Interval Forecasting with Fuzzy Time Series

Petrônio C. L. Silva[1], Hossein Javedani Sadaei[1], Frederico Gadelha Guimarães[2]

***Abstract*— In recent years, the demand for developing low computational cost methods to deal with uncertainty in forecasting has been increased. Interval forecasting is a category of forecasting in which the method provides intervals as outputs of its forecasting. The initial aim of this paper is therefore proposing a new interval forecasting method based on a low cost and accurate forecasting method, namely first order Fuzzy Time Series. In this study, the results of the proposed method are compared with actual data and regular point forecasts using Fuzzy Time Series. The evaluation of the results shows the accuracy and promising performance of the proposed method.**

***Index Terms*— Fuzzy Time Series, Uncertainty, Interval Forecasting**

I. INTRODUCTION

In many practical forecasting problems, such as financial and environmental time series, one can find challenging behaviors and patterns inside the data which can turn its analysis and prediction into a very hard task. In addition to that, the volume of data is another significant challenge when accurate, reliable, and fast forecasting methods are sought for. This scenario motivated, in recent years, the development of new methods of forecasting allying predictive performance and low processing cost. One of these methods is Fuzzy Time Series [1], which has been drawing some attention and getting more relevance in recent years due to many studies reporting its good accuracy compared with other models [2]. Although these FTS methods have received some criticism from the literature, see for instance [3], due to methodological problems, many of these issues have been approached in more recent papers [4].

Nevertheless, all forecasting problems are subject to different types of uncertainties, either related to imprecision in the data due to measurement errors, sensor calibration and other unknown factors, or related to the forecasting activity itself, since not all characteristics of the underlying process that generated the data might be modeled perfectly, leading to errors in the forecast.

This fact led to the development of methods for Probabilistic Forecasting [5] and Interval Forecasting [6], to deal with forecasting uncertainty by estimating distributions of possible values instead of a unique point forecast. In wide sense, the Interval Forecasting can be viewed as a particular case of the Probabilistic Forecasting, if we consider an uniform distribution between the bounds of the interval forecast.

FTS models extend time series with the concept of fuzzy sets, identifying patterns and relationships between these fuzzy sets in the in-sample data. However, most papers on FTS typically produce a point forecast defuzzyfying the forecast value. In the literature of FTS, the methods to produce interval forecasts are usually based on Type 2 Fuzzy Sets, as in [7] and [8]. The main drawback of these techniques is the high computational cost demanded by the Type 2 Fuzzy Sets.

The present study proposes a new approach for Interval forecasting by extending FTS point forecast. The proposed Interval Fuzzy Time Series ([$\mathbb{I}$]FTS) is introduced to produce not only the point forecast based on the midpoints of the fuzzy sets but also an interval forecast based on the support of those fuzzy sets in the fuzzy logical relationship groups (FLRG) and using interval algebra. The proposed method is simple and computationally cheap. To explain the details, the paper is organized as follows: section II presents the related literature, section III introduces the proposed Interval Fuzzy Time Series model as an alternative method for forecasting under uncertainty, section IV provides some experiments to validate the proposed method and compare its performance with that of other methods, and finally, section V concludes the paper.

II. LITERATURE REVIEW

A. *Fuzzy Time Series Models*

Fuzzy Time Series (FTS) are non parametric models introduced by Song and Chissom [1] based on Fuzzy Set theory [9]. These methods are easy to implement and very flexible, affording ways to deal with numeric and non-numeric data. Some of FTS methods produce compact and human readable models of the time series behavior using fuzzy rules which can be used by business experts and researchers.

There are several categories of FTS methods, varying mainly by its order and time-variance. The order indicates how many time-delays (lags) are used in modeling the time series. Given the time series data F, the First Order models use $F(t-1)$ data to predict $F(t)$ and the High Order models use $F(t-1), F(t-2), ..., F(t-k)$ data to predict $F(t)$. Time varying models require updates of the current model with time to produce new forecasts.

Song and Chissom [1] proposed the main steps of all FTS methods but its computation demands many matrix operations for each forecasting making the process computationaly expensive. Chen [10] simplified Song and Chissom's algorithm by creating the Fuzzy Logical Rule Groups (FLRG),

[1]Graduate Program in Electrical Engineering - Federal University of Minas Gerais - Av. Antnio Carlos 6627, 31270-901, Belo Horizonte, MG, Brazil.

[2]Department of Electrical Engineering, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil.

making the forecasting process cheaper by avoiding the use of matrix manipulations. The FLRG represent the rule base of the model and are human readable and easy to interpret. Both methods are known as Conventional FTS models.

The initial step of training an FTS model is the partitioning of the Universe of Discourse U, that is, the data range covered by the training data that must be transformed into fuzzy sets. This is certainly one of the most crucial steps on fuzzy time series methods due to their influence on forecasting accuracy. The number of intervals, the length of these intervals and their midpoints are all parameters that should be chosen carefully or indeed optimized. The partitioning scheme initially proposed in Conventional FTS is just the division of the data range in k equal length intervals, the same method adopted hereafter in this work just for simplicity. However, more accurate methods have been proposed in the literature, we refer to [11] and [12] for further details.

The generation of FLRG from the fuzzyfied data in FTS model has, at least, two drawbacks: the lost of rule's recurrence and their chronological order. Thus at the forecasting process a very recurrent pattern of data has the same importance of a unique occurrence pattern. Moreover, newer and older patterns also have the same weight in the forecast.

To fix these drawbacks Yu [13] proposed the *Weighted Fuzzy Time Series* (WFTS) model by including weights on FLRG's. These weights are monotonically increasing and have a smoothing effect, giving more importance to the most recent data on forecasting process.

The works of [14] and [15] present the *Improved Weighted Fuzzy Time Series* (IWFTS) model and change the way in which the weights are assigned to the RHS rules on Yu's model. The main difference is that the weights are calculated by the recurrence of each rule, discarding the chronological order. The *Exponentialy Weighted Fuzzy Time Series* (EWFTS) method, proposed by Sadaei [16] and [17], replaces the linear weight growth of WFTS model by an exponential growth.

There are hybrid FTS techniques such as Askari and Montazerin [18] that proposes a high-order multi-variable FTS algorithm based on fuzzy clustering, Sadaei et al. [4] combines statistical ARFIMA models with FTS for forecasting of long-memory time series.

All these models have some common drawbacks. First, in the forecasting step just one FLRG is choosen for computing the result, based on the maximum membership between the input value and all the FLRG's. This causes the lost of "smoothing" effect of fuzzy methods, which demands mixing many sets according to their fuzzy memberships . Secondly, these models are point-based forecasters and give no uncertainty measures about their results.

B. Interval Forecasting

Interval Forecasts generalize point forecasts to represent and incorporate uncertainty [19], usually by estimating an inter quantile range.

Parametric approaches include Chatfield [6], Gardner and Everette [20]. These methods use strong statistical assumptions about data that can make it less useful where data is not conforming.

Some non parametric approaches were developed to bypass the normality assumptions, for instance in Koenker [21], Steinwart and Christmann [22] and Takeuchi [23]. However these methods usually involve loss functions optimizations or iterative approaches such as Markov Chain Monte Carlo or Expectation-Maximization.

Quantile and Interval Forecasts have been used in a variety of fields of forecasting including Wind Electrical Generation [24], Supermarket Sales [25] and Power Load [26], [27], [28].

Beyond the probabilistic uncertainty represented by these quantile intervals other kind of uncertainty needs to be represented, the fuzzy uncertainty. Intervals can be seen as fuzzy sets projections on real line and common arithmetic operations can be applied to them. These topological and algebraic properties of intervals are discussed in [29], which also discusses their relationships with fuzzy sets.

In the next section a new method of interval forecasting is proposed. This method represents the fuzzy uncertainty by extending the point-based FTS methods to incorporate the fuzzy bounds on prediction.

III. The Interval Fuzzy Time Series - $[\mathbb{I}]$FTS

Here a simple, fast and effective method is proposed to deal with uncertainty, combining the flexibility of the FTS models with the properties of Interval Forecasts without the need to resort to parametric methods or optimization techniques as in quantile estimation methods. The importance of this is, for example, when the method is used in a high sized data or on a fast streaming with concept drifts, which demands the model to be frequently updated.

The Inverval Fuzy Time Series Model ($[\mathbb{I}]$FTS) aims to produce a prediction interval based of fuzzy bounds but without probabilistic meaning. The method is described below in two separated procedures: the model building procedure and the forecasting procedure. The model training procedure is based on [1] and [10] and aims to construct the FLRG rule base:

Model building procedure:

1) Define the universe of discourse U from data D as $U = [D_{min} - D_1, D_{max} + D2]$;
2) Partition the universe of discourse in k even intervals u_i of size $\frac{(D_{max}+D2)-(D_{min}-D_1)}{k}$, where D_1 and D_2 are just numbers used to round D_{max} and D_{min} to next integer multiple of 10 ;
3) Define the fuzzy sets A_i on the universe U. Each fuzzy set will be related to an interval u_i, will have a midpoint m_{A_i} and will be associated with a triangular fuzzy membership function $\mu_{A_i}(x)$. The vector $\mu_{A_i} = [\mu_{A_i}(u_1), .., \mu_{A_i}(u_k)]$ represents the membership values of fuzzy set A_i with the midpoints of all the u_i intervals;

4) Fuzzify historical data D, generating a new dataset D_f. Each datapoint $d_i \in D$ will be replaced by the fuzzy set A_k which has maximum membership value $\mu_{A_k}(d_i)$;
5) From D_f establish all the Fuzzy Logical Relationship - FLR between two following sets in the format $A_j \rightarrow A_j$ where A_i is a fuzzyfied value in time t and A_j is the fuzzyfied value in time $t+1$. After all the FLR's are generated, eliminate the duplicate rules;

$$\begin{array}{ccc} A_i \rightarrow A_j & A_i \rightarrow A_k & A_i \rightarrow A_l \\ A_j \rightarrow A_j & A_j \rightarrow A_l & A_i \rightarrow A_l \\ A_i \rightarrow A_j & A_j \rightarrow A_l & A_j \rightarrow A_j \end{array} \tag{1}$$

6 Generate the FLRG - Fuzzy Logic Relationship Groups by grouping the FLR' by the left hand side (LHS) of each rule and the right hand side (RHS) sets of each rule will be grouped to form the right hand side of the FLRG. For example, the group of FLR's on Equation 1 will generate the FLRG on Equation 2.

$$\begin{array}{ccc} A_i & \rightarrow & A_j, A_k, A_l \\ A_j & \rightarrow & A_j, A_l \end{array} \tag{2}$$

The FLRG make up the rule base of the model and are human readable and easy to interpret. An FLRG has the form $LHS \rightarrow RHS$ where LHS always has one fuzzy set, representing $F(t-1)$ and the RHS has all the fuzzy sets that followed LHS in the FLR's, representing all possible $F(t)$ coming from $F(t-1)$. The number and format of the FLRG rule base is closely related with the variance and stationarity of the dataset.

The forecasting procedure uses the generated model to build prediction intervals based on the mean interval of the RHS fuzzy sets on each FLRG weighted by their fuzzy membership in relation to input value:

Forecasting procedure:

1) For a given input value $F(t)$, find the membership value μ_i for all A_i fuzzy sets;
2) Choose all FLRGs which LHS $\mu_i > 0$;
3) Each choosen FLRG will generate a interval $\mathbb{I}^i = [\underline{\mathbb{I}^i_{min}}, \overline{\mathbb{I}^i_{max}}]$ where $\mathbb{I}^i_{min}$ is the minimum lower bound of all RHS fuzzy sets of FLRG i and $\mathbb{I}^i_{max}$ is the maximum upper bound of RHS fuzzy sets of FLRG i;

$$\begin{array}{ccc} \mathbb{I}^i_{min} & = & min(A_1, ..., A_k) \\ \mathbb{I}^i_{max} & = & max(A_1, ..., A_k) \end{array} \tag{3}$$

4) The final forecast interval $\mathbb{I}_f$ is calculated as the sum of the FLRGs intervals weighted by the membership value of each FLRG, as showed in Equation 4

$$\mathbb{I}_f = \frac{\sum_{i \in A} \mu_i \mathbb{I}^i}{\sum_{i \in A} \mu_i} = \frac{\sum_{i \in A} [\mu_i \underline{\mathbb{I}^i_{min}}, \mu_i \overline{\mathbb{I}^i_{max}}]}{\sum_{i \in A} \mu_i} \tag{4}$$

The generated interval $\mathbb{I}_f$ is bounded by a composition of the fuzzy sets bounds on the FLRG's which have some membership with the input value $F(t)$ and is expected to contain the true value $F(t+1)$. In next sections we present a demonstration of the method and propose a discussion about its main characteristics.

A. Application example

For clarification of the method, let us take the University of Alabama Enrollments dataset, retrieved from [30]. Using a partitioning scheme with 6 equal-length intervals and triangular membership functions we get the fuzzy sets listed on table I, and the generated FLRG's are listed in Equation 5.

Fuzzy Set	Lower bound	Midpoint	Upper bound
A_1	10612	12008	13404
A_2	12008	13404	14800
A_3	13404	14800	16196
A_4	14800	16196	17592
A_5	16196	17592	18988
A_6	17592	18988	20384

TABLE I
PARTITION SCHEME FOR UNIVERSITY OF ALABAMA ENROLLMENTS DATASET

$$\begin{array}{ccc} A_1 & \rightarrow & A_1 \\ A_2 & \rightarrow & A_2, A_3 \\ A_3 & \rightarrow & A_3, A_4 \\ A_4 & \rightarrow & A_3, A_4, A_5 \\ A_5 & \rightarrow & A_4, A_6 \\ A_6 & \rightarrow & A_6 \end{array} \tag{5}$$

Given an input value $F(t-1) = 16,894$, the membership values for fuzzy sets will be zero except for $\mu_{A_4} = 0.5$ and $\mu_{A_5} = 0.5$. Picking the FLRG's $A_4 \rightarrow A_3, A_4, A_5$ and $A_5 \rightarrow A_4, A_6$ we need to find the intervals I_1 and I_2 from first and second FLRGs. Following the values of the fuzzy sets the intervals will be $I_1 = [\underline{13404}, \overline{18988}]$ and $I_2 = [\underline{14800}, \overline{20384}]$. Now we can calculate the final interval I_f as showed in Equation 6 and the same process is illustrated in Figure 1. In figure 2 it is possible to check the interval forecast along the time series.

$$\begin{array}{ccc} \mathbb{I}_f & = & \frac{[0.5 \cdot \underline{13404} + 0.5 \cdot \overline{14800}\ ,\ 0.5 \cdot \underline{18988} + 0.5 \cdot \overline{20384}]}{0.5+0.5} \\ & = & [\underline{14102}, \overline{19686}] \end{array} \tag{6}$$

B. Method Discussion

This model has three major improvements from the models cited on section II-A: a) use of more than one FLRG on forecasting process; b) weighting the forecast output by the membership values of the input value; c) the interval forecast. The last characteristic put this model at a different category in time-invariant first-order fuzzy time series models. Other non-interval FTS methods can be compared with the [$\mathbb{I}$]FTS by comparing their point forecasts with the midpoint of the interval $\mathbb{I}_f$.

The prediction interval has a different meaning from intervals generated by parametric models and quantile based methods cited on section II-B. The interval produced by this model covers up the uncertainty of future values $F(t+1)$ by their historical fuzzyfied values. First because the learned model, represented by the FLRG's, is a fuzzy description of

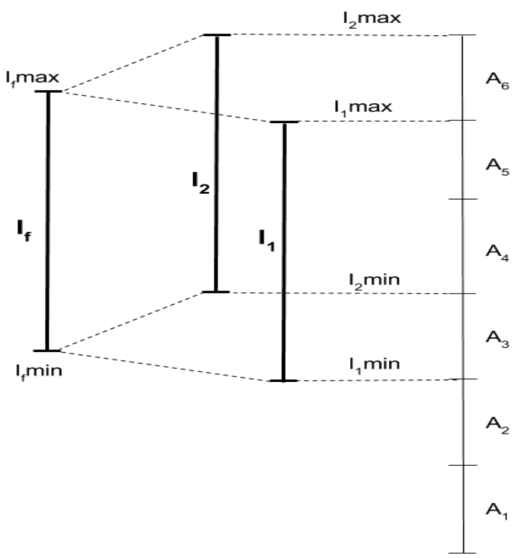

Fig. 1. Interval calculus from two FLRG's

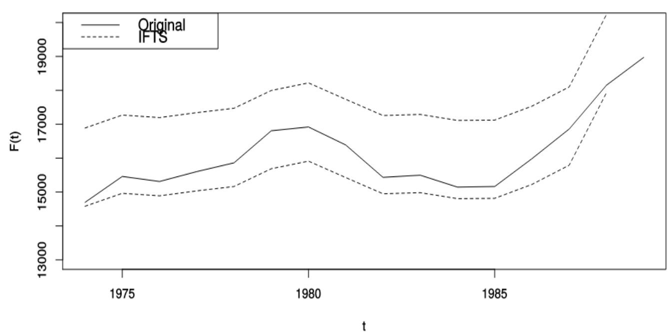

Fig. 2. Interval prediction over the University of Alabama Enrollments dataset

past behaviors of data set assuming time invariance. Each FLRG represents all possibilities of change from the actual value $F(t)$ (the LHS fuzzy set) to future value $F(t+1)$ (one or more of the RHS fuzzy sets). Then, forecasting $F(t+1)$ for this model is to find the lower and upper bounds of the fuzzy uncertainty represented in the RHS fuzzy sets of all FLRG's related with $F(t)$. The meaning of forecasted interval $\mathbb{I}_f$ is the value range which contains the real value given the fuzzyness of data.

There is no need to know the population's statistical parameters but the Universe of Discourse partitioning scheme affects the precision of the model by changing the length of forecast intervals. In Figure 3 and Table II, using same data set as the previous section, the effects of partitioning scheme on the intervals can be seen. It shows that the accuracy of the intervals is adjustable by changing the partitioning scheme of the model. This is indeed the unique required parameter for this method.

In next sections we compare the effectiveness of this proposed model with common FTS point forecasts methods and study the properties of the forecast intervals.

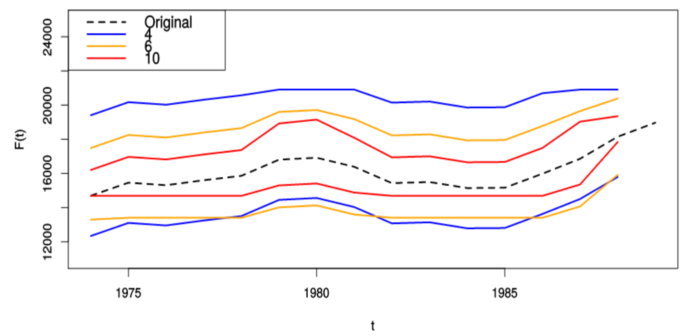

Fig. 3. Effect of partition length on prediction interval length

Partitions	Partition Length	Interval Length
4	4711.5	6608.797
6	2792.0	4860.562
10	1507.7	2475.775

TABLE II

EFFECT OF PARTITION LENGTH ON PREDICTION INTERVAL LENGTH

IV. BENCHMARKS

Comparing point forecasts with interval forecasts is just possible if we consider the midpoint of the interval, which is the transformation of the interval forecast to a point forecast. This is clearly not a good way to evaluate interval forecasts but it makes possible to compare these two different approaches. Further ahead in this section we describe the properties of the proposed method results using interval metrics as coverage, calibration and sharpness.

To measure the performance of the proposed model, two well known financial time series data (the TAIEX and NASDAQ data sets) were selected, where a cross-validation method was applied for training and forecasting on test data. The results were then compared with the methods of Conventional FTS from Chen [10], Weighted FTS of Yu [13], Improved Weighted FTS of Ismail and Efendi [14] and Exponentialy Weighted FTS of Sadaei [3] models, all of them trained with the same methods and data.

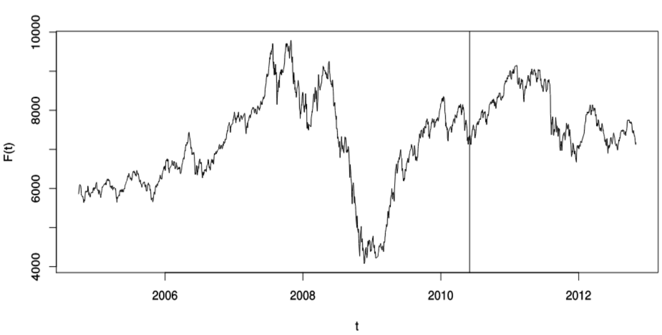

Fig. 4. TAIEX dataset sample used on benchmarks

The Taiwan Stock Exchange Capitalization Weighted

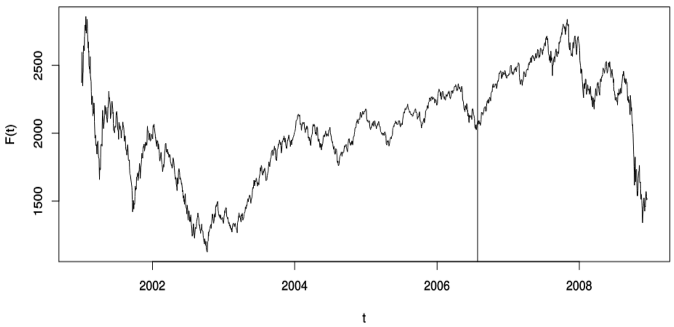

Fig. 5. NASDAQ dataset sample used on benchmarks

Stock Index (TAIEX)[1], a well known economic time series data commonly used in FTS literature ([13], [31], [7], [32], [4], etc) was sampled on 2005 to 2013 time window, and has the averaged daily index by business day, totalizing 2000 instances where 1400 were used in training and the remaining 600 in validation. At Figure 4 the sample time series is shown, and the vertical line separates training and validation sets. The National Association of Securities Dealers Automated Quotations - Composite Index (NASDAQ ÎXIC)[2] was sampled on 2001 to 2008 time window, and has the averaged daily index by business day, also totalizing 2000 instances where 1400 were used in training and the remaining 600 in validation. At Figure 5 the sample data is shown with its train and test sub samples..

The accuracy metrics used to evaluate models are the Mean Average Percent Error (MAPE), described in Equation 7, and Root Mean Squared Error (RMSE), described in Equation 8, where Y means the real data and $\hat{Y}$ the forecasted values. The universe of discourse was partitioned in a grid scheme, where all partitions have the same length. Each model was trained and tested for 15, 20, 25, 30, 35 and 40 partitions and the results are shown in Table III.

$$MAPE = \frac{1}{n}\sum_{i=1}^{n}\left|\frac{Y_i - \hat{Y}_i}{Y_i}\right| \tag{7}$$

$$RMSE = \sqrt{\frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n}} \tag{8}$$

The results make clear the strong impact that the number of partition intervals has on the accuracy of point forecasts. The proposed model has a bit lower performance when compared with the other models, but it is necessary to remind that this comparisons is made over the midpoint of the interval which is not the focus of the interval forecast. When comparing the results in Figures 6 and 8 with all test data, and Figures 7 and 9 with a smaller sample, the desired effect of the interval forecast becomes more visible. The produced intervals were capable of encapsulating the predictions of all other models and delimiting the boundaries of models uncertainty.

Model	**Partitions**	TAIEX		NASDAQ	
		RMSE	**MAPE**	**RMSE**	**MAPE**
Chen	15	122.819	0.0136	36.024	0.0133
	20	121.830	0.0128	36.144	0.0125
	25	86.919	0.0091	40.993	0.0134
	30	81.442	0.0083	38.666	0.0127
	35	85.403	0.0088	46.687	0.0151
	40	62.913	0.0066	44.855	0.0152
Yu	15	122.078	0.0135	37.625	0.0137
	20	93.133	0.0104	35.670	0.0124
	25	72.081	0.0079	38.877	0.0126
	30	57.412	0.0062	34.853	0.0118
	35	51.746	0.0056	42.749	0.0131
	40	46.009	0.0050	47.745	0.0151
Ismail & Efendi	15	123.844	0.0136	39.501	0.0144
	20	93.049	0.0104	35.085	0.0122
	25	72.036	0.0079	36.833	0.0124
	30	56.935	0.0061	39.087	0.0125
	35	50.714	0.0055	47.795	0.0146
	40	45.794	0.0050	49.429	0.0155
Sadaei	15	143.802	0.0146	42.840	0.0150
	20	110.091	0.0116	36.678	0.0124
	25	77.507	0.0080	35.252	0.0120
	30	70.400	0.0073	36.120	0.0120
	35	63.462	0.0065	45.432	0.0144
	40	52.064	0.0053	48.615	0.0156
$[\mathbb{I}]$FTS	15	77.684	0.0072	7.722	0.000
	20	99.734	0.0097	17.213	0.0035
	25	86.351	0.0081	29.061	0.0087
	30	93.559	0.0093	26.268	0.0071
	35	97.293	0.0097	36.609	0.0104
	40	84.125	0.0080	31.743	0.0107

TABLE III
MODEL ERROR COMPARISONS

Now three main properties for interval forecasts must be evaluated: *coverage rate*, *calibration* and *sharpness*, as proposed in [33] and [24]. The *coverage* refers to the statistical consistency between the forecasts and the observations, and measures which proportion of the observations are inside the interval. This can be done by a Indicator Function, developed by [34], as shown in Equation 9. Given a forecasting interval $\mathcal{I} = [\underline{\alpha}, \overline{\beta}]$ and the real value y, the indicator function I verifies if y is covered by $\mathcal{I}$ or not.

$$I(y, \mathcal{I}) = \begin{cases} 1 & \text{if } y \in \mathcal{I} \\ 0 & \text{if } y \ni \mathcal{I} \end{cases} \tag{9}$$

The *coverage rate* is the average value of indicator function between forecasted intervals and the real values, in which the ideal value is 1. The coverage rate is shown at Equation 10 where $y_i \in Y$ are the real values and $\mathcal{I}_i \in \mathcal{I}$ are the predicted intervals for these values.

$$C(Y, \mathcal{I}) = \frac{\sum_{i=1}^{|Y|} I(y_i, \mathcal{I}_i)}{|Y|} \tag{10}$$

The property of *sharpness* and *resolution* refers to the concentration of the predictive distribution, or how wide and variable are the intervals and refers uniquely to the forecasts. *Sharpness*, presented in Equation 11, is the average size of

[1] $http://www.twse.com.tw/en/products/indices/Index_series.php$. Access in 23/05/2016

[2] $http://www.nasdaq.com/aspx/flashquotes.aspx?symbol = IXIC\&selected = IXIC$. Access in 23/05/2016

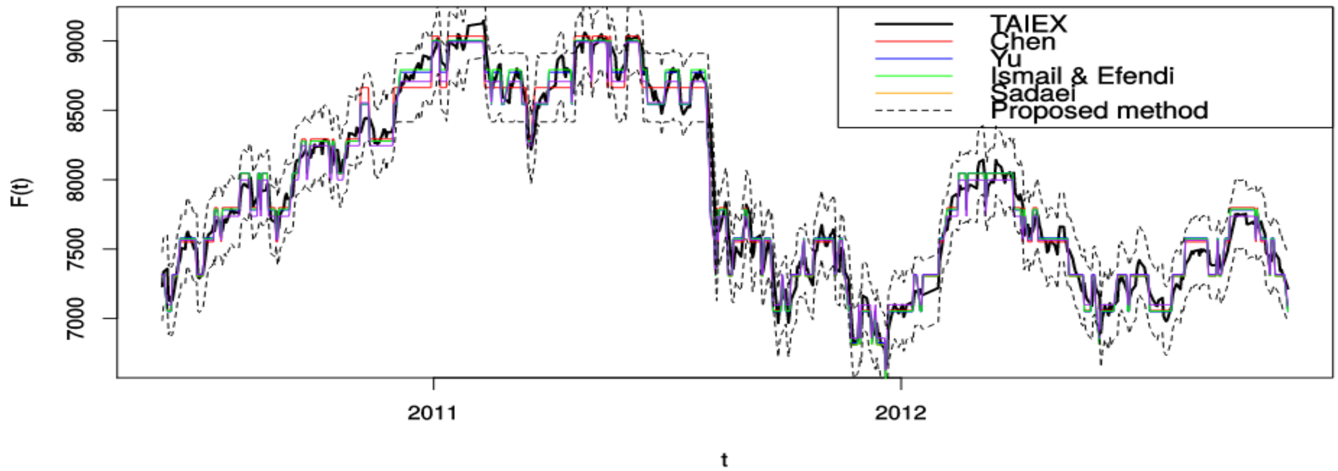

Fig. 6. Forecasts for the whole TAIEX test data

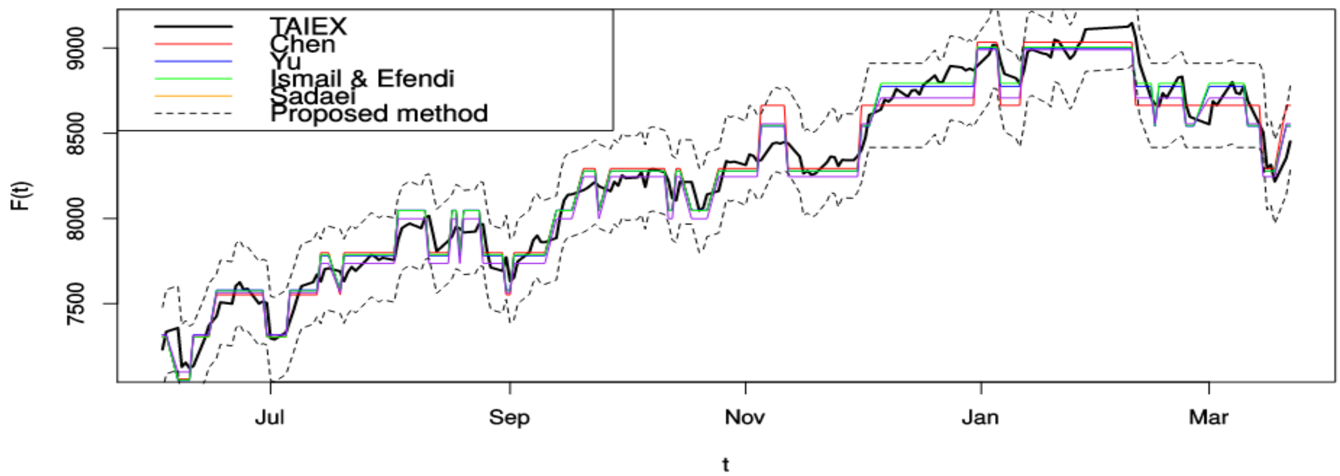

Fig. 7. Forecasts for a small piece of TAIEX test data

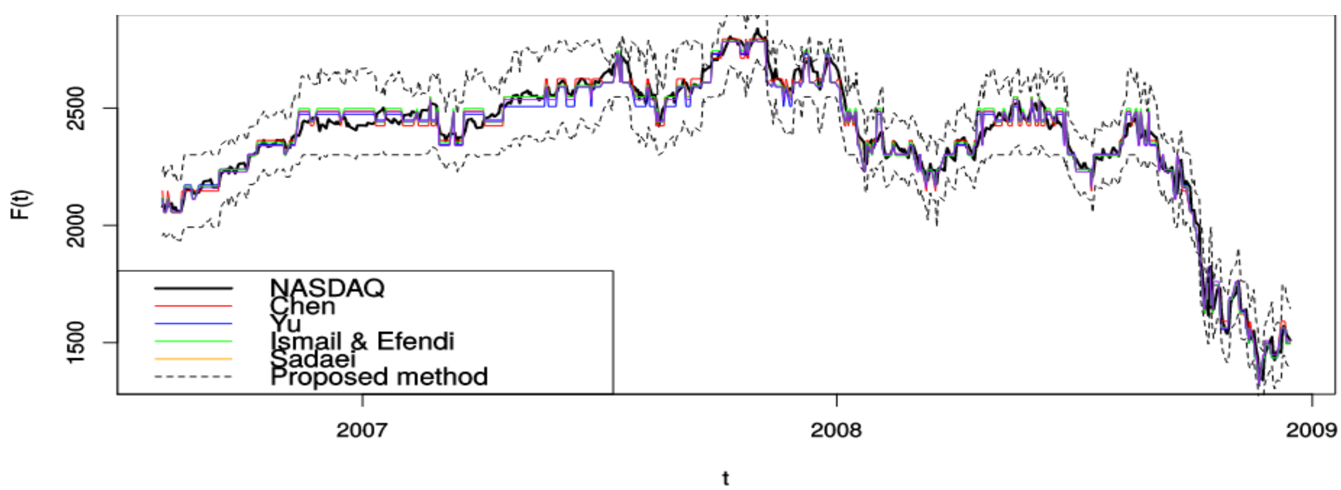

Fig. 8. Forecasts for the whole NASDAQ test data

the intervals and *resolution*, presented in the equation 12, is the variability of the intervals.

$$\bar{\delta_{\mathcal{I}}} = \frac{\sum_{i=1}^{|\mathcal{I}|} \delta_{\mathcal{I}_i}}{|\mathcal{I}|} = \frac{\sum_{i=1}^{|\mathcal{I}|} \overline{\beta_i} - \underline{\alpha_i}}{|\mathcal{I}|} \tag{11}$$

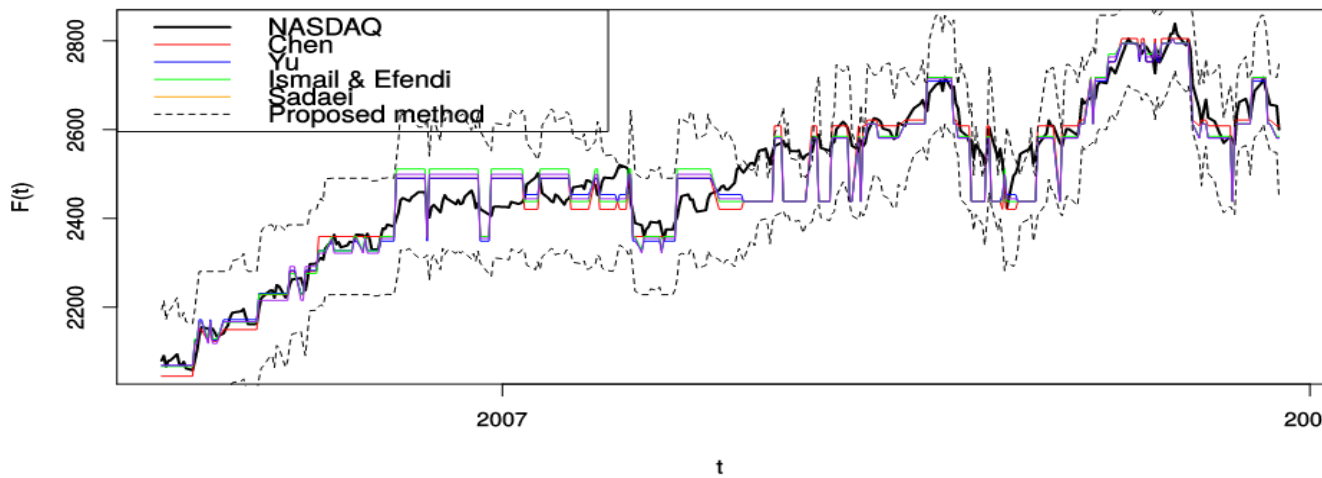

Fig. 9. Forecastes for a small piece of NASDAQ test data

$$\sigma_{\mathcal{I}} = \frac{\sum_{i=1}^{|\mathcal{I}|} |\delta_{\mathcal{I}_i} - \bar{\delta_{\mathcal{I}}}|}{|\mathcal{I}|} \tag{12}$$

While small values of $\bar{\delta_{\mathcal{I}}}$ are desirable, meaning a compact interval, wide values of $\sigma_{\mathcal{I}}$ are best, meaning the capability of the model to adapt the interval with the increase of uncertainty. There's no absolute reference values for sharpness and resolution which depends on the statistical properties of the data.

At Table IV it is possible to compare the coverage rate, sharpness and resolution of the proposed model for each number of partitions and each data set. This table shows that in the worst case the real value is covered by the interval in 94% of time even with thin intervals, by augmenting the number of partitions. The effects of the number of partitions in the length of intervals is also displayed at Figure 10 for TAIEX data set. It is important to observe that the intervals are not always evenly distributed around the real value, as effect to the weighting of the fuzzy membership of input value and the distribution of the intervals along the Universe of Discourse, but will be inside the interval almost all time.

Partitions	TAIEX			NASDAQ		
	Cov.	**Sharp.**	**Res.**	**Cov.**	**Sharp.**	**Res.**
15	1	1728.27	102.14	1	521.71	5.79
20	1	1333.61	88.97	1	385.84	20.39
25	1	1078.45	107.44	1	352.77	41.61
30	1	954.16	106.31	1	287.51	44.61
35	1	827.28	106.76	0.94	251.13	36.15
40	1	787.29	99.48	0.94	210.11	36.99

TABLE IV
COVERAGE, SHARPNESS AND RESOLUTION BY NUMBER OF PARTITIONS

These results show that the proposed method has similar accuracy to the FTS methods with point forecasts and, in reference of its interval properties, has good reliability (or calibration) and adjustable sharpness according to the number of the partitions on the Universe of Discourse.

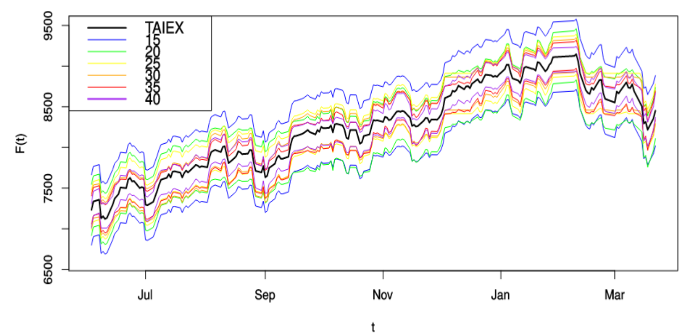

Fig. 10. Forecasted intervals by number of intervals

V. CONCLUSION

Point forecasting methods has as its main general drawback the inability to measure the uncertainty of their results. Depending on the field of application, knowing the uncertainty associated with forecast is indispensable. A way to introduce, and bound, the forecasting uncertainty is using the Interval Forecasting methods such as Quantile Regression and Estimation. In the other hand, Fuzzy Time Series methods have been gaining attention in recent years by their simplicity and accuracy.

In this work a new method for interval forecasting is proposed, using First Order Time Invariant Fuzzy Time Series. This method extends classical model of [10] to produce intervals and include two others innovations: new weighting method based of membership value and the use of multiple FLRG's on the forecasting procedure.

The experiments with benchmarks, comparing some literature FTS models and the proposed method, shows its effectiveness, with accuracy values near the point forecasters, 94% of coverage rate and adjustable sharpness.

A. Method Limitations

It has been observed on the benchmarks that this method has a low resolution, therefore the forecasted intervals have low variation independent of input. It is desirable that the interval length has variations measuring the confidence of that interval due to input value.

B. Future works

For the next works some improvements are planned:

1) Extending the proposed interval forecast to a probabilistic forecast, producing a probability mass distribution as result;
2) Introducing a conditional probability distribution (or Bayesian stochastic model) for calculating the transitions of FLRG's with time;
3) Weight the FLRG rules by their confidence of result based on input values, thus correcting the resolution issue.

ACKNOWLEDGMENT

This work has been supported by the Brazilian agency CAPES. Petrônio C. L. Silva would like to thank to the support given by the PBQS program of the Brazilian institution IFNMG. H. Javedani Sadaei and F. G. Guimarães would like to thank the support given by the Brazilian Agencies CNPq and FAPEMIG.

REFERENCES

[1] Q. Song and B. S. Chissom, "Fuzzy time series and its models," *Fuzzy Sets and Systems*, vol. 54, no. 3, pp. 269–277, 1993.
[2] P. Singh, "A brief review of modeling approaches based on fuzzy time series,"
[3] H. Javedani Sadaei, *Improved models in Fuzzy Time Series for forecasting*. PhD thesis, Universiti Teknologi Malaysia, 2013.
[4] H. Javedani Sadaei, R. Enayatifar, F. G. Guimarães, M. Mahmud, and Z. A. Alzamil, "Combining ARFIMA models and fuzzy time series for the forecast of long memory time series," *Neurocomputing*, vol. 175, pp. 782–796, 2016.
[5] T. Gneiting and M. Katzfuss, "Probabilistic Forecasting," *Annual Review of Statistics and Its Application*, vol. 1, pp. 125–151, 2014.
[6] C. Chatfield, "Calculating Interval Forecasts," *Journal of Business & Economic Statistics*, vol. 11, p. 121, 4 1993.
[7] N. S. Bajestani and A. Zare, "Forecasting TAIEX using improved type 2 fuzzy time series," *Expert Systems with Applications*, vol. 38, no. 5, pp. 5816–5821, 2011.
[8] K. Huarng and H.-K. Yu, "A type 2 fuzzy time series model for stock index forecasting," *Physica A: Statistical Mechanics and its Applications*, vol. 353, pp. 445–462, 2005.
[9] L. A. Zadeh, "Fuzzy Sets," *Infomation and Control*, no. 8, pp. 338–353, 1965.
[10] S.-M. Chen, "Forecasting enrollments based on fuzzy time series," *Fuzzy Sets and Systems*, vol. 81, no. 3, pp. 311–319, 1996.
[11] K. Huarng, "Effective lengths of intervals to improve forecasting in fuzzy time series," *Fuzzy Sets and Systems*, vol. 123, no. 3, pp. 387–394, 2001.
[12] C.-H. H. Cheng, J.-R. R. Chang, and C.-A. A. Yeh, "Entropy-based and trapezoid fuzzification-based fuzzy time series approaches for forecasting IT project cost," *Technological Forecasting and Social Change*, vol. 73, no. 5, pp. 524–542, 2006.
[13] H.-K. K. Yu, "Weighted fuzzy time series models for TAIEX forecasting," *Physica A: Statistical Mechanics and its Applications*, vol. 349, no. 3, pp. 609–624, 2005.
[14] Z. Ismail and R. Efendi, "Enrollment forecasting based on modified weight fuzzy time series," *Journal of Artificial Intelligence*, vol. 4, no. 1, pp. 110–118, 2011.
[15] R. EFENDI, Z. Ismail, and M. M. DERIS, "Improved weight Fuzzy Time Series as used in the exchange rates forecasting of US Dollar to Ringgit Malaysia," *International Journal of Computational Intelligence and Applications*, vol. 12, no. 01, p. 1350005, 2013.
[16] H. J. Sadaei, R. Enayatifar, A. H. Abdullah, and A. Gani, "Short-term load forecasting using a hybrid model with a refined exponentially weighted fuzzy time series and an improved harmony search," *International Journal of Electrical Power & Energy Systems*, vol. 62, no. from 2005, pp. 118–129, 2014.
[17] F. M. Talarposhti, H. J. Sadaei, R. Enayatifar, F. G. Guimarães, M. Mahmud, and T. Eslami, "Stock market forecasting by using a hybrid model of exponential fuzzy time series," *International Journal of Approximate Reasoning*, vol. 70, pp. 79–98, 2016.
[18] S. Askari and N. Montazerin, "A high-order multi-variable Fuzzy Time Series forecasting algorithm based on fuzzy clustering," *Expert Systems with Applications*, vol. 42, no. 4, pp. 2121–2135, 2015.
[19] B. E. Hansen, "Interval forecasts and parameter uncertainty," *Journal of Econometrics*, vol. 135, pp. 377–398, 2006.
[20] Everette S. Gardner, "A Simple Method of Computing Prediction Intervals for Time Series Forecasts," *Management Science*, vol. 34, no. 4, pp. 541–546, 1998.
[21] R. Koenker and K. F. Hallock, "Quantile Regression," *Journal of Economic PerspectivesVolume*, vol. 15, no. 4, pp. 143–156, 2001.
[22] I. Steinwart and A. Christmann, "Estimating conditional quantiles with the help of the pinball loss," *Bernoulli*, vol. 17, no. 1, pp. 211–225, 2011.
[23] I. Takeuchi, Q. V. Le, T. D. Sears, and A. J. Smola, "Nonparametric Quantile Estimation," *Journal of Machine Learning Research*, vol. 7, pp. 1231–1264, 2006.
[24] P. Pinson, G. Kariniotaki, H. Aa Nielsen, T. S. Nielsen, and H. Madsen, "Properties of Quantile and Interval Forecasts of Wind Generation and their Evaluation," in *Proceedings of the European Wind Energy Conference & Exhibition*, (Athens), 2006.
[25] J. W. Taylor, "Forecasting Daily Supermarket Sales Using Exponentially Weighted Quantile Regression," *European Journal of Operational Research*, vol. 178, pp. 154–167, 2007.
[26] T. Hong, J. Wilson, and J. Xie, "Long Term Probabilistic Load Forecasting and Normalization With Hourly Information," *IEEE Transactions On Smart Grid*, vol. 5, no. 1, 2014.
[27] T. Hong and S. Fan, "Probabilistic electric load forecasting: A tutorial review," *International Journal of Forecasting*, vol. 32, pp. 914–938, 2016.
[28] T. Hong, P. Pinson, S. Fan, H. Zareipour, A. Troccoli, and R. J. Hyndman, "Probabilistic energy forecasting: Global Energy Forecasting Competition 2014 and beyond," *International Journal of Forecasting*, vol. 32, pp. 896–913, 2016.
[29] R. Moore and W. Lodwick, "Interval analysis and fuzzy set theory," *Fuzzy Sets and Systems*, vol. 135, pp. 5–9, 2003.
[30] Q. Song and B. S. Chissom, "Forecasting Enrollments with Fuzzy Time Series - part I," *Fuzzy Sets And Systems*, vol. 54, pp. 1–9, 1993.
[31] S. M. Chen and C. D. Chen, "TAIEX forecasting based on fuzzy time series and fuzzy variation groups," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 1, pp. 1–12, 2011.
[32] M. H. Lee, H. J. Sadaei, and Suhartono, "Improving TAIEX forecasting using fuzzy time series with BoxCox power transformation," *Journal of Applied Statistics*, vol. 40, pp. 2407–2422, 11 2013.
[33] T. Gneiting, F. Balabdaoui, and A. E. Raftery, "Probabilistic Forecasts, Calibration and Sharpness," *Source Journal of the Royal Statistical Society. Series B (Statistical Methodology) Journal of the Royal Statistical Society. Series B (Statistical Methodology J. R. Statist. Soc. B*, vol. 69, no. 2, pp. 243–268, 2007.
[34] P. F. Christoffersen, "Evaluating Interval Forecasts," *International Economic Review*, vol. 39, p. 841, 11 1998.