The Closed-Form Solution of the Control Related States of *Deficient Gen-Left k*-th order System (the essential element of non-sharing subnet) of Petri Nets

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***Abstract*—Earlier, Chao pioneered the very first closed-form solution of the number of reachable and other states for *k*-th order system which is the first step that will let the exponential computation time for enumerating reachable states of a particular large Petri Net be reduced within intra-second. This paper progresses one step further on enumerating the Control Related States for the *Deficient Gen-Left k*-th order systems (the initial marking of idle places in one process is less than *k*) which is an essential element of non-sharing subnet in PNs.**

Keywords*—Control systems; discrete event systems; flexible manufacturing systems; Petri nets

I. Introduction

In the light of the rapid innovation of the Internet of Things (IoT), robot systems and the cloud computing system, we need an efficient methodology to model real-time resource-allocation systems for bottleneck situations, deadlock avoidance/prevention policy and other system control related problems.

As the widely applied instrument for modeling and analyzing flexible manufacturing (or resource allocation) systems (FMS) (or RAS) [1-9], the persist problem of using Petri net (PN) for modeling various systems is the large number of states generated (called the state explosion problem). It has been shown that the complexity of the reachability problem of PN is EXPSPACE-hard [10] and is NP-complete for a live and safe Free Choice net (LSFC) [10]. Even using mixed integer programming(MIP) [11] to solve the reachability problem, it still is an NP-hard problem [12]. Hence the bottleneck is how can generate the reachability related information efficiently, which will consume the exponential computation time, so that we can apply the information for the resource allocation/deallocation decision-making for a large real-time, dynamic resource allocation system.

To overcome this barrier, applying graph theory Chao [13] constructed the concept of complete reachability graph; split the reachability graph of the control net into *reachable*, *live*, *forbidden*, *deadlock*, *non-reachable* and *empty-siphon+non-reachable states* (we call all of different types of control states as Control Related States, denoted as CRSs) and pioneered the very first closed-form solution of the number of CRSs for *k*-th order system (defined in Definition 1) (e.g. Fig.1). The most important contribution is that the exponential computation time of deriving a particular and very large PN's CRSs information can be reduced within intra-second. We have also extended and applied Chao's [13] key methodology to construct the closed-form formulas of the CRSs of various *k*-th order systems with a non-sharing resource r^* in different specific locations [14-24], respectively. However, the research domain is limited on the issue of the effect of non-sharing resource in *k*-th order system.(In this paper, **the closed-form formulas /solution mean the closed-form formulas/solution of CRSs for PNs**)

The deadlock avoidance/prevention policy for PNs presently is focus on how to find critic first-met bad markings (FBMs) for maximally permissive control purpose which is the policy based on the net structure [25]. Chao [26] showed that for the deadlocks prevention policy of a *k*-th order system with *k*=5, it needs additional 10 controllers. Hence it will be heavy cost in modeling large real-time resource allocation systems and adopting full deadlocks prevention policy for both the space and time implementation problems of the controllers allocation. Furthermore, modeling the collision problems of robots collaborative manufacturing systems, we cannot add additional controllers to prevent collision but need find a mechanism to avoid collision problems.

Based on the contributions of closed-form solution, we propose a new concept - the moment to launch resource allocation (MLR) for a partial deadlock avoidance\prevention policy to save the cost of controllers implementation for large real-time systems [23, 27]. In [28]applying the closed-form solution of *Gen-Left* and *Gen-Right*, the MLR concept can be completed for the non-sharing resource allocation loading balance in whole variant *k*-th order system; regarding the deadlock thread-holder (denoted as *DTH*) as a non-sharing waiting dummy resource that can provide process holding this *DTH* and wait for the other process's work flow, a simple deadlock avoiding algorithm is proposed. The decision-making of a *DTH* allocation is based on the maximum value of the reachable states of *Gen-Left* or *Gen-Right* (defined in Definition 1) subnet and the loading balance of processes.

However extending the application of MLR to the variant *k*-th order system containing a non-sharing subnet, the first problem to be solved is to construct the closed-form solution of the Deficient *k*-th order system with a non-sharing resource. (e.g. Fig.2). As shown in Fig.3, unless there is a mechanism of dynamic assigning the tokens of init marking, otherwise the process of non-sharing subnet (NNS) will share the tokens that flow into the *k*-th order like system. Hence the *Deficient k*-th order system is the essential element of non-sharing subnet in variant *k*-th order system. This paper will report the construction of the closed-form solution of a *Deficient Gen-Left* system. Due to the closed-form solution we can enhance the capability of dynamic modeling a PN containing a non-sharing subnet and resource extended from variant *k*-th order systems.

The approach is explained as follows. Chao [13] has proved that in a *k*-th order system the number of non-reachable states is the number of total states – Ŗ (Ŗ is the number of reachable states); *F=Ŗ – Ļ* where *F* (resp. *Ļ*) is the number of forbidden (resp. live) states. Let *Ŗ(k,h)* (resp. *Ļ(k,h)*) be the number of reachable states of a *Deficient k*-th order system with *h* ($h \leq k$) tokens in the left idle place. We have shown the states patterns of the "**A**dditional **R**estricted **R**eachable (resp. **L**ive) **S**tates derived from non-reachable (resp. non-live) states of the equivalent (defined in Definition 3) caused by a non-sharing resource", ARRS *ΔŖ(* resp. ARLS *ΔĻ*) [23], so that the number of reachable (resp. live) states of *Deficient Gen-Left* will be *Ŗ(k,h)* + *Ŗ(k,h−1)* +*ΔŖ*(resp. *Ļ(k,h)* + *Ļ(k,h−1)*+*ΔĻ*), where the value of "*h−1*" of the part of *Ŗ(k,h−1)* (resp. *Ļ(k,h−1)*) is caused by the situation when the holder of non-sharing resource contains the token will consume one token of *h*.

II. LITERATURE REVIEW

Definition 1: *A variant Deficient k-th order system is a subclass of S^3PR with k resource places r_1, r_2, ..., r_k shared between two processes N_L and N_R and one non-sharing resource place r'_{pos} (=r*) used by an operation place p^* in P_L* [23].

1. $\forall r \in P_R$, $M_0(r)=1$ [13].

2. N_L (resp. N_R) uses r_1, r_2, ..., r_k (resp. r_k, r_{k-1}, ...r_2, r_1) in that order [13].

3. $M_0(p_0)=h$, $1 \leq h<k$, $M_0(p'_0)=k$, where p_0 and p'_0 are the idle places in processes N_L and N_R, respectively. The system is denoted as a k_h system [31].

4. When $M_0(p_0)=M_0(p'_0)=k$ and non-sharing resource does not exist, the system is called a k-th order system [23].

5. Holder places of r_j in N_L and N_R are denoted as p_j and p'_j respectively [13].

6. The compound circuit containing r_i, r_{i+1}, ..., r_{j-1}, r_j is called (r_i-r_j)-region [13].

7.There are 3 possibilities for the token initially at r_i to sit at: p_i (N_L), r_i and p'_i (N_R). The corresponding token or r_i state is denoted by 1, 0 and −1 respectively [13].

8.d^n means r_{pos} is at d state (d=1,0, −1) and r'_{pos} is at n state (n=1, 0), where pos is the location of non-sharing resource being used by an operation place p^*, $1 \leq pos \leq k-1$ [23]. *The system is denoted as a (Deficient) Gen-Left k-th order system when p* in P_L* [31].

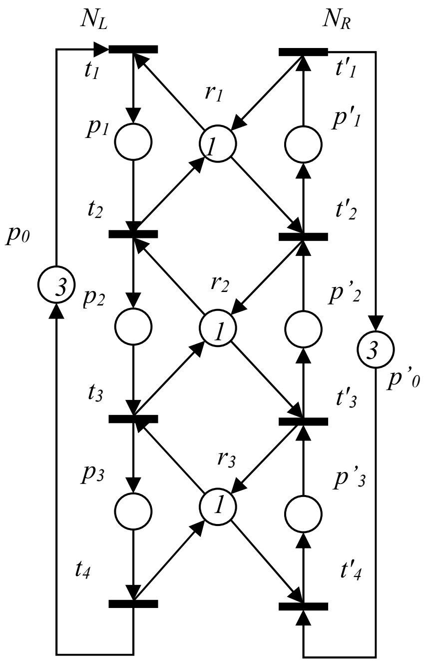

Fig. 1.3rd order system N^k.

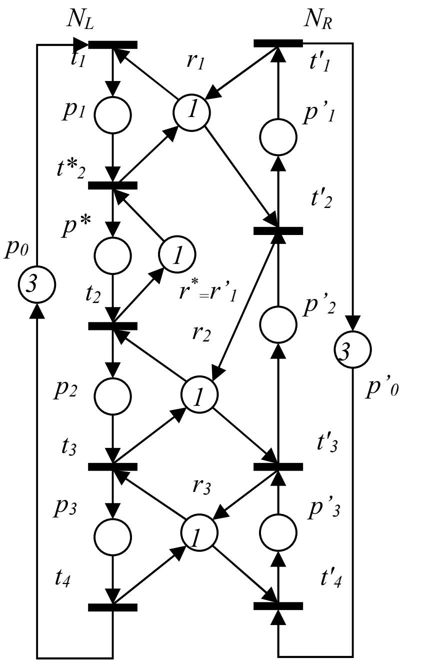

Fig. 2 Deficient) Gen-Left system where *pos* =1.

Examples are shown in Figs. 1, 2.

Definition 1[13]: *s = ($d_1 d_2$ ... d_k), x_i=1, 0 or −1, i = 1 to k, is a state for a k-th order system N, d_i is the token at r_i to sit at: p_i (N_L), r_i or p'_i (N_R) respectively. ($d_i d_{i+1}$ ... $d_q d_{q+1}$), $k \geq i \geq 1$, $k \geq q \geq i \geq 1$ (embedded in s) is a substate of s.*

Let N^r be the reverse net of a PN N. Chao [13] constructed the concept of a complete reachability graph and split the reachability graph of the PN N into *reachable* (from the initial state), *forbidden* (reachable to deadlock states only), *deadlock* (reachable to none state), *non-reachable* (from the initial state), and *empty-siphon+non-reachable states* (both non-reachable states in N and N^r)

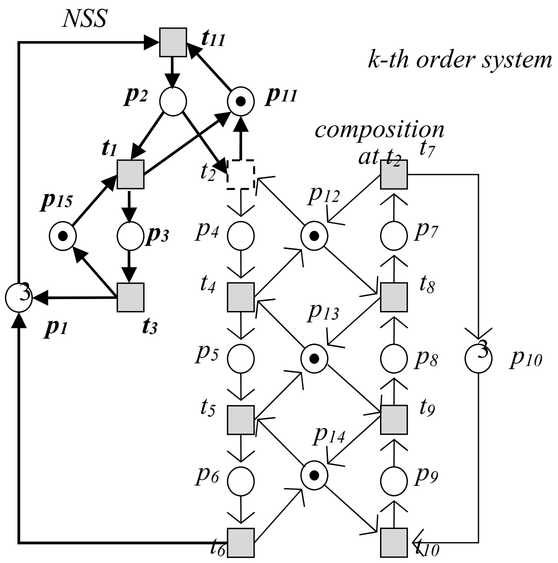

Fig. 3[29]. *a*-net :composed by *Deficient* *k*-th order system and non-sharing subnet(NSS)

According to graph theory, Chao found *Lemma 1* and *Lemma 2.*

Lemma 1 *[13]*: *Any forbidden state in N is non-reachable in* N^r.

Lemma 2 *[13]: Any non-reachable state s in N is a forbidden one or a non-reachable one in* N^r.

Lemma 3 *[13]: 1) s is a live state if and only if (iff)* $s = \{(y_1 \ldots y_k) | \, y_i = -1 \text{ or } 0\}$, *or* $s = \{(x_1 \ldots x_k) | \, x_i = 1 \text{ or } 0\}$. *2) The set of live states* $L_k = \{(x_1 \ldots x_k) | x_i = 1 \text{ or } 0\} \cup \{(y_1 \ldots y_k) | \, y_i = -1 \text{ or } 0\} = L_a \cup L_b$. 3) *The total number of live states is* $2^{k\,1} - 1$.

Lemma 4 *[13]: The possible reachable states are* $s = \{(x_1 x_2 \ldots x_j y_{j+1} \ldots y_k) | 0 \le j \le k\} = \{(x_1 \ldots x_j\, 1\, y_{j+2} \ldots y_k) | 1 \le j \le k \} \cup \{(y_1 \ldots y_k)\}$, *where* $x_i = 1$ *or* 0 $(i = 1$ *to* $j)$ *and* $y_p = 0$ *or* -1 $(p = j+2$ *to* $k) = L_c \cup L_d$.

Let *Ŗ, Ļ, F, Ŭ, Ɓ* and *Ď* be the functions of the total number of *reachable*(*Ŗ*), *live*(*Ļ*), *forbidden*(*F*), *non-reachable*(*Ŭ*), *empty-siphon+non-reachable*(*Ɓ*) and *deadlock*(*Ď*) states of a variant *k*-th order system, *respectively.*

In a *k*-th order system [13], the total number of states is 3^k; *Ļ(k)*$=2^{k+1}-1$; *Ŗ(k)*$=(k+2)2^{(k-1)}$; $F(k)=(k-2)2^{(k-1)}+1$; *Ŭ(k)*$=3^k-(k+2)2^{(k-1)}$; *Ɓ(k)*$=3^k-k2^k-1$; *Ď(k)*$=k-1$.

Definition 3: *The equivalent* $N^e = (P^e \cup P^e{}_R, T^e, F^e)$ *of a net* $N = (P \cup P_R, T, F)$ *(*P_{NR} *is the set of non-sharing places) is defined as*

1. $P^e{}_R = P_R \setminus P_{NR}$;
2. $P^e = P \setminus \bigcup_{r \in P_{NR}} H(r)$;
3. $T^e = T \setminus \bigcup_{r \in P_{NR}} r^{\bullet}$;

$$F^e = (F \bigcup_{r \in P_{NR}} ({}^{\bullet}r, r^{\bullet\bullet}) \cup ({}^{\bullet}(r^{\bullet}), {}^{\bullet}r) \setminus \bigcup_{r \in P_{NR}} [(H(r), H(r)^{\bullet}) \cup$$

$$({}^{\bullet}H(r), H(r)) \cup ({}^{\bullet}r, r) \cup (r, r^{\bullet}) \cup (r^{\bullet}, r^{\bullet\bullet}) \cup ({}^{\bullet}(r^{\bullet}), r^{\bullet})];$$

We can say that the net in Fig. 1 is the equivalent of the net in Fig. 2.

Let N^G be the *Gen-Left* system; N^k be the *k*-th order system; N^{rk} be the reverse net of N^k; *Ŗ(k,h)* (resp. *Ļ(k,h)*) be the number of reachable states of a *Deficient k*-th order system with *h* ($h \leqq k$) tokens in the left idle place. In [23], we showed the states pattern and the number of "**A**dditional **R**estricted **R**eachable (resp. **L**ive) **S**tates derived from non-reachable (resp. non-live) states of the equivalent (defined in Definition 3) caused by a non-sharing resource", ARRS *ΔŖ(* resp. ARLS *ΔĻ*) of *Gen-Left* system in ***Lemma*** 5 (resp. ***Lemma*** 7 and 8); the number of reachable (resp. live) states is *2Ŗ(k)+ΔŖ* (resp. *2Ļ(k)+ΔĻ*), where 2 is the token distribution of the existing one non-sharing resource under the condition that there are sufficient tokens in the left idle place. In *Deficient Gen-Left* the number of reachable (resp. live) will be *Ŗ(k,h)* + *Ŗ(k,h−1)* +*ΔŖ*$_D$ (resp. *Ļ(k,h)* + *Ļ(k,h−1)*+*ΔĻ*$_D$) where *ΔŖ*$_D$(resp. *ΔĻ*$_D$) is the number of ARRS(resp. ARLS) of *Deficient Gen-Left*. As shown in Fig.2, when *p* does not* contains the token, the *equivalent* of Fig.2 will be k_3 system, where the number of reachable states is *Ŗ(k,3)*; *p** contains the token, the *equivalent* will be k_2 system, where the number of reachable states is *Ŗ(k,2)*.

Lemma 5 [23]*: Let* $s = (d_1 \ldots -1_m \ldots 0^0 \ldots 1_n\, d_{n+1}\, d_{n+2} \ldots d_k)$ *where* $1 \le m \le pos$; $pos+1 \le n \le k$, *such that only the* r_m-r_n *siphon in* N^{rk} *is unmarked. M is non-reachable in* N^k, $M^* = M + r^*$ *is reachable in* N^G, *the total number of such M* is*$(2^{(k-n)})$*Ŗ(m−1).*

Lemma 6 [23]*: The total number of reachable states in* N^G *is* $Ŗ(k, pos) = 2Ŗ(k) + \left(\sum_{m=1}^{pos} Ŗ(m-1)\right)\left(\sum_{n=pos+1}^{k} \left(2^{(k-n)}\right)\right)$.

In ***Lemma 5***, $d_i=0$ or -1, $m+1 \le i \le k$ which is the tokens distribution of right hand side process; $(d_1 \ldots d_{m-1})$ is a reachable states pattern of an $(m-1)$-th order system. Hence The total number of such *M** in *Deficient Gen-Left* will be $(2^{(k-n)})$*Ŗ(m−1,h−1)* due to the fact that 1_n consume one token of *h*.

Hence we have the total number of reachable states in *Deficient Gen-Left* will be

$$Ŗ(k,h,pos) = Ŗ(k,h) + Ŗ(k,h-1) + \sum_{m=1}^{pos}\left(\sum_{n=pos+1}^{k} Ŗ(m-1,h-1)\left(2^{(k-n)}\right)\right).$$

Lemma 7 [23]*: Let $s=(d_1\ d_2\ \ldots\ 1_m\ \ldots\ 0^0\ 0\ \ldots\ -1_n\ \ldots d_k)$, $1 \le m \le pos$, $pos+1 \le n \le k$ correspond to Marking M such that there are unmarked siphons in only the$(r_m\text{-}r_n)$-region in N^k. The total number of possible live markings under M is $2^{(k-n)}$.*

Lemma 8 [23]*: Let $s=(d_1\ d_2\ \ldots -1_m\ 0_{m+1}\ldots\ 0^0\ \ldots 0_{n-1}\ 1_n\ \ldots d_k)$ $1 \le m \le pos$, $pos+1 \le n \le k$ correspond to non-reachable marking M such that there are unmarked siphons in only the r_m-r_n siphon in N^{rk}. The total number of possible live markings under M is $2^{(m-1)}$.*

Lemma 9 : *The total number of live states of Gen-Left is $Ļ(k,pos)=2(2^{(k+1)}-1)+(pos)(2^{(k-pos)}-1)+(k-pos)(2^{(pos)}-1)$, where $(pos)(2^{(k-pos)}-1)+(k-pos)(2^{(pos)}-1)$ is the total number of ARLS of Gen-Left.*

In ***Lemma 7*** and ***8*** the total number of possible live markings of *Gen-Left* is the value of the tokens distribution of right hand side process, hence it is independent to the tokens distribution of left hand side process, the number of ARLS of *Deficient Gen-Left* will be equal to *Gen-Left*. Hence the total number of live states in *Deficient Gen-Left* is

$$Ļ(k,h,pos) = Ļ(k,h) + Ļ(k,h-1) + (pos)\left(2^{(k-pos)}-1\right) + (k-pos)\left(2^{(pos)}-1\right).$$

III. COMPUTATION OF THE REACHABLE AND LIVE STATES OF *DEFICIENT GEN-LEFT* K-TH ORDER SYSTEM

Let k_h system be a *Deficient* k-th order system with h ($h \leqq k$) tokens in the left idle place; i.e., $M(p_0)=h$ and $M(p'_0)=k$.

Theorem 1 [31]*: Let Ŗ(k) be the reachable states of k-th order system, Ŗ(k,h) be the number of reachable states of k_h, a Deficient k-th order system.*

$$Ŗ(k,h) = Ŗ(k) - \sum_{l=h+1}^{k}\left(\sum_{i=0}^{l-h-1} C(l-1,l-1-i)\right)\left(2^{k-l}\right),$$

where $C(l-1,l-1-i) = (l-1)!/[(l-1-i)!i!]$ *is a binomial coefficient.*

Theorem 2 [31]*: Let Ļ(k) be the live states of k-th order system, Ļ(k,h) be the number of live states of k-th order system with token number of left idle place being h.*

$$Ļ(k,h) = Ļ(k) - \sum_{l=h+1}^{k}\left(\sum_{i=0}^{l-h-1} C(l-1,l-1-i)\right), \quad \textit{where}$$

$C(l-1,l-1-i) = (l-1)!/[(l-1-i)!i!]$ *is a binomial coefficient.*

Theorem 3 [31]:

$$\begin{aligned} F(k,h) &= Ŗ(k,h) - Ļ(k,h) \\ &= \left(Ŗ(k) - Ļ(k)\right) \\ &\quad - \sum_{l=h+1}^{k}\left(\sum_{i=0}^{l-h-1} C(l-1,l-1-i)\right)\left(2^{k-l}-1\right). \end{aligned}$$

Table 1 shows the total reachable, live and forbidden states of k_h system, where k=7 and 8.

Please refer to the above table for *Ŗ(k,h)*, *Ļ(k,h)* and *(k,h)*, where *k =7 and 8; h =1* to *8*; they are consistent with the reachability analysis using the INA [30] (Integrated Net Analyzer) tool.

Lemma 10*: Let $s = (x_1\ \ldots -1_m\ 0 \ldots 0^0\ \ldots 0\ 1_n\ x_{n+1}\ x_{n+2} \ldots\ x_k)$ where $1 \le m \le pos$; $pos+1 \le n \le k$, such that only the r_m-r_n siphon in N^{rk} is unmarked. M is non-reachable in N^k, $M^*=M+r^*$ is reachable in Deficient Gen-Left, the total number of such M^* is $(2^{(k-n)})Ŗ(m-1,h-1)$.*

TABLE I. STATES FOR K_H- SYSTEM, WHER K=7 AND 8.

h	Ŗ(7,h)	Ļ(7,h)	• (7,h)	Ŗ(8,h)	Ļ(8,h)	F(8,h)
1	255	135	120	511	264	247
2	375	156	219	758	292	466
3	474	191	283	977	348	629
4	538	226	312	1140	418	722
5	567	247	320	1233	474	759
6	575	254	321	1270	502	768
7	576	255	321	1279	510	769
8	576	255	321	1280	511	769

Theorem 4*: The total number of reachable states in Deficient Gen-Left is*

$$R(k,h,gen) = R(k,h) + R(k,h-1) + \sum_{m=1}^{pos}\left(\sum_{n=pos+1}^{k} R(m-1,h-1)\right)\left(2^{k-n}\right).$$

Theorem 5*:The total number of live states in Deficient Gen-Left is*

$$Ļ(k,h,pos) = Ļ(k,h) + Ļ(k,h-1) + (pos)(2^{(k-pos)}-1) + (k-pos)(2^{(pos)}-1).$$

Theorem 6:

$$\begin{aligned} F(k,h,pos) &= R(k,h,pos) - L(k,h,pos) \\ &= \left(R(k,h) + R(k,h-1) - L(k,h) - L(k,h-1)\right) \\ &\quad + \sum_{m=1}^{pos}\left(\sum_{n=pos+1}^{k} R(m-1,h-1)\right)\left(2^{k-n}\right) \\ &\quad - (pos)\left(2^{(k-pos)}-1\right) - (k-pos)\left(2^{(pos)}-1\right). \end{aligned}$$

According to ***Theorem 4,*** *Ŗ(8,7,3)*=*Ŗ(8,7)*+*Ŗ(8,6)*+ $\sum_{i=1 \text{ to } 3}$ ($\sum_{j=4 \text{ to } 8}$(*Ŗ(i−1,h−1)*$(2^{(k-j)})$))=1279+1270+(*Ŗ(0,6)*+*Ŗ(1,6)*+*Ŗ(2,6)*)($2^4+2^3+2^2+2^1+2^0$)=2921; according to ***Theorem 5,*** *Ļ(8,7,3)*=*Ļ(8,7)*+*Ļ(8,6)*+$(pos)(2^{(k-pos)}-1)+(k-pos)(2^{(pos)}-1)$=

1140. The values of *Ŗ(8,7,3)* and *Ļ(8,7,3)* are consistent with the reachability analysis using the INA.

IV. Conclusion

The characteristic of real-time resource allocation PNs model is that the net structure or the tokens of resources are changed dynamically. By MLR concept we report the very first method to compute in closed-form the number of reachable states of *Deficient Gen-Left k*-th order system (a simple version of S^3PR) without constructing a reachability graph. This helps to realize the MLR partial deadlock avoidance concept to a variant *k*-th order system with non-sharing subnet as shown in Fig.3 due to that we can dynamically and real-time allocate the deadlock thread-holder based on the maximum reachable states number by closed-form formula and avoid the dire situation of mid-run abortion of reachability analysis due to exhausted memory. Future work should extend the research on MLR concept to *Deficient k*-th order system with multi-tokens.

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APPENDIX

Index of terms

Table A1. The index of the notation of PN.

$H(r)$	The set of holder places that use r. $H(r) = {}^{\bullet\bullet}r \cap P$
M	Marking
M_0	Initial marking
$M_0(p)$	All tokens initially in p
N^G	Petri net (P, T, F, W), used in Section II for a *Gen-Left* k-th order system.
N_i	An S^2PR in N, i = 1, 2
(N, M_0)	Called a Petri net, a marked net, or a net system
P	The set of places
$p^\bullet$	The set of output transitions of a place p
${}^\bullet p$	The set of input transitions of a place p
P_i	Set of operation places of N_i, i = L, R
p_i	An operation place, i = 1, 2, …, n
$R(N,M_0)$	The set of reachable markings in the net (N, M_0)
S^2PR	Simple sequential process with resources
S^3PR	Systems of simple sequential process with resources
siphon and *trap*	For a Petri net (N, M_0), a non-empty subset S (resp. τ) of places is called a *siphon* (resp. *trap*) if ${}^\bullet S \subseteq S^\bullet$(resp. $\tau^\bullet \subseteq {}^\bullet\tau$)