An adaptive constraint-handling approach for optimization problems with expensive objective and constraints

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***Abstract*—In this work, an adaptive constraint-handling approach is developed to improve the efficiency of surrogate-based optimization (SBO). Similar to other SBO methods, the proposed approach is a sequential updating process, whereas two candidate points considering the significance of objective and constraints are generated respectively in each cycle. In detail, the candidate point of objective is obtained through the penalized lower confidence bounding (PLCB) infill criterion. Additionally, an infill criterion of the constraints (called MLCB) which can accurately characterize the boundaries of the constraints is developed to determine the candidate point of constraints. Then, a selection algorithm is developed to select one or two candidate point(s) as the new training point(s) adaptively according to the current optimal value and the accuracy of the constraint boundaries. The selection algorithm is composed of three phases. In the first phase, the candidate point of constraints is selected to find a feasible solution. Two candidate points of objective and constraints are added to speed up the convergence in the second phase. In the third phase, the candidate point of objective is chosen to improve the quality of the feasible optimal solution. The proposed approach is tested on seven numerical functions and compared with state-of-the-art methods. Results indicate that the proposed approach has excellent global optimization ability, meanwhile, it reduces significantly computational resources.**

Keywords*—Expensive optimization problems, Surrogate-based optimization, Kriging model, Sequential process

I. INTRODUCTION

The high fidelity simulation models are broadly employed to improve the reliability and quality of engineering products. However, these simulations (e.g. the Finite Elements Method and Computational Fluid Dynamics analysis) take hours or days to obtain the response of a design scheme [1]. In that case, the simulation-based optimization is hardly practical because tremendous samples are needed for most traditional optimization algorithms to search the optimal solution. Hence, surrogate-based optimization (SBO) has gained extensive attention because it significantly reduces the resources compared with simulation-based optimization approaches[2, 3].

The SBO methods refine the surrogates dynamically by certain infill criteria to obtain the optimal solution in the design space. Representatively, Jones et al. [4] proposed an efficient global optimization (EGO) method which updates the Kriging model sequentially by the expected improvement (EI) infill criterion. During the dynamic updating process, the point with the maximum value of EI is selected to supplement the sample set, which can effectively balance the global exploration and local exploitation. Recently, many related articles are published to improve the efficiency of the EI criterion through assigning different weights of the exploration and exploitation terms [4]. Another popular infill criterion of the EGO method is the lower confidence bounding (LCB) criterion [5] which combines the predicted value and variance of the Kriging model directly to search for potential optimal value. However, the LCB criterion has a large risk to fall into the local valley because of the constant weights between exploration and exploitation terms. Many works aim to get rid of this shortcoming by different weight assigned strategies [6, 7]. Apart from the above approaches, Gutmann [8] utilized the bumpiness of the radial basis function (RBF) model to guide the sequential process based on an RBF model. Dong et al. [9] proposed the multi-start space reduction strategy to solve expensive black-box objective optimization problems. These approaches have appealing performances on handling the expensive optimization problems without constraints, or the problems with cheap constraints. In fact, most optimization problems of engineering are subjected to a number of black-box constraints which are also computationally expensive. Without loss of generality, the optimization problem with inequality constraints is defined by

$$\begin{aligned} \min \quad & f(\boldsymbol{x}) \\ s.t. \quad & g_j(\boldsymbol{x}) \le 0, j = 1,2,...,J \\ & \boldsymbol{lb} \le \boldsymbol{x} \le \boldsymbol{ub} \end{aligned} \tag{1}$$

where $f(\boldsymbol{x})$ is the objective function, $g_i(\boldsymbol{x})$ denotes the j^{th} constraint function, $\boldsymbol{lb}$ and $\boldsymbol{ub}$ are the lower and upper bounds of the design variables, respectively. To save computational resources, both the objective and constraints need to be approximated by surrogates during the optimization process on these issues with expensive objective and constraints. In this situation. the commonly used approach, i.e., the penalty function method [10], may decrease the efficiency of the sequential process to a large extent because of the uncertainty of the constraint surrogates.

To address this bottleneck, many works have been

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developed to tackle the optimization problem with expensive objective and constraints over the past years. For instance, the EI criterion is extended to the constrained EI (CEI) by multiplying the probability of feasibility (PoF) [11], where the PoF is an index to measure the satisfaction degree of the constraints. Due to its excellent ability to solve simple problems, the CEI algorithm is regarded as the baseline algorithm to verify the performance of other approaches [12]. Fonseca et al.[13] attempted to make a better trade-off between exploration and exploitation by treating the EI and PoF as two optimization targets. Then, it selects new training points from the Pareto front. Li et al. [14] proposed a Kriging-based bi-objective constrained optimization method where the CEI and the uncertainties of surrogates are set to be the optimization objectives. Parr et al. [15] reviewed the distinctions of these approaches and developed a new criterion to choose multiple points from the Pareto front. In addition, Regis has promoted SBO methods to deal with expensive constrained problems. Two of his representative works, i.e., constrained optimization by radial basis function interpolation (COBRA) [16] and constrained local stochastic (constrLMSRB) [16], are composed of two phases. Specifically, the feasible solution is the primary target in phase one and the second phase aims to improve the quality of the current optimal value. Liu et al. [17] developed the DIRECT-type method which can handle the feasible and infeasible cells separately without any user-defined parameter. Dong et al. [18] proposed a new Kriging-based constrained global optimization algorithm, which reduces the computational resources through the space reduction technology and local surrogate models. In general, the constraints in the constrained optimization divide the design space into feasible and infeasible regions. Therefore, for the constraints, the constrained optimization concerns the accuracy at the constraint boundaries instead of global accuracy. However, some approaches concern the global accuracy of constraint surrogates, which may waste computational resources when the constraint boundaries have been well approximated. Other methods only consider the constraints in the first phase while ignore the objective, which may influence the convergence speed.

To overcome this conflict, an adaptive constraint-handling approach is developed to make full use of the surrogate models and enhance the efficiency of the dynamic updating process for finding the feasible optimal solution. Besides, a ratio factor used to estimate the accuracy of the Kriging models of constraints at constraint boundaries is developed. The proposed approach can decide whether to consider the influence of constraint surrogates through the ratio factor or not. Specifically, two candidate points, which are determined by considering the significance of objective and constraints respectively, are generated in each cycle. The candidate point determined by minimizing the penalized LCB (PLCB) criterion is used to find the feasible optimal solution. On the other hand, a modified LCB criterion (MLCB) is developed to accurately characterize the constraint boundaries. Meantime, the candidate point of constraints is obtained by minimizing the MLCB criterion. A selection algorithm is developed to select one or two candidate point(s) as the new training point(s) to refine the Kriging models in a cycle, which aims to balance the accuracy and the convergence speed. The selection algorithm consists of three phases combining the ratio factor and the current optimal solution. In detail, only the candidate point determined by the MLCB is selected when no feasible solution exists, two candidate points are selected after a feasible solution is found which speeds up the iterative process, only the candidate point determined by the PLCB criterion is chosen when the accuracy of Kriging models of constraints around constraint boundary is at a high level. To verify the accuracy and effectiveness of the proposed approach, seven numerical problems with different complexities are tested. Results show that the proposed approach has competitive performance compared with both the classic and state-of-the-art methods, especially in reducing computational resources.

The remainder of this work is organized as follows, Section 2 reviews the basic theories of the Kriging model and efficient global methods for constrained optimization problems. The proposed approach is elaborated in Section 3. In Section 4, the demonstrations are presented. Some conclusions and future works are provided in Section 5.

II. Background

A. Kriging model

Suppose that there is a sample set with n points $\{\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n\}$ and their responses $\{y_1, y_2, ..., y_n\}$, and the dimension of the design variables is d. The Kriging model proposed by Krige [19] approximates the input-output relationship of the black-box system through the information of these points. The Kriging model can be expressed as

$$f(\boldsymbol{x}) = \mu + Z(\boldsymbol{x}) \tag{2}$$

where μ is the mean of the Gaussian process which represents the overall tendency, $Z(\boldsymbol{x})$ is a Gaussian process which denotes the local deviation. Meanwhile, $E[Z(\boldsymbol{x})] = 0$ and the covariance of two-points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ is $\mathrm{cov} = \sigma^2 \varphi(Z(\boldsymbol{x}_i), Z(\boldsymbol{x}_j))$, where

$$\varphi(Z(\boldsymbol{x}_i), Z(\boldsymbol{x}_j)) = \exp(-\sum_{k=1}^{d} \theta_k (\boldsymbol{x}_i^k - \boldsymbol{x}_j^k)^{p_k}) \tag{3}$$

where $\boldsymbol{\theta} = [\theta_1, \theta_2, ..., \theta_d]$ and $\boldsymbol{p} = [p_1, p_2, ..., p_d]$ are two modeling parameters that are utilized to control the smooth of the Kriging model and adjust the correlation between samples, respectively. Consequently, there are 2*d*+2 unknown modeling parameters, i.e. $\mu, \sigma, \boldsymbol{\theta}, \boldsymbol{p}$, to decide a Kriging model, which can be determined by the maximum likelihood estimation. A detailed introduction of parameter estimation can be found in [20].

Compared with other surrogates, the Kriging model can provide both the predicted value and uncertainty. The predicted value of an un-sampled point $\boldsymbol{x}^*$ can be expressed as

$$\hat{f}(\boldsymbol{x}^*) = \hat{\mu} + \mathbf{1}^T \boldsymbol{\Phi}^{-1} (\boldsymbol{y} - \mathbf{1}\hat{\mu}) \tag{4}$$

where

$$\boldsymbol{\Phi} = \begin{bmatrix} \varphi(Z(\boldsymbol{x}_1), Z(\boldsymbol{x}_1)) & \cdots & \boldsymbol{x}_1), Z(\boldsymbol{x}_n)) \\ \vdots & \ddots & \vdots \\ \varphi(Z(\boldsymbol{x}_n), Z(\boldsymbol{x}_1)) & \cdots & \boldsymbol{x}_n), Z(\boldsymbol{x}_n)) \end{bmatrix} \tag{5}$$

$$\hat{\mu} = \mathbf{1}^T \boldsymbol{\Phi}^{-1} \boldsymbol{y} / \mathbf{1}^T \boldsymbol{\Phi}^{-1} \mathbf{1} \tag{6}$$

The estimated variance is determined by minimizing the root square error, which is expressed as

$$\hat{s}^2(\boldsymbol{x}^*) = \hat{\sigma}^2 \left[1 - \boldsymbol{\varphi}^T \boldsymbol{\Phi}^{-1} \boldsymbol{\varphi} + (1 - \mathbf{1}^T \boldsymbol{\Phi}^{-1} \boldsymbol{\varphi})^2 / \mathbf{1}^T \boldsymbol{\Phi}^{-1} \mathbf{1} \right] \tag{7}$$

where

$$\hat{\sigma}^2 = \frac{(\boldsymbol{y} - \mathbf{1}\boldsymbol{\mu})^T \boldsymbol{\Phi}^{-1} (\boldsymbol{y} - \mathbf{1}\boldsymbol{\mu})}{n} \tag{8}$$

$$\boldsymbol{\varphi} = \left[\varphi(Z(\boldsymbol{x}_i), Z(\boldsymbol{x}^*)) \quad \cdots \qquad \boldsymbol{x}_n), Z(\boldsymbol{x}^*)) \right] \tag{9}$$

B. The constrained expected improvement criterion

Because the prediction of Kriging model obeys the normal distribution $\hat{y} \sim \qquad , \hat{s}(\boldsymbol{x}))$, the improvement of a sample is defined by

$$I(\boldsymbol{x}) = \max(y_{\min} - \hat{y}, 0) \tag{10}$$

The expected improvement is the expected value of $I(\boldsymbol{x})$, which can be obtained by

$$E[I(\boldsymbol{x})] = (y_{\min} - \hat{y})\Phi\left(\frac{y_{\min} - \hat{y}}{s(\boldsymbol{x})}\right) + \hat{s}(x)\phi\left(\frac{y_{\min} - \hat{y}}{\hat{s}(\boldsymbol{x})}\right) \tag{11}$$

where Φ and ϕ are the cumulative density function and probability density function of Gaussian distribution, respectively.

The first term of $E[I(\boldsymbol{x})]$ is utilized for local exploitation and global exploration is realized by the second term. However, the EI criterion cannot deal with constrained optimization problems. To get out of this dilemma, the EI criterion is modified combining the probability of feasibility (PoF), defined by

$$PoF(\boldsymbol{x}) = \prod_{i=1}^{J} \left[\Phi\left(\frac{0 - \hat{g}_i(\boldsymbol{x})}{\hat{s}_{gi}(\boldsymbol{x})} \right) \right], \quad i = 1, 2, ..., J \tag{12}$$

where $\hat{g}(\boldsymbol{x})$ and $\hat{s}_g(\boldsymbol{x})$ denote the predicted value and uncertainty of constraint Kriging models respectively, J is the number of constraints. The values of $PoF(\boldsymbol{x})$ range from 0 to 1, where a larger value indicates higher satisfaction of constraints.

To this end, the CEI criterion is expressed as

$$CEI(\boldsymbol{x}) = E(I(\boldsymbol{x})) \times PoF(\boldsymbol{x}) \tag{13}$$

C. The penalized lower confidence bounding criterion

The LCB criterion is utilized as the guideline for the updating process of objective Kriging model. The LCB criterion can be defined by

$$lcb(\boldsymbol{x}) = \hat{f}(\boldsymbol{x}) - 1.96\hat{s}_f(\boldsymbol{x}) \tag{14}$$

According to (13), the LCB criterion makes a trade-off between exploration and exploitation by combining the predicted value and variance simply. To guarantee the feasibility of the new training point selected by LCB, the LCB function is revised by adding a penalty. In this case, the revised LCB function for constrained optimization can be expressed as

$$\begin{aligned} plcb(\boldsymbol{x}) &= lcb(\boldsymbol{x}) + \alpha \max\left\{ \hat{g}_j(\boldsymbol{x}) - a\hat{s}_{gj}(\boldsymbol{x}), 0 \right\} \\ j &= 1, 2, \ldots, J \end{aligned} \tag{15}$$

where α is a penalty factor, a is the relaxation of constraints. Usually, $a = 0$ or $a = 1.96$.

The PLCB criterion can search for a feasible optimal solution in each cycle by minimizing (14).

III. PROPOSED ADAPTIVE CONSTRAINT-HANDLING APPROACH

Generally, the constraints of the constrained optimization problem provide the estimation of the feasibility, which depends on the sign of the constraints. Therefore, it is not mandatory to consider the influence of Kriging models of constraints when the accuracies of constraint boundaries are at a high degree. Because of this, an MLCB criterion for constraint Kriging models, which can accurately characterize the constraint boundaries, is proposed. Meanwhile, a ratio factor is proposed to estimate the relative accuracy of Kriging models of constraints at the constraint boundaries. The proposed MLCB criterion, ratio factor, and original PLCB criterion can be integrated into the proposed adaptive updating process to find the feasible optimal solution while reducing computational resources. The framework of the proposed approach is shown in Fig.1, which mainly consists of several essential parts including the construction of the initial Kriging model, the determination of the candidate points, the procedure of selecting new training points, and the optimization process of searching the feasible optimal value.

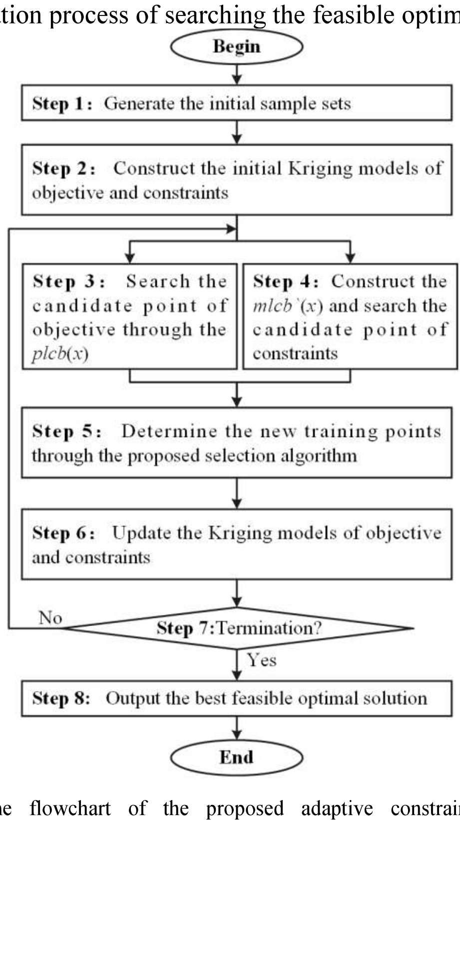

Fig. 1. The flowchart of the proposed adaptive constraint-handling approach

A. The infill criterion of the constraint Kriging models

The goal of the infill criterion for the constraint Kriging models is to provide an accurate feasibility estimation. In that case, the constraint Kriging models are expected to be accurate around the constraint boundaries. Therefore, the LCB criterion is modified to identify the boundary of a constraint, which can be expressed as

$$mlcb_j(\boldsymbol{x}) = \left|\hat{g}_j(\boldsymbol{x}) - z_j\right| - 1.96\hat{s}_{g_j}(\boldsymbol{x}), \quad j = 1,2,\ldots,J \tag{16}$$

where z_j is the threshold for each constraint, which equals 0 in this research.

The proposed criterion of constraints can make a trade-off between exploration and exploitation, because the point near the boundary of a constraint or with larger uncertainty is selected by minimizing (16). Theoretically, a new point can be determined for each constraint according to (16). If all these points are evaluated by the expensive model to update the Kriging models in each cycle, it may waste the computational burden. In fact, the feasibility of a solution in the current iteration is dominated by the activated constraints, which is defined by

$$\hat{g}'(\boldsymbol{x}) = \max\left\{\hat{g}_j(\boldsymbol{x})\right\} \; j = 1,2,...,J \tag{17}$$

According to (17), a design scheme is feasible when $\hat{g}'(\boldsymbol{x}) < 0$, vice versa. Therefore, the MLCB criterion can be applied to select a new point for the activated constraints which can improve the efficiency of the iterative process.

To fabricate the MLCB criterion more suitable for the dynamic optimization process, an index associating with the repetitions of the current optimal solution is used to further revise (17). The final version of the MLCB criterion is defined by

$$mlcb'(x) = \left|\hat{g}'(\boldsymbol{x})\right| - (1.96 + \ln(flag))\hat{s}'_g(\boldsymbol{x}) \tag{18}$$

where $\hat{g}'(\boldsymbol{x})$ and $\hat{s}'_g(\boldsymbol{x})$ are the predicted value and variance of the activated constraint, respectively. $flag$ is an index that represents the repetitions of the current optimal solution. When the value of $flag$ enlarges, it means that the iterative process has a large risk of falling in a local valley. In that case, the $mlcb'(x)$ is dominated by the $\hat{s}'_g(\boldsymbol{x})$ so that it can get rid of the local valley by searching a new feasible region.

B. The selection algorithm for selecting new training points

In steps 3 and 4 of the procedure, two candidate points are ascertained by minimizing the PLCB and MLCB infill criteria respectively. To reduce the computational resources, a selection algorithm is developed to determine the new training points for the sequential process of SBO. Therein, a ratio factor used to estimate the accuracy of Kriging models of constraints at the constraint boundaries is developed. The derivation of the ratio factor is elaborated as follows:

The points in the design space can be divided into two situations due to the prediction uncertainty. One situation is that the points would not change their feasibility even though considering their uncertainties. The other is that their feasibility would change due to the uncertainties. The points in the latter situation should be concerned during the sequential process. Because the prediction of Kriging obeys the normal distribution, the points with the probabilities of wrong sign prediction larger than 95% for can be expressed as

$$\begin{aligned} \mathbf{X}_1 &= \{\boldsymbol{x} \mid \hat{g}'(\boldsymbol{x}) < 0; \hat{g}'(\boldsymbol{x}) + 1.96\hat{s}'(\boldsymbol{x}) > 0\} \\ \mathbf{X}_2 &= \{\boldsymbol{x} \mid \hat{g}'(\boldsymbol{x}) > 0; \hat{g}'(\boldsymbol{x}) + 1.96\hat{s}'(\boldsymbol{x}) < 0\} \end{aligned} \tag{19}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are sample sets of two situations for wrong sign prediction under the active constraints. Generally, the accuracy of the activated constraints is higher when the number of points in $\mathbf{X}_1$ and $\mathbf{X}_2$ is less. In order to quantify the relative accuracy of the active constraints, the ratio is defined by

$$R = \frac{N_{X_1} + N_{X_2}}{N_{test}} \tag{20}$$

where N_{test} is the number of tested points generated by the Latin Hypercube sampling (LHS) [21] method, N_{X_1} and N_{X_2} are the numbers of points in $\mathbf{X}_1$ and $\mathbf{X}_2$ among the tested points, respectively. Considering that the value of R is extremely small when the size of the feasible region is small in the design space, the (19) can be revised by

$$R = \frac{N_{X_1} + N_{X_2}}{N_{X_f}} \tag{21}$$

where N_{X_f} is the number of infeasible points. The value of R indicates the relative accuracy of the constraint boundaries, where a smaller value represents higher accuracy. If the value of R equals 0, it means that there is no wrong sign prediction of the constraint Kriging models under the uncertainties.

Combining R and the current feasible optimal solution, a selection algorithm used to select new training points is developed as listed in Algorithm I.

TABLE I. ALGORITHM I: SELECTION ALGORITHM OF CHOOSING NEW TRAINING POINTS

Input: candidate point of objective $\boldsymbol{x}'_{obj}$, candidate point of constraints $\boldsymbol{x}'_{cons}$, ratio factor R, trained sample set $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{g})$	
Begin	
If $sum\max\{\boldsymbol{g}, 0\} \neq 0$ then $\boldsymbol{x}'_{new} = \boldsymbol{x}'_{cons}$	←The initial phase to find a feasible solution
elseif $R \leq 0.05$ then $\boldsymbol{x}'_{new} = \boldsymbol{x}'_{obj}$	←The final phase where the accuracy of the constraints at a high level.
elseif Euclidean distance $(\boldsymbol{x}'_{obj}, \boldsymbol{x}'_{cons}) \leq \gamma$ then $\boldsymbol{x}'_{new} = \boldsymbol{x}'_{obj}$	←To avoid the cluster of new training points
else $\boldsymbol{x}'_{new} = \left(\boldsymbol{x}'_{obj} \cup \boldsymbol{x}'_{cons}\right)$	←The accelerating phase, two candidate points are selected as the new training points
End	
Output: the new training points $\boldsymbol{x}'_{new}$	

According to Algorithm I, the proposed approach can balance the objective and constraints simultaneously by considering the current optimal solution, the relative accuracy at the boundary of the constraint, and the distribution of samples.

IV. DEMONSTRATIONS

To validate the effectiveness and accuracy of the adaptive constraint-handling approach, twofold demonstrations are devised: (1) comparison with the typical methods, (2) comparison with the state-of-the-art methods.

A. Comparison with the typical approaches

In this section, the proposed approach is compared with the CEI infill criterion, the PLCB criteria on two toy examples to show the details of the proposed approach. The tested functions are the Brainin functions which can be expressed in (22) and (23), respectively.

The Branin functions are representative benchmark test functions in constrained optimization problems. In detail, the Branin 1 function is simple with only one huge continuous feasible region while the Branin 2 function is complex with three discontinuous feasible subregions. To show the dynamic process of the proposed approach intuitively, the sampling processes of the two functions are shown in Fig. 2 and Fig. 3 respectively.

Branin 1 function

$$\begin{aligned} f(\boldsymbol{x}) &= \left(15x_2 - \frac{5.1}{4\pi^2}(15x_1-5)^2 + \frac{5}{\pi}(15x_1-5) - 6\right)^2 \\ &\quad + 10\left((1-\frac{1}{8\pi})\cos(15x_1-5)+1\right) + 5(15x_1-5) \\ g(\boldsymbol{x}) &= -x_1x_2 + 0.2 \le 0 \\ &0 \le x_1, x_2 \le 1 \end{aligned} \quad (22)$$

Branin 2 function

$$\begin{aligned} f(\boldsymbol{x}) &= \left(15x_2 - \frac{5.1}{4\pi^2}(15x_1-5)^2 + \frac{5}{\pi}(15x_1-5) - 6\right)^2 \\ &\quad + 10\left((1-\frac{1}{8\pi})\cos(15x_1-5)+1\right) + 5(15x_1-5) \\ g(\boldsymbol{x}) &= 6 - (4 - 2.1x_1^2 + \frac{1}{3}x_1^4)x_1^2 - x_1x_2 - (-4+4x_2^2)x_2^2 \\ &\quad - 3\sin(6-6x_1) - 2\sin(6-6x_2) \le 0 \\ &0 \le x_1, x_2 \le 1 \end{aligned} \quad (23)$$

According to Fig. 2, the zero contour of the constraint function divides the design space into the feasible and infeasible regions. Meanwhile, the feasible region is a continuous area and has a larger feasible size among the design space. The boundary of the constraint is smooth so only one new training point is added in iteration 1 through the $mlcb'(\boldsymbol{x})$ to update the Kriging model among the 8 iterations. In the last 7 iterations, only the candidate points of objective are selected by Algorithm I. However, the Branin 2 function has 3 discontinuous feasible regions and the best feasible solution is in the lower-right corner of the design space. The constraint boundaries is hard to be well approximated so the traditional methods have large risks to fall into the local optimal region. As shown in Fig. 3, there is a feasible solution in the initial samples, therefore, the algorithm goes to the accelerating phase and adds new training points considering the objective and constraint simultaneously. Specifically, the $plcb(\boldsymbol{x})$ exploits the existing feasible region during the iterations and the $mlcb'(\boldsymbol{x})$ explores the design space for new possible regions. As shown in Fig. 3, the $mlcb'(\boldsymbol{x})$ has identified two feasible regions and the red new training point in iteration 8 is in the vicinity of the final un-identified feasible region. It indicates the excellent global exploration ability of the $mlcb'(\boldsymbol{x})$. To this end, the proposed adaptive constraint-handling approach is efficient combining the two infill criteria and the proposed algorithm I for determining the new training points adaptively.

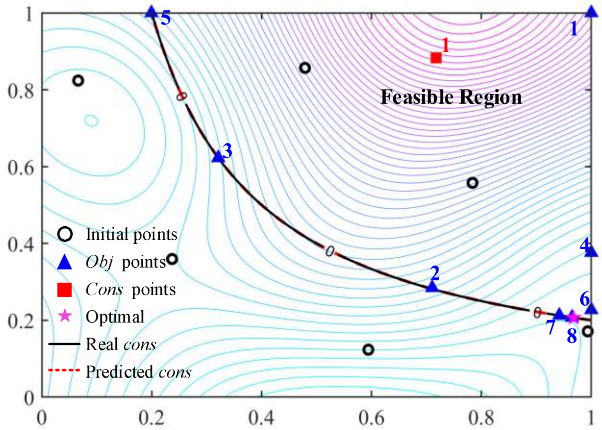

Fig. 2. The sequential process of proposed approach of the Branin 1 function

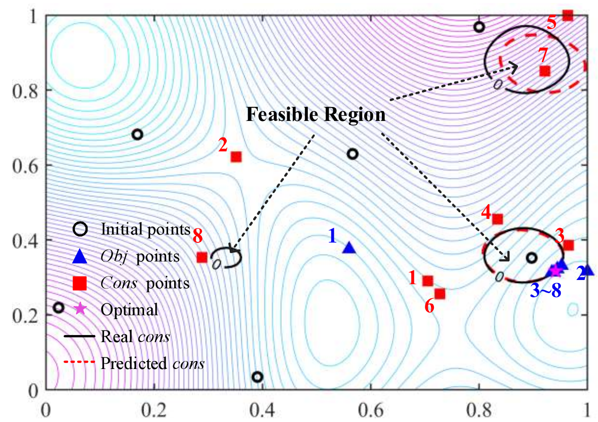

Fig. 3. The sequential process of proposed approach of the Branin 2 function

Table II lists the means of the approximated optimal value (f_{min}) and the number of function evaluations (NFE) of different methods over 30 independent runs. The maximum evaluations of the two tested functions are set to be 50, the real optimal solution and predetermined values to reach are the values beside and in the parentheses below the function names in Table II. The symbols of "−", "+ ", and "≈" indicate the results of the T-tests at a significant level of α=0.05, which represent the performance of the proposed approach on NFE is worse than, better than and similar to that of other approaches.

TABLE II. RESULTS OF DIFFERENT APPROACHES ON THE BRANIN FUNCTIONS

Functions	Branin1 5.5757 (5.6318)			Branin2 12.001 (12.1210)		
Mean	f_{min}	NFE	*T-test*	f_{min}	NFE	*T-test*
CEI	5.5899	17.77	≈	12.0718	34.67	+
PLCB1 ($a = 0$)	5.5855	21.40	+	55.4904 (1)	35.79	+
PLCB2 ($a = 1.96$)	5.6714	41.33	+	15.8176 (3)	32.93	+
Proposed	**5.5840**	**17.07**		**12.0660**	**29.77**	

As we know, the CEI has high efficiency on the simple functions, however, the proposed approach is better than the CEI on both the f_{min} and NFE. It indicates the effectiveness of the proposed approach. In general, the performance of the proposed approach on the NFE is better than other approaches according to the T-test results. Concerning the PLCB infill criteria with $a = 0$ and $a = 1.96$, they have evident shortcomings when dealing with the constraint optimization problems directly. When $a = 0$, the PLCB is tended to exploit the local region during the optimization process, therefore, the PLCB is prone to fall into the local optimal. Meanwhile, the f_{min} of the Branin 2 function is 55.4904 which is larger than those of the other approaches and there is one trail that has even not found a feasible solution as listed in the parentheses beside the f_{min}. On the other hand, in the case $a = 1.96$, the ability to allocate the optimal solution is influenced by relaxing the constraints because the NFE is 41.33 on Branin 1 function. Meanwhile, three trials have not found a feasible solution for the Branin 2 function. However, it has a better global exploration ability to some extent, where the value f_{min} is significantly smaller than that of the case $a = 0$.

To show the effectiveness of different approaches intuitively, Fig. 4 and Fig. 5 plot the box chart of results of different approaches of the Branin functions respectively.

As shown in Fig. 4 and Fig. 5, the black boxes represent the range of NFE, the blue boxes denote the range of the f_{min}. It is noting that the infeasible cases of PLCB are not included. The case of PLCB1 cannot get out of the valley because most of its cases in Fig. 5 find the other two local optimal values. On the contrary, the case of PLCB2 has a poor ability to converge to the optimal value under limited resources. The CEI has a better performance than the cases of PLCB both on the NFE and approximated f_{min}. However, in general, the proposed approach has better stability on the NFE and f_{min} compared with the listed methods.

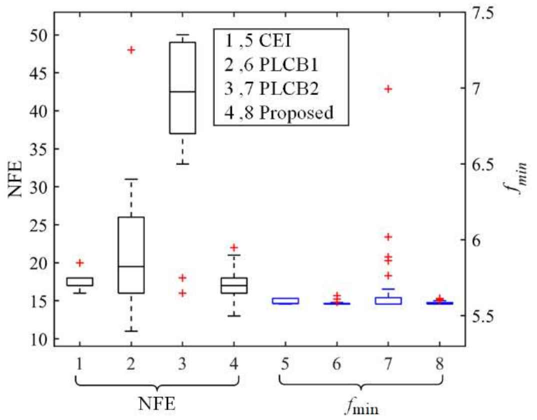

Fig. 4. Box chart of different approaches of Branin 1 function

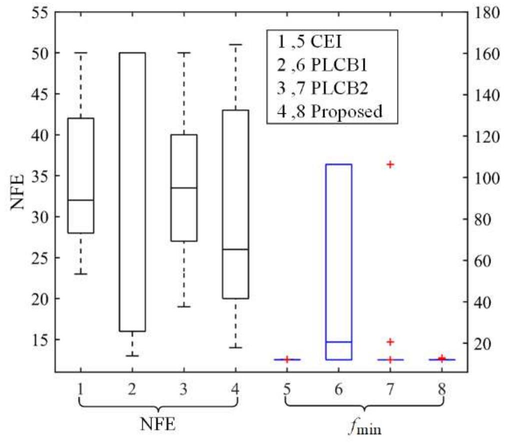

Fig. 5. Box chart of different approaches of Branin 2 function

B. Comparison with the state-of-the-art approaches

To further test the merits of the proposed adaptive constraint-handling approach, several state-of-the-art approaches including the KBCO [14], SCGOSR [7], and BOAEF [15] are utilized. The tested functions are G2, G4, G6, G7, G8, and G24 from the CEC 2006 test function set [23] and a speed reducer design problem [18]. Because of the lack of the original codes of these approaches, the results reported in these researches are cited. The initial number of sample points is $2(d+3)$, the detailed information of these tested functions are illustrated in Table III, where d denotes the dimension of the test function, Cons is the number of constraints, PV is the predetermined value to reach, ME denotes the maximum evaluations of the real function, R is the ratio of the feasible region. Besides, the results of these tested functions are listed in Table IV.

TABLE III. TERMINATION CONDITIONS OF THE TESTED FUNCTIONS

Case	*d*	Cons	Optimum	PV	ME	*R* (%)
G4	5	6	-30665 .539	-30665	60	52.123 0
G6	2	2	-6961 .8139	-6961.5	50	0.0067
G7	10	8	24.3062	25	150	0.0003
G8	2	2	-0.095825	-0.0957	50	0.8560
SR7	7	11	2994.42	2995	100	0.0978
G24	2	2	-5.5080	-5.5080	50	44.212
G2	10	2	-0.67	-0.20	150	99.687

As illustrated in Table IV, the proposed approach can save remarkable computational resources compared with the other methods on most of these test functions. Specifically, the reduction of the resource is proportional to the complexity of the problem in the cases of G4, G6, G8, SR7. Concerning the accuracy, the proposed approach obtains smaller value both on the mean and standard deviation of f_{min} in the cases of G4, SR7. In these cases, there is no feasible solution in the initial sample set, however, the proposed approach can find a feasible solution through the MLCB criterion with several iterations. The mean and standard deviation of f_{min} is competitive to those of other approaches in the cases of G6 and G8. In the case of G7, the proposed approach costs slightly higher resources with a larger value. The complexity of G7 is higher than other tested functions because the dimension and number of constraints are 10 and 8 respectively. Besides, the ratio of the feasible region is only 0.0001%. It is noting that the proposed approach only uses an average of 16.6 iterations to find the feasible region, in this situation convergence speed is influenced by the PLCB criterion. Therefore, the proposed approach has a large potential to further improve the efficiency with another efficient global optimization algorithm. Meanwhile, the best, the median, and the worst optimal values of G4, G6, G8, SR7, and G24 have little difference, while that of G2 and G7 remain apparently due to the high complexity. Finally, the results of the T-tests on the NFE show that the computational burden of the proposed approach is significantly small than other approaches on most test functions.

TABLE IV. RESULTS OF THE TESTED FUNCTIONS ON DIFFERENT APPROACHES

Cases	Methods	NFE_{Mean}	f_{min} [mean ± std]	Best	Median	Worst	T-test
G4	Proposed	**34.0**	**-30665.51** ± 0.055	-30665.539	-30665.539	-30665.067	/
	KBCO	43.5	-30665.47 ± 0.063	N/A	N/A	N/A	+
	SCRGOSR	53.9	-30665.46 ± 0.064	N/A	N/A	N/A	+
	BOAEF	47.1	-30665.47 ± **0.045**	N/A	N/A	N/A	+
G6	Proposed	**27.2**	-6961.80 ± **0.006**	-6961.81	-6961.80	-6961.80	/
	KBCO	41.7	**-6961.81** ± 0.012	N/A	N/A	N/A	+
	SCRGOSR	75.1	-6961.80 ± 0.016	N/A	N/A	N/A	+
	BOAEF	42.3	-6961.80 ± 0.015	N/A	N/A	N/A	+
G7	Proposed	134.8	24.8409± **0.386**	24.3215	24.8059	25.9336	/
	KBCO	**121.6**	**24.5047** ± 0.183	N/A	N/A	N/A	-
	SCRGOSR	178.2	24.6559 ± 0.314	N/A	N/A	N/A	+
	BOAEF	133.4	24.5883 ± 0.321	N/A	N/A	N/A	≈
G8	Proposed	**38.8**	**-0.0958** ± 0.000057	-0.09582	-0.09580	-0.09561	/
	KBCO	45.8	-0.0958 ± 0.000021	N/A	N/A	N/A	+
	SCRGOSR	51.8	-0.0958 ± **0.000013**	N/A	N/A	N/A	+
	BOAEF	46.2	-0.0958 ± 0.000014	N/A	N/A	N/A	+
SR7	Proposed	**53.1**	**2994.47 ± 0.0005**	2994.47	2994.47	2994.48	/
	KBCO	76.3	2995.24 ± 0.89	N/A	N/A	N/A	+
	SCRGOSR	88.1	2996.21 ± 1.73	N/A	N/A	N/A	+
	BOAEF	81.9	2996.18 ± 1.16	N/A	N/A	N/A	+
G24	Proposed	16.37	-5.5079 ± 0.0002	-5.5080	-5.5080	-5.5073	/
G2	Proposed	144.33	-0.2006 ±0.0633	-0.3367	-0.1928	-0.1175	/

* N/A: the results are not available in their researches

V. CONCLUSIONS

In this work, an adaptive constraint-handling approach is proposed to solve the optimization problem with expensive objective and constraints with high efficiency. The proposed approach follows a sequential process where two candidate points are generated in each iteration. The candidate points of the PLCB criterion is utilized to exploit the feasible region. Meanwhile, the candidate point generated by the developed MLCB criterion, which can accurately characterize the constraint boundaries, is regarded as the exploration component. To devise the sequential process efficiently and make full use of the computational resources, a selection algorithm used to determine the new training points is proposed. The proposed algorithm is composed of three phases which can be recognized automatically. In detail, the MLCB criterion is used to find a feasible solution in the first phase. In the second phase, two candidate points of objective and constraints are selected to accelerate the sequential process. In the final phase, the LCB criterion is used to improve the quality of the current feasible optimal solution.

To verify the performance of the proposed approach, twofold comparisons are devised. First, the proposed approach is compared with the traditional constraint-handling approach under the Branin functions, results show the proposed approach has compensated for the exploration ability of the PLCB criterion. Concerning the computational burden, the proposed approach is less than the CEI, which indicates the high effectiveness of the proposed approach on the simple functions. Second, in comparison with the recently reported approaches, the proposed approach can converge to the real optimal solution with fewer NFEs in most cases. Simultaneously, the quality of the approximated optimal solutions is better than that of the compared approaches. Consequently, the proposed approach is a promising approach for the computationally expensive constrained optimization problem.

As for future works, the proposed approach can be extended to update the Kriging models in a parallel way.

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