Riesz s-energy-based Reference Sets for Multi-Objective Optimization

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***Abstract*—Currently, reference sets, which are a collection of feasible or infeasible points in objective space, are the backbone of several multi-objective evolutionary algorithms (MOEAs) and quality indicators (QIs). For both MOEAs and QIs, an important question is how to construct the reference set regardless of the dimensionality of the objective space, preserving well-diversified solutions. The Simplex-Lattice-Design method (SLD) that constructs a set of convex weights in a simplex, has been usually used to define reference sets. However, it is not a good option since Pareto fronts with irregular geometries cannot be completely intersected by the weight vectors. In this paper, we propose a tool based on the Riesz s-energy to generate reference sets exhibiting good diversity properties. Our experimental results support the Riesz s-energy-based reference sets as a better option due to their invariance to the Pareto front shape and the objective space dimensionality.**

***Index Terms*—Multi-objective optimization, reference sets, Riesz s-energy**

I. Introduction

Regarding optimization theory, a reference point is a feasible or infeasible point in the objective space that reasonably fulfills the desires of a decision maker [1]. In evolutionary multi-objective optimization (EMOO), it is usual to employ a set of reference points, also known as a reference set, in two main directions. First, in the context of multi-objective evolutionary algorithms (MOEAs), reference sets have been used to guide the population towards the Pareto front [2]. According to Li *et al.* [2], this class of MOEAs uses reference sets based on examined points (i.e., nondominated solutions gathered during the evolutionary process) or virtually generated points in the objective space, using, for instance, the method of Das and Dennis [3]. The Nondominated Sorting Genetic Algorithm III (NSGA-III) [4] is a well-known reference set-based MOEA that uses a set of virtual points as its reference set. On the other hand, reference sets play an important role in the assessment of MOEAs. Some quality indicators (QIs) such as the inverted generational distance (IGD) [5] and the IGD$^+$ indicator [6] require a reference set for its computation. The idea of these QIs is to determine how close and similar is an approximation set, generated by an MOEA, to the reference set on the basis of a distance function.

Regardless of where the reference set is applied, a critical issue is how to construct it. Currently, the Das and Dennis method, denoted as the Simplex-Lattice-Design (SLD), that generates a set of convex weight vectors[1] fitting the shape of a simplex, has been widely employed to generate reference sets [4]. However, the generation of a simplex is the main drawback of the SLD method. Ishibuchi *et al.* [7] empirically showed that the performance of MOEAs using convex weight vectors strongly depends on the Pareto front shape of the MOP being tackled. If the weight vectors completely intersect the Pareto front, the MOEA will have a good performance. Otherwise, the MOEA will not be able to completely cover the Pareto front shape and will not be able to produce well-diversified solutions. Concerning the IGD and IGD$^+$ indicators, if a set of convex weight vectors is used as their reference set, both QIs will reward similar approximation sets. Hence, both QIs will produce misleading results since they prefer approximation sets similar to the set of convex weight vectors [8], [9]. Another difficulty of the SLD method is that the cardinality of the set is the combinatorial number $N = \binom{H+m-1}{m-1}$, where m is the dimension of the objective space and $H \in \mathbb{N}$ is a user-supplied parameter that controls the number of divisions of the objective space. In the case of high-dimensional objective spaces, the SLD method will generate a number of reference points that, from a practical point of view, is not feasible to handle.

Recently, the Riesz s-energy (E_s) [10] has been employed to improve the diversity of MOEAs [11], [12]. This measure arises from the problem of distributing N points on the unit sphere S^d in $\mathbb{R}^{d+1}$, having the influence of potential theory and the distribution of charges. A relevant application of the Riesz s-energy is the discretization of manifolds (e.g., Pareto fronts). According to Hardin and Saff [10], [13], if a manifold has the d-dimensional Hausdorff measure, the minimization of the Riesz s-energy leads to asymptotically uniformly distributed solutions. Due to these nice mathematical properties, the Riesz s-energy can be used as a diversity indicator and, hence, as

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[1]A vector $\vec{w} \in \mathbb{R}^m$ is called a convex weight vector if and only if $\sum_{i=1}^{m} w_i = 1$ and $w_i \geq 0$ for all $i = 1, \ldots, m$.

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part of an MOEA's selection mechanism, aiming to generate well-diversified Pareto front approximations. In this regard, Falcón-Cardona *et al.* [12] have empirically shown that the use of E_s helps MOEAs to avoid the performance dependence on specific MOPs as pointed out by Ishibuchi *et al.* [7].

In this paper, we propose a tool to generate reference sets of benchmark problems, using the Riesz s-energy. The underlying idea is to exploit the invariance of E_s to produce discretizations of manifolds with a high degree of diversity. Consequently, we provide reference sets of classical benchmark problems in the EMOO field such that researchers can use them either to guide MOEAs or for the assessment of MOEAs on the basis of QIs such as IGD and IGD$^+$. To this aim, we performed several experiments that show the superiority of our Riesz s-energy-based reference sets in terms of diversity for MOPs having from 2 up to 10 objectives.

The remainder of this paper is organized as follows. Section II briefly describes related studies to the Riesz s-energy and some techniques to generate reference sets. Section II outlines our proposed E_s-based method to generate reference sets. The experimental results are shown in Section IV. Finally, our conclusions as well as some future research topics are provided in Section V.

II. Related Work

A. Riesz s-energy

Hardin and Saff [10], [13] proposed the discrete Riesz s-energy to measure the evenness of a set of points in d-dimensional manifolds. Mathematically, given an approximation set $\mathcal{A} = \{\vec{a}^1, \ldots, \vec{a}^N\}$, where $\vec{a}^i \in \mathbb{R}^m$, the Riesz s-energy is given as follows:

$$E_s(\mathcal{A}) = \sum_{\vec{x} \in \mathcal{A}} \sum_{\substack{\vec{y} \in \mathcal{A} \\ \vec{y} \neq \vec{x}}} k_s(\vec{x}, \vec{y}), \tag{1}$$

where

$$k_s(\vec{x}, \vec{y}) = \begin{cases} ||\vec{x} - \vec{y}||^{-s}, & s > 0 \\ -\log ||\vec{x} - \vec{y}||, & s = 0 \end{cases} \tag{2}$$

The function k_s is the Riesz s-kernel, $||\cdot||$ denotes the Euclidean distance, and $s \geq 0$ is a parameter that controls the emphasis on the uniform distribution. As $s \to \infty$, a more uniform distribution is rewarded. It is worth noting that $s \geq 0$ is independent of the geometry of the underlying manifold of $\mathcal{A}$. According to Hardin and Saff [13], the minimization of E_s is related to the solution of the best-packing problem. There are several applications of the Riesz s-energy such as the discretization of a manifold (statistical sampling), quadrature rules, starting points for Newton's method, computer-aided design, interpolation schemes, finite element tessellations, among others [13].

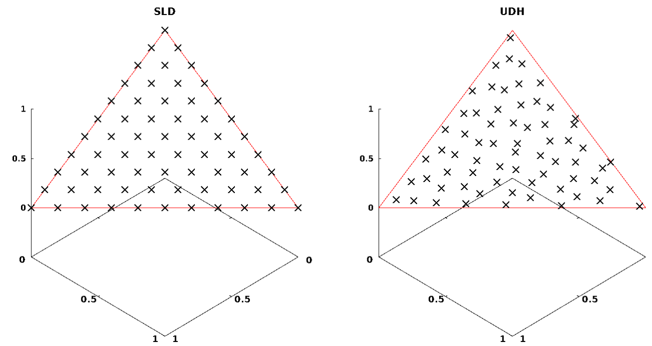


Fig. 1: Weight vectors generated by SLD and UDH in a three-dimensional space. The contour of the simplex is shown in red.

B. Reference Sets based on Weight Vectors

In the EMOO field, the Simplex-Lattice-Design method to generate weight vectors has been widely used [3], [4], [11]. Its authors, Das and Dennis, proposed to generate uniformly distributed weight vectors in the simplex lattice, where each weight vector $\vec{w} \in \mathbb{R}^m$ has $\sum_{i=1}^{m} w_i = 1$, and $w_i \in \{0, \frac{1}{H}, \frac{2}{H}, \ldots, \frac{H}{H}\}$, i.e., it is a convex weight vector. $H \in \mathbb{N}$ is a user-supplied parameter that determines the number of divisions in each axis. The SLD method generates $N = C_{m-1}^{H+m-1}$ weight vectors in the simplex. This combinatorial number of vectors is an important drawback since as m increases, N grows in an exponential fashion which, from a practical point of view, is not desirable for MOEAs or even to evaluate Pareto front approximations using QIs. To generate reference sets using SLD, the usual way is to determine the best relationship between the set of points from the Pareto front and the weight vectors via a scalarizing function $u : \mathbb{R}^m \to \mathbb{R}$ [1]. For instance, a good scalarizing function is the achivement scalarizing function (ASF) that is defined as follows:

$$u_{\vec{w}}^{\text{ASF}}(\vec{x}, \vec{z}) = \max_{i=1,\ldots,m} \left\{ \frac{|x_i - z_i|}{w_i} \right\}. \tag{3}$$

where $\vec{x}, \vec{z} \in \mathbb{R}^m$ are the solution vector to evaluate and a reference point[2], respectively.

Another approach to generating evenly distributed weight vectors is the uniform design using the Hammersley method (UDH) [14]. UDH aims to tackle the three main drawbacks of SLD: (1) the diversity of weight vectors, (2) the generation of too many vectors in the boundary of the simplex, and (3) the nonlinear increase of the set cardinality. Uniform design generates uniformly scattered points in the space. According to Molinet Berenguer and Coello Coello [14], in uniform design, a set of points is considered uniformly spread throughout the entire domain if it has a small discrepancy, where the discrepancy is a numerical measure of scatter. Unlike SLD, the combination of the uniform design and the Hammersley method produces more uniform solutions and the cardinality

[2]In multi-objective optimization, $\vec{z}$ is usually the ideal point that has the minimum values of all objective functions.

of the set is not subject to a formula. We refer readers to [14] to obtain more details of this method. Figure 1 compares the distribution of points between SLD and UDH. It is clear that UDH does not generate several solutions in the boundary of the simplex which is a good property in the case of high-dimensional objective spaces. Similarly to SLD, to generate reference sets using UDH, the best relationship between each weight vector and a point from the Pareto front is found using a scalarizing function.

III. Our Proposed Approach

In the following, we consider $\mathcal{A} = \{\vec{a}^1, \ldots, \vec{a}^N\}$ as a finite subset of the Pareto front. To generate reference sets based on the Riesz s-energy, a subset $\mathcal{Z}$ of size $\mu < N$ has to be constructed by solving the so-called Riesz s-energy subset selection problem:

$$\mathcal{Z} = \underset{\substack{\mathcal{Z}' \subset \mathcal{A} \\ |\mathcal{Z}'| = \mu}}{\arg\min} \quad E_s(\mathcal{Z}'). \tag{4}$$

However, the size of the search space of the above problem is $\binom{N}{\mu}$. Hence, solving the Riesz s-energy subset selection problem requires a lot of computational effort. To overcome this difficulty, we follow a heuristic approach to iteratively reduce the cardinality of $\mathcal{A}$ until getting the desired set size. To this aim, we compute the individual contribution C of each solution $\vec{a} \in \mathcal{A}$ to the Riesz s-energy as follows:

$$C(\vec{a}, \mathcal{A}) = \frac{1}{2}[E_s(\mathcal{A}) - E_s(\mathcal{A} \setminus \{\vec{a}\})]. \tag{5}$$

Finally, to reduce the cardinality of $\mathcal{A}$, the worst-contributing solution $\vec{a}_{\text{worst}} = \arg\max_{\vec{a} \in \mathcal{A}} C(\vec{a}, \mathcal{A})$ is deleted.

The cost of computing E_s and $C(\vec{a}, \mathcal{A})$ is $\Theta(N^2)$. In consequence, the cost of computing all N individual contributions is $\Theta(N^3)$, following a naïve approach. In Figure 2, we propose a memoization structure that allows us to reduce the computational cost of computing the individual contributions to E_s. When $E_s(\mathcal{A})$ is calculated, we take advantage of the dissimilarity matrix by storing all $k_{ij} = k(\vec{a}^i, \vec{a}^j), i \neq j$. The memoization structure is a vector $\vec{r} \in \mathbb{R}^N$, where each $r_t = \sum_{j=1}^{N} k_{tj}$. Based on the components of $\vec{r}$, it is possible to compute $E_s(\mathcal{A})$ as shown in Figure 2. To compute $C(\vec{a}^i, \mathcal{A})$, we only need to substract k_{it} from each $r_t, t \neq i$ such that $E_s(\mathcal{A} \setminus \{\vec{a}^i\}) = \sum_{t=1, t\neq i}^{N} r_t$. This update process allows to compute every $C(\vec{a}, \mathcal{A})$ in $\Theta(N)$ and, thus, all individual contributions are computed in $\Theta(N^2)$. Algorithm 1 sketches the above described process.

Algorithm 1 Riesz s-energy steady state selection

Require: Pareto front approximation $\mathcal{A}$; size of the desired reference set μ
Ensure: Reference set
Compute dissimilarity matrix
while $|\mathcal{A}| > \mu$ **do**
  $\vec{a}_{\text{worst}} = \arg\max_{\vec{a} \in \mathcal{A}} \frac{1}{2}[E_s(\mathcal{A}) - E_s(\mathcal{A} \setminus \{\vec{a}\})]$
  From the dissimilarity matrix, delete the row and column associated to $\vec{a}_{\text{worst}}$
  Update the memoization structure $\vec{r}$
  $\mathcal{A} = \mathcal{A} \setminus \{\vec{a}_{\text{worst}}\}$
return $\mathcal{A}$

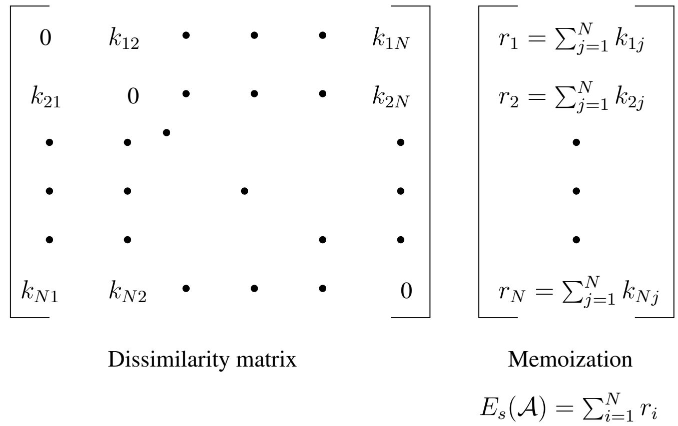


Fig. 2: Memoization structure that takes advantage of the dissimilarity matrix to reduce the cost associated to the computation of all the individual contributions of a set.

IV. Experimental Results

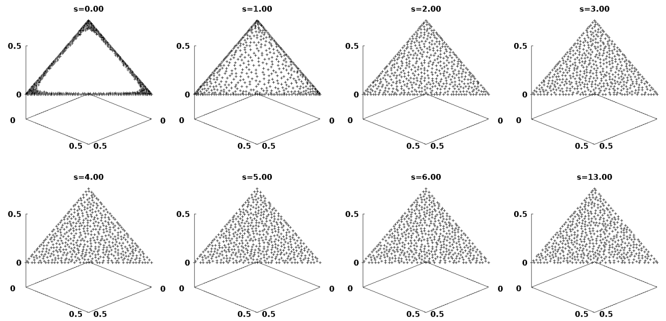
In this section, we compare our Riesz s-energy-based reference sets with reference sets constructed by the SLD, UDH, and a random selection. For both SLD and UDH, we use ASF to find the best relationship between the weight vectors and the solutions in the Pareto fronts. For our experiments, we employed MOPs from the benchmarks: Deb-Thiele-Laumanns-Zitzler (DTLZ) [15], Walking-Fish-Group (WFG) [16], Imbalanced MOPs (IMOPs) [17], and the Viennet MOPs (VIE) [18]. We employed PlatEMO 2.0 [19] to obtain the Pareto fronts, i.e., the sets $\mathcal{A} = \{\vec{a}^i\}_{i=1,\ldots,N}$. To show the properties of the Riesz s-energy-based reference sets, we performed the following experiments:

1) We studied the influence of the parameter s in the distribution of solutions, aiming to determine which is its best value.
2) A diversity comparison of the reference sets produced by the four methodologies was performed based on the hypervolume indicator (HV), the Solow-Polasky Diversity (SPD), IGD, and IGD$^+$.
3) We analyzed the effect of all the reference set schemes for the optimal μ-distributions of the IGD and IGD$^+$ indicators.
4) A set of MOEAs is evaluated by IGD and IGD$^+$ using the four types of reference sets to analyze the difference in preferences.

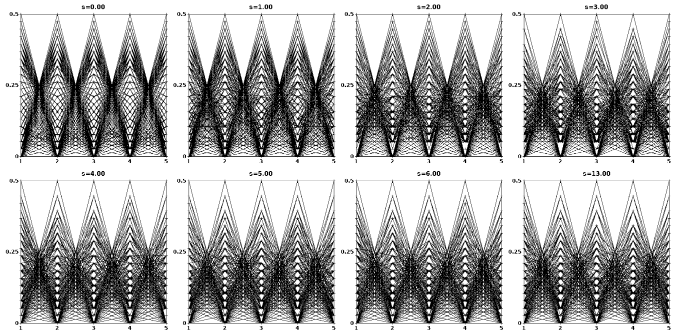
It is worth noting that the source code of the Riesz s-energy steady state selection that implements the fast computation of the individual contributions and the complete numerical results of the proposed experiments are available at http://computacion.cs.cinvestav.mx/~jfalcon/ReferenceSets.html.

A. Influence of the Parameter s

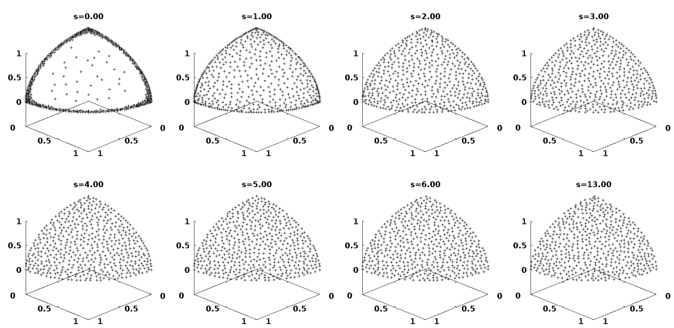
The Riesz s-energy depends on the parameter $s \geq 0$ that controls the uniformity of solutions in the E_s-based optimal distribution. Regarding MOEAs, s has been usually set to $m-1$, where m is the number of objective functions. However,



(a) DTLZ1 3D



(b) DTLZ1 5D



(c) DTLZ2 3D

Fig. 3: Approximated Riesz s-energy optimal distributions, varying the value of the parameter s.

we do not completely know what is the effect of s in the Riesz s-energy optimal μ-distributions. In consequence, we varied the value of s for different MOPs to observe some distribution properties. Figure 3 shows the distributions related to $s = 0, 1, 2, 3, 4, 5, 6, 13$ for DTLZ1 3D, DTLZ1 5D, and DTLZ2 3D. From the distributions, there is evidence that the closer s is to zero, the stronger the preference of the Riesz s-energy for boundary solutions is. Regarding the three-objective DTLZ1 and DTLZ2, we can see well-diversified distributions with a slight emphasis on the boundaries when $s = 2$ and there is not a clear difference between the distributions with $s > 2$. Hence, this supports the election of $s = m - 1$. Although the interpretation of the parallel coordinates of DTLZ1 5D is difficult, when the "peaks" are crowded, this means that the density of solutions in the boundary is high. It is worth emphasizing that for large values of s, the numerical values of the

TABLE I: Cardinality of the reference sets where H is the parameter of the SLD method.

m	N_1	N_2	N_3	N_4	N_5
2	$50_{H=49}$	$100_{H=99}$	$200_{H=199}$	$300_{H=299}$	$500_{H=499}$
3	$66_{H=10}$	$105_{H=13}$	$210_{H=19}$	$300_{H=23}$	$496_{H=30}$
4	$56_{H=5}$	$120_{H=7}$	$220_{H=9}$	$364_{H=11}$	$560_{H=13}$
5	$35_{H=3}$	$126_{H=5}$	$210_{H=6}$	$330_{H=7}$	$495_{H=8}$
6	$56_{H=3}$	$126_{H=4}$	$252_{H=5}$	-	$462_{H=6}$
7	$28_{H=2}$	$84_{H=3}$	$210_{H=4}$	-	$462_{H=5}$
8	$36_{H=2}$	$120_{H=3}$	-	$330_{H=4}$	$792_{H=5}$
9	$45_{H=2}$	$165_{H=3}$	-	-	$495_{H=4}$
10	$55_{H=2}$	-	$220_{H=3}$	-	$715_{H=4}$

Riesz s-kernel grow greatly which can produce representation problems in a computer. Hence, for the experiments in the following sections, we employed $s = m - 1$ as suggested in [11], [12].

B. *Assessing Reference Sets*

This experiment aims to provide evidence that our proposed approach produces reference sets having high diversity properties. To this purpose, we generated reference sets of the problems DTLZ1, DTLZ2, DTLZ5, DTLZ6, DTLZ7, WFG1-WFG4 for 2 to 10 objective functions and we also considered the test instances IMOP1-IMOP8 and VIE1-VIE3 that have a fixed number of objective functions. We adopted these MOPs since all of them cover linear, concave, convex, degenerate, disconnected, and mixed Pareto front shapes. For all test instances, we produced reference sets of different cardinalities that are shown in Table I (N_1: about 50, N_2: about 100, N_3: about 200, N_4 about 300, and N_5: about 500). As previously mentioned, the Pareto fronts were obtained from PlatEMO 2.0. Tables II, III, IV, and V show the average ranking results for the HV, SPD, IGD, and IGD$^+$ comparisons, respectively. SPD employs $\theta = 10$. IGD and IGD$^+$ employ reference sets of size 2,000 for their computation in each test instance, where these reference sets were directly obtained from PlatEMO. Our proposed reference sets are in all cases the best-ranked approaches considering the SPD indicator which is a pure-diversity QI. On the other hand, HV, IGD, and IGD$^+$, which are convergence-diversity QIs, mostly prefer the Riesz s-energy reference sets. The differences with respect to SPD are due to their own preferences properties, .e.g., HV prefers solutions around the Pareto front's knee. Consequently, there is strong empirical evidence that the Riesz s-energy indicator is able to produce well-diversified reference sets regardless of the geometry of the Pareto fronts and their dimensionality. Figure 4 shows a graphical comparison of the four methodologies for the five-objective DTLZ5, WFG1, and WFG3 problems. The Riesz s-energy-based reference sets exhibit better coverage and diversity of solutions in comparison to the random SLD, UDH, and random selection.

C. *IGD and IGD$^+$ optimal μ-distributions*

Our aim is to determine which is the effect of the Riesz s-energy-based reference sets to approximate the IGD and IGD$^+$ optimal μ-distribution. Based on the results of the previous

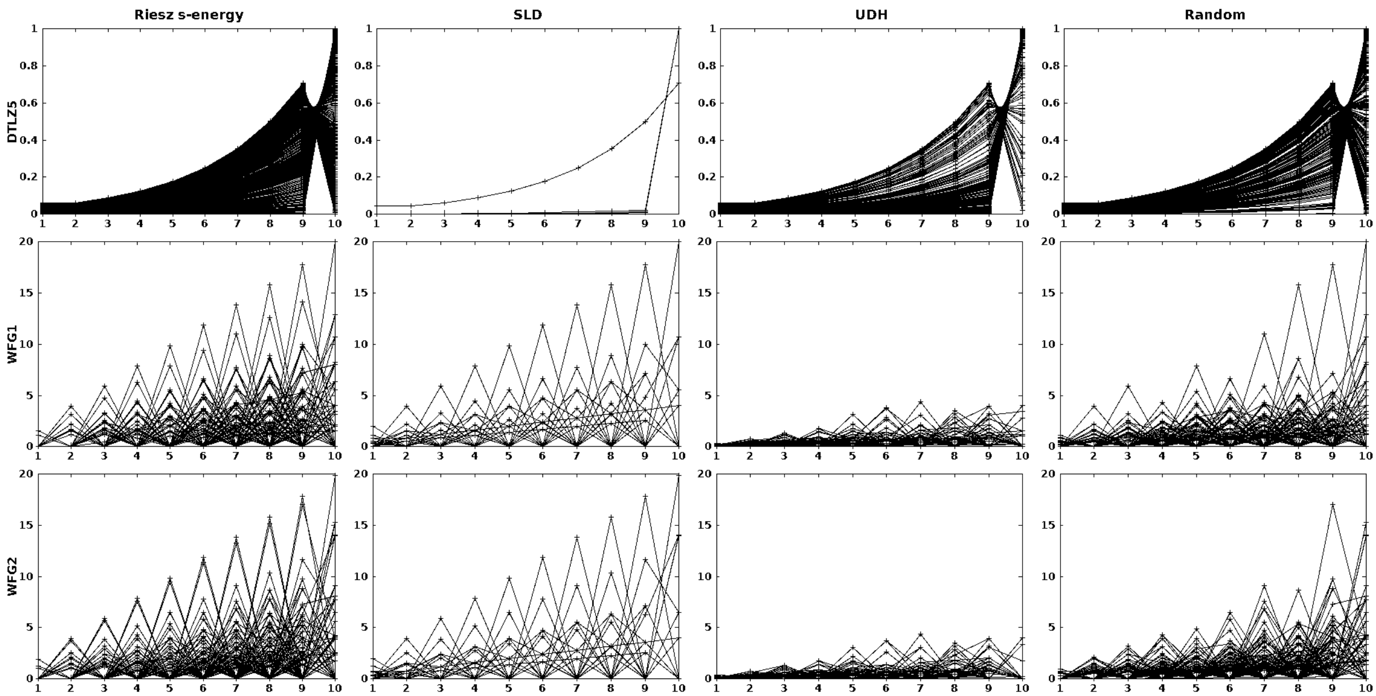


Fig. 4: Examples of reference sets for the DTLZ5, WFG1, and WFG2 problems with five objective functions.

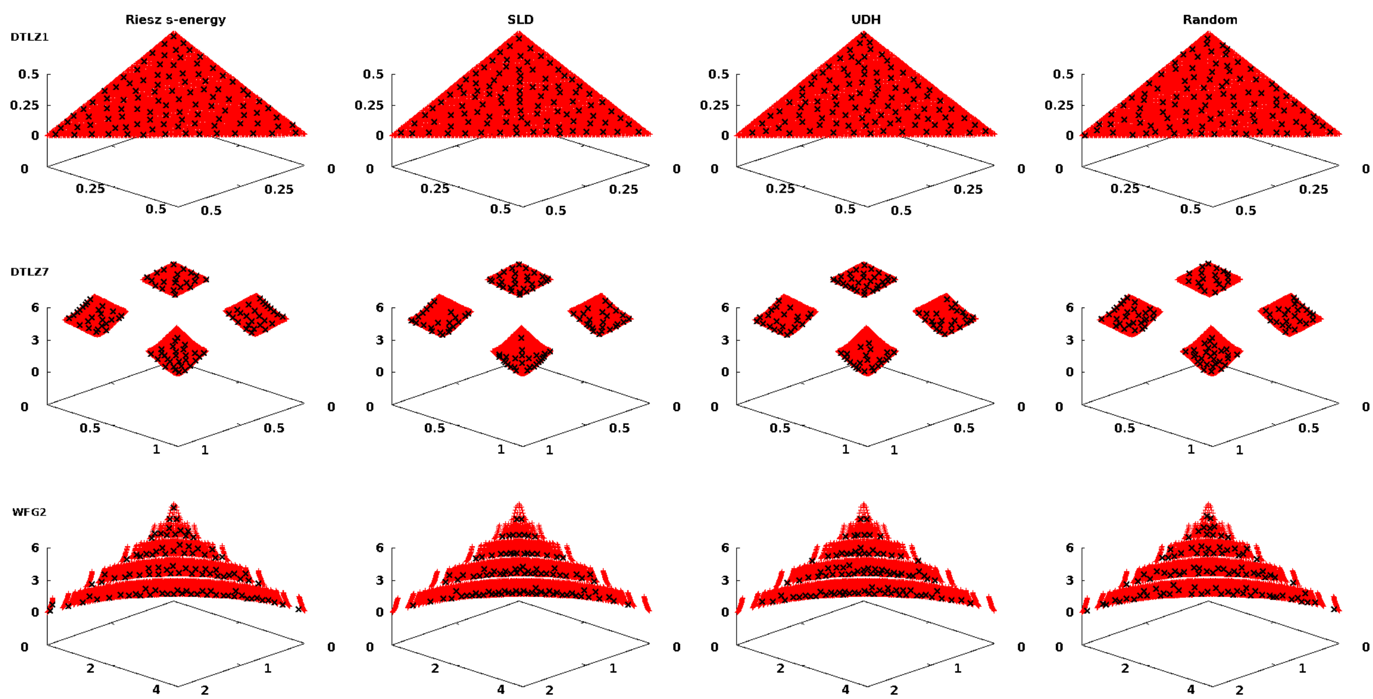


Fig. 5: Approximated IGD$^+$ optimal distributions of size 100 of the three-dimensional DTLZ1, DTLZ7, and WFG2 problems. The true Pareto front is shown in red.

TABLE II: Average ranking for Hypervolume comparison.

MOP	Riesz s-energy	Random	SLD	UDH
DTLZ1	1.184	2.789	2.289	3.736
DTLZ2	1.973	3.710	1.710	2.605
DTLZ5	1.000	2.421	3.8425	2.736
DTLZ6	1.000	2.263	3.842	2.894
DTLZ7	1.531	3.312	2.562	2.593
WFG1	2.500	2.421	1.657	3.421
WFG2	2.684	2.552	1.447	3.315
WFG3	1.000	2.210	3.736	3.052
WFG4	1.552	3.736	2.210	2.500
IMOP1	1.000	2.200	3.600	3.200
IMOP2	1.000	3.600	2.800	2.600
IMOP3	1.000	2.200	3.600	3.200
IMOP4	1.000	3.200	3.400	2.400
IMOP5	1.000	4.000	2.800	2.200
IMOP6	1.200	1.800	3.600	3.400
IMOP7	1.000	4.000	2.600	2.400
IMOP8	1.000	3.800	3.200	2.000
VIE1	1.000	2.666	3.333	3.000
VIE2	1.000	2.000	3.666	3.333
VIE3	1.333	2.666	2.000	4.000

TABLE III: Average ranking for Solow-Polasky Diversity comparison.

MOP	Riesz s-energy	Random	SLD	UDH
DTLZ1	1.473	2.921	1.763	3.842
DTLZ2	1.184	3.131	2.421	3.263
DTLZ5	1.000	2.289	3.894	2.815
DTLZ6	1.000	2.263	3.842	2.894
DTLZ7	1.000	2.218	3.500	3.281
WFG1	1.000	2.631	2.684	3.684
WFG2	1.000	2.342	2.894	3.763
WFG3	1.000	2.342	3.815	2.842
WFG4	1.000	2.552	2.815	3.631
IMOP1	1.000	2.000	3.800	3.200
IMOP2	1.000	3.600	2.600	2.800
IMOP3	1.000	3.000	2.800	3.200
IMOP4	1.000	2.200	4.000	2.800
IMOP5	1.000	4.000	3.000	2.000
IMOP6	1.000	2.800	3.000	3.200
IMOP7	1.000	4.000	2.000	3.000
IMOP8	1.000	3.200	3.000	2.800
VIE1	1.000	2.000	4.000	3.000
VIE2	1.000	2.000	3.666	3.333
VIE3	1.000	2.000	3.000	4.000

TABLE IV: Average ranking for IGD comparison.

MOP	Riesz s-energy	Random	SLD	UDH
DTLZ1	2.921	3.078	2.657	1.342
DTLZ2	1.763	3.342	3.026	1.868
DTLZ5	1.263	2.368	3.605	2.763
DTLZ6	1.263	2.421	3.605	2.710
DTLZ7	1.343	2.125	3.562	2.968
WFG1	3.342	2.526	2.078	2.052
WFG2	2.921	2.263	2.210	2.605
WFG3	1.263	2.368	3.605	2.763
WFG4	1.526	2.947	3.263	2.263
IMOP1	1.000	2.000	3.800	3.200
IMOP2	2.200	4.000	2.200	1.600
IMOP3	1.000	3.800	2.400	2.800
IMOP4	1.000	2.000	4.000	3.000
IMOP5	1.000	2.200	3.800	3.000
IMOP6	1.000	2.000	3.200	3.800
IMOP7	1.000	2.400	3.000	3.600
IMOP8	1.000	2.800	3.400	2.800
VIE1	1.000	2.666	2.666	3.666
VIE2	1.000	2.000	3.666	3.333
VIE3	1.000	2.000	3.000	4.000

TABLE V: Average ranking for IGD$^+$ comparison.

MOP	Riesz s-energy	Random	SLD	UDH
DTLZ1	2.894	2.842	3.000	1.263
DTLZ2	2.131	4.000	1.368	2.500
DTLZ5	1.000	2.342	3.763	2.894
DTLZ6	1.000	2.342	3.763	2.894
DTLZ7	2.218	2.718	3.187	1.875
WFG1	3.763	3.105	1.973	1.157
WFG2	3.421	3.289	1.921	1.368
WFG3	1.236	2.342	3.605	2.815
WFG4	2.473	3.921	1.184	2.421
IMOP1	1.000	2.200	3.000	3.800
IMOP2	1.000	4.000	2.200	2.800
IMOP3	1.000	3.000	3.200	2.800
IMOP4	1.000	2.000	4.000	3.000
IMOP5	1.200	2.000	4.000	2.800
IMOP6	1.000	2.000	3.200	3.800
IMOP7	1.000	2.600	2.600	3.800
IMOP8	1.800	3.600	2.400	2.200
VIE1	1.000	2.333	4.000	2.666
VIE2	1.000	2.000	3.666	3.333
VIE3	1.000	2.333	2.666	4.000

section, we hypothesize that their use could improve the diversity proporties of these distributions. For this experiment, we implemented a steady-state MOEA (based on the framework of the SMS-EMOA [20]) that uses a density estimator based on IGD and IGD$^+$. We denoted such algorithms as IGD-MaOEA and IGD$^+$-MaOEA. Both algorithms employed a fixed reference set whose cardinality is given by the column N_5 of Table I. We approximated the optimal μ-distributions of the problems DTLZ1, DTLZ2, DTLZ5, DTLZ6, DTLZ7, and WFG1-WFG4 with 2 to 10 objective functions, where μ is the population size of the algorithms and we used $\mu = 20, 50, 100$. To correctly approximate the distributions, we turned off all the difficulties of the problems. For each test instance, the stopping criterion of both algorithms was 100,000 function evaluations.

Tables VI and VII show the average ranking results for IGD-MaOEA and IGD$^+$-MaOEA regarding SPD, IGD, and IGD$^+$ values. Similarly to the previous section, the SPD indicator shows that the use of the Riesz s-energy-based reference sets allows both MOEAs to achieve better distributions in comparison to the use of reference sets based on SLD, UDH, and the random selection. However, the results of both IGD and IGD$^+$ are different from the SPD results, biasing the preference to the UDH-based reference sets. This behavior can be explained as follows. For the calculation of IGD and IGD$^+$, we employed the reference sets produced by the PlatEMO that uses weight vectors to find Pareto optimal solutions. Hence, there is a correlation between such reference sets and the results using UDH. Figure 5 presents a comparison of distributions for the DTLZ1, DTLZ7, and WFG2 problems with three-objective functions. As the SPD, in Tables VI and VII, indicates, the distributions created using the Riesz s-energy-based reference set have better diversity of solutions. Based on these discussions, we can claim that the use of the Riesz s-energy to generate reference sets could help referece set-based MOEAs to generate Pareto front approximations with a higher degree of diversity.

TABLE VI: SPD, IGD, and IGD$^+$ average ranking results of the IGD optimal μ-distributions using the four types of reference sets.

QI	MOP	Riesz s-energy	Random	SLD	UDH
SPD	DTLZ1	1.962	2.296	1.777	3.962
	DTLZ2	2.333	3.481	2.074	2.111
	DTLZ5	1.370	2.148	3.777	2.703
	DTLZ6	1.592	1.851	3.740	2.814
	DTLZ7	1.380	2.428	3.190	3.000
	WFG1	1.222	2.037	2.962	3.777
	WFG2	1.407	1.814	3.037	3.740
	WFG3	1.296	2.296	3.037	3.370
	WFG4	1.148	2.148	3.481	3.222
IGD	DTLZ1	2.629	3.037	3.148	1.185
	DTLZ2	1.740	3.259	3.370	1.629
	DTLZ5	1.259	2.222	3.703	2.814
	DTLZ6	1.370	2.111	3.703	2.814
	DTLZ7	2.904	2.761	2.476	1.857
	WFG1	3.703	2.814	2.185	1.296
	WFG2	3.629	2.629	1.888	1.851
	WFG3	1.222	2.222	3.666	2.888
	WFG4	2.481	2.629	3.259	1.629
IGD$^+$	DTLZ1	2.851	2.888	3.111	1.148
	DTLZ2	2.592	3.814	1.185	2.407
	DTLZ5	1.148	2.111	3.740	3.000
	DTLZ6	1.370	1.888	3.740	3.000
	DTLZ7	1.428	2.095	3.619	2.857
	WFG1	3.851	3.074	1.925	1.148
	WFG2	3.666	3.333	1.851	1.148
	WFG3	1.296	2.148	3.666	2.888
	WFG4	3.222	3.296	1.333	2.148

TABLE VII: SPD, IGD, and IGD$^+$ average ranking results of the IGD$^+$ optimal μ-distributions using the four types of reference sets.

QI	MOP	Riesz s-energy	Random	SLD	UDH
SPD	DTLZ1	1.777	2.481	1.777	3.962
	DTLZ2	1.925	2.333	2.925	2.814
	DTLZ5	1.518	2.555	3.740	2.185
	DTLZ6	1.555	2.296	3.777	2.370
	DTLZ7	1.428	1.809	3.380	3.380
	WFG1	1.259	1.925	3.037	3.777
	WFG2	1.185	1.851	3.074	3.888
	WFG3	1.296	2.444	2.851	3.407
	WFG4	2.000	2.148	2.925	2.925
IGD	DTLZ1	2.481	3.074	3.074	1.370
	DTLZ2	2.296	1.777	3.111	2.814
	DTLZ5	2.333	2.555	3.444	1.666
	DTLZ6	2.370	2.444	3.481	1.703
	DTLZ7	1.952	1.761	3.666	2.619
	WFG1	1.555	1.592	3.185	3.666
	WFG2	1.444	1.629	3.148	3.777
	WFG3	1.185	2.222	3.777	2.814
	WFG4	2.296	1.296	3.407	3.000
IGD$^+$	DTLZ1	2.666	3.074	3.074	1.185
	DTLZ2	2.333	3.629	2.148	1.888
	DTLZ5	1.296	2.296	3.629	2.777
	DTLZ6	1.370	2.185	3.666	2.777
	DTLZ7	2.666	3.190	2.428	1.714
	WFG1	3.740	3.222	1.592	1.444
	WFG2	3.740	3.037	1.925	1.296
	WFG3	1.185	2.185	3.740	2.888
	WFG4	3.074	3.777	1.592	1.555

D. Assessment of MOEAs

In this section, we aim to analyze the preference of the IGD indicator using the four types of reference sets in the IGD-based comparison of different MOEAs. For this aim, we executed several MOEAs on MOPs having 2 to 6 objective functions. Each MOEA was executed 30 independent times on each test instance. Table VIII shows the numerical results of the comparison only for the WFG2 problem due to lack of space. We employed the reference sets of cardinality equal to the column N_5 of Table I. The use of different methods to generate reference sets does not radically change the preference of IGD as it is clear in Table VIII. Further research is required in this direction to determine the effect of the Riesz s-energy-based reference on the IGD comparison.

V. Conclusions and Future Work

Currently, reference sets are widely used to guide the population of an MOEA towards the Pareto front or for the assessment of MOEAs through quality indicators. An important issue is how to construct the reference set, having good diversity properties regardless of the dimensionality of the objective space and the geometry of the Pareto front. In this paper, we proposed to use the Riesz s-energy to construct reference sets due to its nice properties that allow to uniformly sample a d-dimensional manifold. Our experimental results that include the diversity assessment of our reference sets and other three construction methodologies of reference sets indicate the superiority of the Riesz s-energy. This superiority was shown in the construction of approximated optimal μ-distributions for the IGD and IGD$^+$ indicators and for the assessment of state-of-the-art MOEAs. As part of our future work, we want to further analyze the mathematical properties of the Riesz s-energy and its use to dynamically guide reference set-based MOEAs.

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TABLE VIII: Mean and, in parentheses, standard deviation of the IGD comparison, for the WFG2 problem, using the four methods to generate reference sets. The two best values are shown in grayscale, where the darker tone corresponds to the best algorithm. The superscript indicates the rank of the algorithm.

Method	Dim.	IGD^{+}-MaOEA	MOMBI3	AR-MOEA	RVEA	GrEA	SPEA2SDE	Two_Arch2
Riesz s-energy	2	3.234381e-01^{7} (5.975547e-02)	1.580167e-01^{1} (1.562619e-01)	2.425274e-01^{6} (1.389113e-01)	1.628697e-01^{2} (1.133657e-01)	2.116046e-01^{4} (1.402733e-01)	2.054706e-01^{3} (1.545234e-01)	2.407449e-01^{5} (1.416473e-01)
	3	4.184759e-01^{6} (1.524529e-01)	3.353822e-01^{1} (1.640826e-01)	4.106785e-01^{5} (1.896221e-01)	3.597037e-01^{3} (1.858185e-01)	3.934160e-01^{4} (1.304571e-01)	4.731361e-01^{7} (1.426240e-01)	3.387827e-01^{2} (1.946614e-01)
	4	9.748887e-01^{7} (2.014461e-01)	6.478702e-01^{3} (2.399085e-01)	6.169584e-01^{1} (2.333210e-01)	6.909600e-01^{4} (2.427336e-01)	8.551820e-01^{5} (1.974260e-01)	9.490244e-01^{6} (1.679712e-01)	6.207841e-01^{2} (2.577176e-01)
	5	1.556550e+00^{7} (2.180828e-01)	8.046459e-01^{1} (2.501754e-01)	8.353550e-01^{2} (2.332823e-01)	1.237189e+00^{5} (3.040188e-01)	1.152309e+00^{4} (1.611949e-01)	1.530695e+00^{6} (2.194192e-01)	8.896438e-01^{3} (2.928290e-01)
	6	2.147859e+00^{6} (2.503432e-01)	1.265947e+00^{3} (3.529989e-01)	1.233485e+00^{1} (3.172725e-01)	1.963559e+00^{5} (3.526986e-01)	1.686628e+00^{4} (2.801769e-01)	2.278093e+00^{7} (2.432840e-01)	1.252698e+00^{2} (3.298790e-01)
SLD	2	2.956513e-01^{7} (5.532228e-02)	1.426335e-01^{1} (1.445513e-01)	2.198279e-01^{5} (1.299147e-01)	1.574546e-01^{2} (1.007099e-01)	1.894829e-01^{4} (1.318221e-01)	1.869568e-01^{3} (1.424213e-01)	2.212709e-01^{6} (1.290839e-01)
	3	2.790953e-01^{3} (1.789936e-01)	2.749698e-01^{2} (1.981554e-01)	3.770123e-01^{7} (2.174194e-01)	3.242739e-01^{6} (1.958300e-01)	2.749419e-01^{1} (1.567955e-01)	3.217098e-01^{5} (1.847323e-01)	3.153545e-01^{4} (2.156497e-01)
	4	8.156440e-01^{7} (3.160402e-01)	6.355970e-01^{4} (3.662658e-01)	5.993264e-01^{1} (3.674012e-01)	6.045240e-01^{2} (3.345032e-01)	7.095258e-01^{5} (3.111989e-01)	7.208153e-01^{6} (2.656273e-01)	6.244147e-01^{3} (3.755997e-01)
	5	1.333039e+00^{7} (4.444634e-01)	7.221917e-01^{1} (4.213339e-01)	7.631819e-01^{2} (4.415437e-01)	1.028465e+00^{5} (4.727804e-01)	8.547878e-01^{3} (3.092179e-01)	1.213322e+00^{6} (4.193114e-01)	8.743622e-01^{4} (5.053471e-01)
	6	1.687462e+00^{6} (4.759212e-01)	1.109226e+00^{1} (5.243480e-01)	1.166047e+00^{3} (5.466763e-01)	1.668561e+00^{5} (5.778571e-01)	1.413500e+00^{4} (5.366980e-01)	1.859156e+00^{7} (4.830788e-01)	1.133397e+00^{2} (5.686533e-01)
UDH	2	2.919313e-01^{7} (5.461468e-02)	1.408839e-01^{1} (1.426805e-01)	2.171267e-01^{5} (1.281298e-01)	1.563367e-01^{2} (9.891542e-02)	1.872846e-01^{4} (1.299533e-01)	1.847227e-01^{3} (1.404722e-01)	2.185182e-01^{6} (1.273169e-01)
	3	2.708864e-01^{1} (1.694095e-01)	2.722777e-01^{3} (1.868100e-01)	3.654160e-01^{7} (2.079050e-01)	3.177317e-01^{6} (1.848709e-01)	2.709690e-01^{2} (1.475962e-01)	3.122282e-01^{5} (1.743504e-01)	3.073239e-01^{4} (2.058473e-01)
	4	7.117344e-01^{7} (3.442947e-01)	6.117706e-01^{6} (3.712138e-01)	5.899830e-01^{2} (3.619661e-01)	5.401684e-01^{1} (3.481553e-01)	6.079171e-01^{5} (3.394134e-01)	5.969660e-01^{4} (2.934897e-01)	5.908195e-01^{3} (3.861430e-01)
	5	1.064249e+00^{7} (5.660817e-01)	6.771401e-01^{2} (4.442194e-01)	7.016092e-01^{3} (4.810877e-01)	8.099523e-01^{5} (5.401982e-01)	5.844477e-01^{1} (3.717637e-01)	9.007483e-01^{6} (5.296794e-01)	7.878792e-01^{4} (5.560114e-01)
	6	1.113797e+00^{5} (7.729060e-01)	9.476411e-01^{1} (6.455016e-01)	1.065796e+00^{3} (7.050877e-01)	1.334352e+00^{6} (8.255314e-01)	1.066176e+00^{4} (7.819136e-01)	1.375711e+00^{7} (7.944234e-01)	9.829118e-01^{2} (7.250366e-01)
Random	2	3.774162e-01^{7} (7.073652e-02)	1.815207e-01^{2} (1.851105e-01)	2.799561e-01^{6} (1.666496e-01)	1.756049e-01^{1} (1.387162e-01)	2.414035e-01^{4} (1.689005e-01)	2.380349e-01^{3} (1.826005e-01)	2.795021e-01^{5} (1.668415e-01)
	3	3.982291e-01^{5} (1.543673e-01)	3.310771e-01^{1} (1.628366e-01)	4.045044e-01^{6} (1.907077e-01)	3.548877e-01^{3} (1.845288e-01)	3.856882e-01^{4} (1.287517e-01)	4.542812e-01^{7} (1.439465e-01)	3.340118e-01^{2} (1.954608e-01)
	4	8.949975e-01^{7} (2.080101e-01)	6.294227e-01^{3} (2.479606e-01)	6.009273e-01^{2} (2.392728e-01)	6.336071e-01^{4} (2.428153e-01)	7.841559e-01^{5} (2.059607e-01)	8.579769e-01^{6} (1.697523e-01)	6.007684e-01^{1} (2.679241e-01)
	5	1.340366e+00^{7} (2.577005e-01)	7.561532e-01^{1} (2.745982e-01)	7.724436e-01^{2} (2.635507e-01)	1.042620e+00^{5} (3.284797e-01)	9.630800e-01^{4} (1.826575e-01)	1.290533e+00^{6} (2.536782e-01)	8.413921e-01^{3} (3.215030e-01)
	6	1.785146e+00^{6} (2.413379e-01)	1.162767e+00^{3} (3.138277e-01)	1.147089e+00^{2} (2.690098e-01)	1.634505e+00^{5} (3.288890e-01)	1.405960e+00^{4} (2.730746e-01)	1.901939e+00^{7} (2.322959e-01)	1.124687e+00^{1} (3.105391e-01)

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