition of the Belief degree is that it can be considered as a lower probability of a in a way similar to [21]. In order to define this measure, we consider *Belief degrees of Fuzzy sets derived from non-fuzzy focal elements*. More precisely, we consider subsets of the set of $\mathcal{W}$, as non-fuzzy subsets. Therefore, a Dempster-Shafer interpretation assigns a mass degree on each subset. Any mass degree value greater than zero, results in a *non-fuzzy focal element*, or a *focal-set-possible world*. This can be derived by considering a minimization linear programming problem, as it has been defined in [21] and presented in Section IV.

Before proceeding into the minimization process, we introduce the sets $\mathcal{W}_\tau$ and $\models_\tau$ of a Dempster-Shafer Fuzzy axiom τ:

Definition 12. The $\mathcal{W}_\tau$ set of a Dempster-Shafer Fuzzy axiom is defined as a fuzzy set $\{\mu_{\mathcal{I}_1}/\mathcal{I}_1, \mu_{\mathcal{I}_2}/\mathcal{I}_2, \ldots\}$, where each $\mu_{\mathcal{I}_i}, i = 1, 2, \ldots$ is the membership degree of τ_{fuzzy} under interpretation $\mathcal{I}_i$.

Definition 13. The $\models_\tau$ set of a Dempster-Shafer Fuzzy axiom is defined as a fuzzy set $\{\mu_{\mathcal{I}_a}/\mathcal{I}_a, \mu_{\mathcal{I}_b}/\mathcal{I}_b, \ldots\}$, where each $\mu_{\mathcal{I}_i}, i = a, b, \ldots$ entails τ_{fuzzy}, i.e $\mathcal{I}_i \models \tau_{fuzzy}$.

In order to apply this minimization process in our Dempster-Shafer Fuzzy DL, we consider the following associations:

- The fuzzy set $\models_\tau$ is regarded as a Dempster-Shafer Fuzzy subset B.
- A focal-set possible world $\mathcal{T}_j$ is regarded as a non-fuzzy focal element A_i.
- $\mathcal{T}_j \models \tau_{fuzzy}$ iff $\mathcal{T}_j \subseteq \models_\tau$

By using the correspondence above, the minimization problem is defined as:

$$\sum_{\mathcal{I}_i \in \models_\tau} \sum_j m(\mathcal{I}_i : \mathcal{T}_j)$$

Also, $m(\mathcal{I}_i : \mathcal{T}_j)$, where $\mathcal{T}_j$ is focal-set possible world, is constrained by the following:

$$m(\mathcal{I}_i : \mathcal{T}_j) \geq 0, j = 1, \ldots, l \quad (7)$$
$$m(\mathcal{I}_i : \mathcal{T}_j) = 0, \forall \mathcal{I}_i \notin \mathcal{T}_j \quad (8)$$
$$m(\mathcal{I}_i : \mathcal{T}_j) = m(\mathcal{T}_j), \forall j = 1, 2, \ldots, l \quad (9)$$

The optimal solutions of the aforementioned problem are denoted as, $m_*(\models_\tau : \mathcal{T}_j)$ and $m^*(\models_\tau : \mathcal{T}_j)$. The Belief and Plausibility measures are computed by adding the optimal solutions for all $\mathcal{T}_j$, $j = 1, \ldots, l$ and assigning all the mass of $\mathcal{T}_j$ to the element of $\mathcal{T}_j$ that has the lowest or highest membership degree in $\models_\tau$:

$$m_*(\models_\tau : \mathcal{T}_j) = m(\mathcal{T}_j) \times inf_{\mathcal{I} \in \mathcal{T}_j} \mu_{\models_\tau}(\mathcal{I})$$
$$m^*(\models_\tau : \mathcal{T}_j) = m(\mathcal{T}_j) \times sup_{\mathcal{I} \in \mathcal{T}_j} \mu_{\models_\tau}(\mathcal{I})$$

Adding these optimal solutions, we get the Belief and Plausibility degree of $\models_\tau$:

$$Bel(\models_\tau) = \sum_{T_j \subseteq \mathcal{W}} m(\mathcal{T}_j) \times inf_{x \in \mathcal{T}_j} \mu_{\models_\tau}(x)$$
$$Pl(\models_\tau) = \sum_{T_j \subseteq \mathcal{W}} m(\mathcal{T}_j) \times sup_{x \in \mathcal{T}_j} \mu_{\models_\tau}(x)$$

Finally, we make the following assumptions:

- $Bel(\models_\tau) \equiv Bel(\tau_{fuzzy})$
- $Pl(\models_\tau) \equiv Pl(\tau_{fuzzy})$
- $T_j \subseteq \mathcal{W} \equiv \mathcal{T} \models \tau$, since $m(\mathcal{I}_i : \mathcal{T}_j) = 0, \forall \mathcal{I}_i \notin \mathcal{T}_j$

By considering the assumptions , we get the formulas of Belief and Plausibility as defined above.

Definition 14. A Dempster-Shafer Fuzzy interpretation $\mathcal{I}_{\mathcal{DS}}$ is a model (or satisfies) a Dempster-Shafer Fuzzy assertion $< \mathcal{E} \quad : k >$ iff $Bel(\mathcal{E}) \geq k$.

Definition 15. A Dempster-Shafer Fuzzy interpretation $\mathcal{I}_{\mathcal{DS}}$ is a model of a set of Dempster-Shafer Fuzzy assertions Ψ iff it satisfies each $\epsilon \in \Psi$.

Definition 16. A Dempster-Shafer Fuzzy assertion $\mathcal{E}$ is a logical consequence of a Dempster-Shafer Fuzzy Knowledge Base $\mathcal{K}$ iff every model of $\mathcal{K}$ satisfies $\mathcal{E}$.

VI. A MATCHMAKING CASE STUDY

Matchmaking problems can be considered as ontology applications for the Semantic Web. In its typical form, a matchmaking problem consists of two groups, denoted as ”sellers” and ”buyers”. Each seller and buyer defines a set of constraints, as *requirements* and *preferences*. A very common situation in constraint setting is the vagueness that describe them [6] [27]. As an example let us consider a job recruitment process, with the following constraints:

- *Job Seeker Constraints*:
  - Job with salary no less than $25,000$ per annum
  - Ideal job salary $30,000$ per annum
- *Job Advertisement Constraints*:
  - Job with salary no more than $26,000$ per annum
  - Ideal job salary $23,000$ per annum

The constraints are defined in a way that an ideal value exists and as the value increases or decreases the satisfaction of the seeker/recruiter goes down. In a formal way, the constraints are defined through the following membership functions:

$$\mu_{Seeker}(x) = \left\{ \begin{array}{ll} 0, & \text{for } 0 \leq x \leq 25000 \\ \frac{x-25000}{5000}, & \text{for } 25000 \leq x \leq 30000 \\ 1, & \text{for } 30000 \leq x \end{array} \right\}$$

$$\mu_{Advertisement}(x) = \left\{ \begin{array}{ll} 1, & \text{for } 0 \leq x \leq 23000 \\ \frac{26000-x}{3000}, & \text{for } 23000 \leq x \leq 26000 \\ 0, & \text{for } 26000 \leq x \end{array} \right\}$$

The first function is called a *right-shoulder membership function*, whereas the second is called a *left-shoulder membership function*. So, the Job Advertisement salary constraint is represented through the *left-shoulder* membership function, with its value denoted as f_1, whereas the the Job Seeker constraint is represented through the *right-shoulder* membership function, with its value denoted as f_2.

This means that for a job individual, j, we have the following axioms:

$$\tau 1_{fuzzy} :< j \quad f_1 >$$
$$\tau 2_{fuzzy} < j \quad f_2 >$$

Also, we define a set of weights considering Seeker and Advertisement requirements, in a way similar to [6], denoting the *credibility* of the Seeker and Advertisement. This is defined by regarding the constraints as a set $\mathcal{C}$ of the following form:

$$\mathcal{C} = \{s_1, \dots, s_k, a_1, \dots, a_l\}$$

with s_i denoting a Seeker constraint and a_i denoting an Advertisement constraint. Then, weights are defined through a basic probability assignment m_{weight}:

$$m_{weight} : 2^{\mathcal{C}} \rightarrow [0,1]$$

In our case study, we have the following set:

$$\mathcal{C} = \{s_1, a_1\}$$

as we have one Seeker and one Advertisement constraint.

The fuzziness describing constraints along with weights definitions pave the way for the application of our Dempster-Shafer Fuzzy DL in the matchmaking procedure.

If we consider Seeker and Advertisement as two fuzzy interpretations, $\mathcal{I}_{Seeker}$ and $\mathcal{I}_{Adverstisement}$, each of them being a model of $\tau 1_{fuzzy}$ and $\tau 2_{fuzzy}$, as follows:

$$\mathcal{I}_{Seeker} \models \tau 1_{fuzzy}$$
$$\mathcal{I}_{Advertisement} \models \tau 2_{fuzzy}$$

Formally, these two interpretations are represented as:

$$\mathcal{I}_{Seeker} = \{\Delta^{\mathcal{I}_{Seeker}}, \cdot^{\mathcal{I}_{Seeker}}\}$$
$$\mathcal{I}_{Advertisement} = \{\Delta^{\mathcal{I}_{Advertisement}}, \cdot^{\mathcal{I}_{Advertisement}}\}$$

where

$$\Delta^{\mathcal{I}_{Seeker}} \equiv \Delta^{\mathcal{I}_{Advertisement}} \equiv \mathbb{N}$$

and $\cdot^{\mathcal{I}_{Seeker}}$, $\cdot^{\mathcal{I}_{Advertisement}}$ are defined based on the membership functions μ_{Seeker} and $\mu_{Advertisement}$. Also, $\mathbb{N}$ is the Salary value domain.

Now, let us suppose that we have a job posting, with salary $25,000$. This, in an ontology context, is represented as an individual $j : Job \sqcap (hasSalary.\{25000\})$.

Also, let have weights of 0.8 for the Seeker and 0.2 for the Advertisement formally represented as $m_{weight}(\{s_1\}) = 0.8$ and $m_{weight}(\{a_1\}) = 0.2$. These weights have been arbitrarily chosen in order to describe our case example.

Our goal is to compute a *matchmaking degree* that depicts the satisfaction value of the job individual, based on fuzzy constraints and uncertainty. In order to do this, we consider the following fuzzy axiom:

$$\tau_{fuzzy} :< j \quad f >$$

where f is defined as $min\{f_1, f_2\}$.

Then, $\mathcal{I}_{Seeker}$ and $\mathcal{I}_{Adverstisement}$ are models of

$$\tau_{fuzzy} :< j \quad f >$$

i.e, $\mathcal{I}_{Seeker} \models \tau_{fuzzy}$ and $\mathcal{I}_{Advertisement} \models \tau_{fuzzy}$.

To sum up, each job individual is related to two fuzzy constraint degrees:

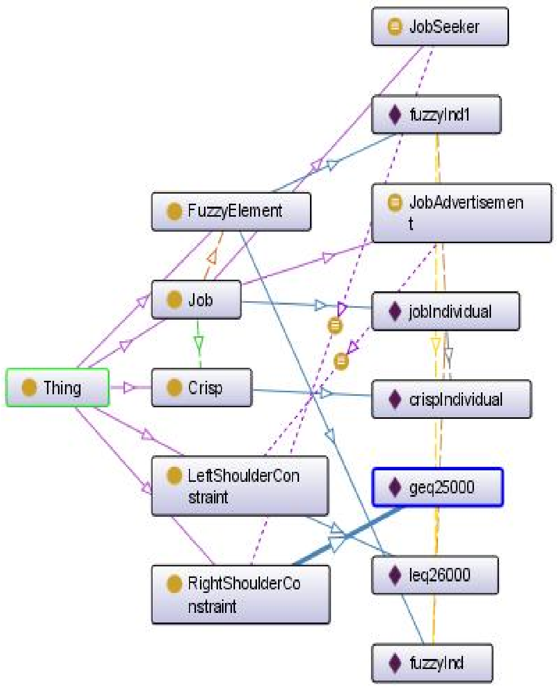

Fig. 1. Matchmaking Ontology

- Fuzzy constraint degree of Seeker
- Fuzzy constraint degree of Advertisement

Also, by considering a basic probability assignment modelled as weights on Seeker and Advertisement constraints, we have each fuzzy degree associated to a mass degree. This, paves the way for the application of our *Dempster-Shafer DL*.

So, the matchmaking degree is computed through the belief degree based on Definition 10 and represented as a Dempster-Shafer Fuzzy axiom.

Following, we overview our matchmaking ontology, depicted in Fig 1. Our ontology is based on the one defined in [27].

In order to represent our world, we consider the following classes:

- *Job Seeker*: It is defined as an OWL class and represents the part who searches for a job position
- *Job Advertisement*: It is defined as an OWL class and represents the part who posts a job position
- *Job*: It is defined as an OWL class and represents the jobs of interest
- *Crisp*: It is defined as OWL class and relates a Job individual with a salary value and an uncertainty value
- *LeftShoulderConstraint*: It is defined as an OWL class and defines a Left-Shoulder membership function
- *RightShoulderConstraint*: It is defined as an OWL class and defines a Right-Shoulder membership function

The constraints are defined in the following way:

$$JobSeeker \equiv Job \sqcap$$
$$hasSalary.RightShoulderConstraint$$

$$JobAdvertisement \equiv Job \sqcap$$
$$hasSalary.LeftShoulderConstraint$$

Each LeftShoulderConstraint and RightShoulderConstraint is related to a fuzzy degree through the *hasFuzzy* data property, whereas each job individual is related to an uncertainty degree through the *hasUncertainty* data property.

The matchmaking processing derives a *matchmaking degree factor*, defined as:

$$Matchmaking \equiv JobSeeker \sqcap JobAdvertisement$$

This *degree factor* is a combination of *fuzzy constraints* and *uncertainty* of salary crisp value.

Considering the constraints, we define for each salary crisp value a membership degree for each constraint. The definition of membership degrees is performed through the rules plugin in Protégé. A rule variable is defined by the symbol $?x$, where x is a user defined variable. The Left Shoulder Constraint is defined through the following set of rules:

$$LeftShoulderConstraint(?l), hasCrisp(?j, ?c),$$
$$hasElement(?f, ?l),$$
$$hasFuzzy(?j, ?f), hasIdealValue(?l, ?i),$$
$$hasThresholdValue(?l, ?t),$$
$$hasValue(?c, ?v), lessThanOrEqual(?v, ?i)$$
$$- > hasFuzzyFactor(?f, 1.0)$$

$$LeftShoulderConstraint(?l), hasCrisp(?j, ?c),$$
$$hasElement(?f, ?l),$$
$$hasFuzzy(?j, ?f), hasIdealValue(?l, ?i),$$
$$hasThresholdValue(?l, ?t),$$
$$hasValue(?c, ?v), greaterThanOrEqual(?v, ?t)$$
$$- > hasFuzzyFactor(?f, 0.0)$$

$$LeftShoulderConstraint(?l), hasCrisp(?j, ?c),$$
$$hasElement(?f, ?l),$$
$$hasFuzzy(?j, ?f), hasIdealValue(?l, ?i),$$
$$hasThresholdValue(?l, ?t),$$
$$hasValue(?c, ?v), divide(?d, ?s1, ?s2),$$
$$greaterThan(?v, ?i),$$
$$lessThan(?v, ?t), subtract(?s1, ?t, ?v),$$
$$subtract(?s2, ?t, ?i)$$
$$- > hasFuzzyFactor(?f, ?d)$$

A Right Shoulder Constraint is defined in an analogous way.

We model the Belief Degree of the job individual through the following rule:

$$
\begin{aligned}
&FuzzyElement(?f1), FuzzyElement(?f2),\\
&LeftShoulderConstraint(?l),\\
&RightShoulderConstraint(?r), hasCrisp(?j, ?c),\\
&hasElement(?f1, ?l),\\
&hasElement(?f2, ?r), hasFuzzy(?j, ?f1),\\
&hasFuzzy(?j, ?f2), hasFuzzyFactor(?f1, ?fa1),\\
&hasFuzzyFactor(?f2, ?fa2), hasWeight(?f1, ?u1),\\
&hasWeight(?f2, ?u2), add(?s, ?m1, ?m2),\\
&multiply(?m1, ?fa1, ?u1),\\
&multiply(?m2, ?fa2, ?u2) -> hasBel(?j, ?s)
\end{aligned}
$$

In our case example, we derive a matchmaking degree of 0.12 for the job posting. This value is the *belief degree* of *Job* $?j$ being a job that matches Seeker and Advertisement constraints. The derivation of the belief degree value comes as a result of our rules definition.

In addition, we have developed a matchmaking application for job recruitment, integrating fuzzy logic, Dempster-Shafer and ontologies, which is presented in [27]. In this application, the *Seeker* and *Advertisement preferences* are represented as concepts and roles in our ontological model. A set of data (job postings) is considered as a real-world case example of our method. In order to draw a matchmaking degree, a set of rules has also been defined.

VII. Conclusion

In our approach, we have defined a unified framework for representing uncertainty and vagueness in a DL environment. Our model considers the assertional part of the DL, i.e the *ABox*. We annotate fuzzy formulas belief conditions and consider the interpretations as a set of possible worlds. We have defined our framework in $\mathcal{ALC}$ DL. Semantics of our DL is defined through a Dempster-Shafer Fuzzy interpretation. As a case study of our method, we have considered a matchmaking problem regarding job offerings. As a next step, we will consider the annotation of *TBox* and *RBox* axioms, along with a rule framework for reasoning upon them. We have to mention that, currently, we work in an application of our model into web size data.

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