A Novel Grid-based Crowding Distance for Multimodal Multi-objective Optimization

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***Abstract*—Preserving diversity in decision space plays an important role in Multimodal Multi-objective Optimization problems (MMOPs). Due to the lack of mechanisms to keep different solutions with the same fitness value, most of the available Multi-objective Evolutionary Algorithms (MOEAs) perform poorly when applied to MMOPs. To deal with these problems, this paper proposes a novel method for diversity preserving in the decision space. To this end, the concept of grid-based crowding distance for decision space is introduced. Furthermore, to keep a good diversity of solutions in both decision and objective spaces, we propose different frameworks by combining this method with crowding distance in decision space, crowding distance in objective space, and the weighted sum of both crowding distances. In order to evaluate the performance of these frameworks, we integrate them into the diversity preserving part of the NSGA-II algorithm, and compare them with the NSGA-II (as the baseline algorithm) and the state-of-the-art multimodal multi-objective optimization algorithms on ten different MMOPs with different levels of complexity.**

***Index Terms*—Grid-based Crowding distance, Multi modality, Evolutionary Algorithms, Non-dominated Sorting Genetic Algorithm, Multi-objective Optimization.**

I. Introduction

There are many real-world multi-objective optimization problems (MOPs) with conflicting objectives which are supposed to be optimized at the same time. In these problems, the improvement of one objective may lead to the deterioration of the other objectives. Therefore, it is required to find an optimal set of solutions to make a trade-off between these conflicting objectives. A solution that is not dominated by any other solution in objective space is called Pareto optimal solution. The set of these optimal solutions in decision space is known as Pareto-Set (PS). The image of these solutions in objective space is called Pareto Front (PF).

In some of the practical multi objective optimization problems, there may exist multiple equivalent PSs in decision space corresponding to the same PF in the objective space. These problems are defined as Multimodal Multi-objective Optimization Problems (MMOPs) [1].

If the decision-makers are informed about these dissimilar optimal solutions with the same quality, they could choose the final optimal solution according to their preferences. These solutions can be located by keeping a high diversity of solutions in the decision space.

Over the last decades, several Multi-objective Evolutionary Algorithms (MOEAs) have been designed to solve MOPs, mainly focused on providing a better approximation of the PF in terms of convergence and diversity (in objective space) [2]. However, the distribution of solutions in decision space has not received much attention. In order to discover as many equivalent PSs as possible in MMOPs, it is necessary to provide a proper distribution of solutions in decision space. Since most available MOEAs are mainly focused on increasing the diversity of solutions in objective space (e.g., the classical crowding distance in NSGA-II algorithm), they could approximate the PF by locating one of these equivalent PSs. To fill this gap, it is important to modify the available MOEAs to improve the distribution of solutions in decision space while not deteriorating the approximation of PF.

To deal with this challenge, we modify the original concept of crowding distance (i.e. the secondary selection criteria) used in Non-dominated Sorting Genetic Algorithm II (NSGA-II) [2], by proposing a novel NSGA-II-Grid-based crowding distance algorithm (NSGA-II-Gr). With the implementation of this method, the solutions selected to be transferred to the next generation are located in the neighborhood of sparse areas in decision space, leading to an improved distribution of solutions in the decision space. The proposed method enables the algorithm to keep the solutions that are far from each other in the decision space, while they may remain near to each other in objective space. To assess the performance of the proposed algorithm, we compared it with some of the available Multimodal Multi-Objective Evolutionary Algorithms (MMOEAs) on a number of MMOPs. Furthermore, we present three additional algorithms with the combination of Grid-based method and crowding distance in decision space (NSGA-II-Gr-CDdec), objective space (NSGA-II-Gr-CDobj), and weighted sum of both values (NSGA-II-Gr-CDws).

The proposed NSGA-II-Gr-CDdec is designed to augment the distribution of solutions in decision space, while NSGA-II-Gr-CDobj is proposed to avoid neglecting the diversify of solutions in objective space. To make a trade-off between the diversity of both spaces, we introduced NSGA-II-Gr-CDws.

This paper is structured as follows. Section 2 gives a brief summary of related works. In Section 3, the proposed algorithms are introduced. The experimental settings and the analysis of the results are covered in Sections 4 and 5,

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respectively. In the final section, conclusions of this work and future research goals are presented.

II. Related Work

Research regarding MOEAs have an extensive background e.g., [3], [4]. The most famous algorithm in this field is NSGA-II [2], which looks for optimal solutions in two steps: first, it uses a non-dominated sorting method to divide the population into fronts of non-dominated solutions, and then it uses a crowding distance technique to sort the solutions in these fronts according to the distance between their nearest neighbors on each dimension. The main focus of the MOEAs relies on providing a better approximation of the PF in the objective space. Nevertheless, in order to deal with MMOPs, we need to consider the approximation of solution in the decision space.

In recent years, there have been a few papers addressing MMOPs by specifically developing methods to find and preserve diverse solutions in the decision space. The first detailed study on MMOPs was performed by Deb et al. [5], proposing the Omni-optimizer algorithm. They modified the concept of crowding distance in NSGA-II, in which, for each solution the crowding distance values in both the objective and the decision spaces are considered. Then, their mean values on each space are compared and the larger value of both distances is assigned to each solution.

In [1] and [6], two other dominance-based MMOEAs, which are extended versions of NSGA-II, are proposed: Decision-Based Niching NSGA-II (DN-NSGA-II) and Double Niched Evolutionary Algorithm (DNEA). In DN-NSGA-II, the crowding distance value is measured in decision space instead of in objective space. In DNEA, Omni-optimizer is extended by adding two sharing functions in both decision and objective spaces.

One of the state-of-the-art algorithms often cited in research on MMOEAs, is an extended version of Multi-Objective Patrice Swarm Optimization algorithm (MOPSO). This algorithm proposed by Yue et al. [7] was called Multi-objective Particle Swarm Optimization using Ring topology by applying Special crowding distance (MO-Ring-PSO-SCD). In this algorithm, an index-based ring topology is used to explore the search space and a special crowding distance method is adopted and amended from the Omni-optimizer algorithm to preserve the obtained diverse solutions in decision space.

Another mechanism called Self-organizing Multi-objective Particle Swarm Optimization (SMPSO-MM) was proposed by Liang et al. [8]. Their proposed self-organizing mechanism, along with an adopted special crowding distance method, helps the mapping of similar solutions to the same neighborhood, and moreover, to maintain the distribution of the obtained solutions in decision space.

Recently, two extensions of NSGA-II have been proposed for MMOPs: NSGA-II-WSCD-NBM [9], [10], and NSGA-II-MDCD [11]. The secondary selection criteria in NSGA-II-WSCD-NBM is based on the weighted sum of crowding distance values in both decision and objective spaces, leading them into a trade-off between the distribution of the solutions in both decision and objective spaces. The weighted factors for both of the spaces are imposed equally. Furthermore, a novel Neighborhood polynomial Mutation (NBM) operator was proposed to explore a wider range of the search space. These two mechanisms together support a better approximation of PS in terms of the distribution of optimal obtained solutions.

NSGA-II-MDCD aims to preserve the solutions far from each other in decision space, even if they are located close to each other in objective space and avoid getting trapped into local optima. In these method, the concept of Manhattan distance was employed in the partitioned decision space, where the sum of the grid difference between each solution and the rest of solutions in the related front is computed.

III. Grid-based crowding distance for MMOEAs

In this section, the proposed Grid-based crowding distance method in decision space (Gr_{dec}) is described, followed by a pseudocode along with a description of the relevant details.

As already mentioned, one of the main concerns of MMOPs is to provide a relatively good diversity of solutions in the decision space. In order to deal with this problem, it is important to select the solutions that are located in less crowded neighborhoods. Therefore, it is required to reformulate the diversity preserving selection criteria (e.g. crowding distance in population-based algorithms), which is responsible for selecting and transferring a number of so far non-dominated solutions to the next generation. In this paper, the crowding distance technique in NSGA-II is modified with the goal to select the solutions from the sparse areas in the decision space.

The original computation of crowding distance for each solution in NSGA-II is as follows: the solutions are separately ordered on each dimension in the objective space. Then, for each of these dimensions, the distance between the two nearest neighbors of each solution is calculated and the normalized sum of the obtained value is assigned as the crowding distance value for the solution.

Because of the natural capability of grids for representing the distribution of solutions [12], we modified this crowding distance method by partitioning the decision space and introducing a Grid-based mechanism to measure distances between solutions. The initial idea of this method comes from the modified Manhattan distance method proposed in [11], where the decision space is partitioned into grids. The grid-distance ($GD(S_i, S_j)$) between two solutions S_i and S_j are then measured by counting the number of the grids between them using Manhattan distances. The final crowding distance value for each solution is calculated by summing the pairwise grid-distances between the solution and the other solutions on the same front.

The crowding distance values help to identify the crowded areas. However, it is important to additionally consider the density of the solutions in the neighborhood of each solution. In order to address the density in the crowding distance measurement, a new definition of neighbourhood is proposed in Grid-Based Evolutionary Algorithm (GrEA) [12], in which different techniques are used to handle many objective optimization problems. A Grid crowding distance (GCD) is proposed to increase the selection pressure towards the PF, by increasing the diversity of individuals in the objective space. This method limits the neighborhood of each solution in the objective space by a maximum grid difference of M (the number of objective functions). In this paper, we modify and adapt this neighborhood from GrEA into the decision space.

For each solution S_i, the neighborhood contains the solutions S_j with the grid difference less than D to S_i (D denotes the number of decision variables) as follows:

$$NB(S_i) = \{S_j | S_i \neq S_j \wedge GD(S_i, S_j) < D\} \tag{1}$$

where $S = (S_1, ..., S_N)$ is the current front of solutions and NB denotes the set of solutions which are in the neighborhood of solution S_i. $GD(S_i, S_j)$ is the grid-distance between pairs of solutions S_i and S_j.

In order to favor solutions located in sparse areas, the Grid-based crowding distance (Gr$_{dec}$) value for each solution is calculated according to the following equation:

$$Gr_{dec}(S_i) = \sum_{S_j \in NB(S_i)} (D - GD(S_i, S_j)) \tag{2}$$

In this way, a solution located in a crowded area will be assigned a large Gr_{dec} value. In our proposed approach, we favor the solutions with small Gr_{dec} values. The proposed method for Grid-based crowding distance in decision space is described in Algorithm 1. First, after the initial setting of parameters (Lines 1 to 6), the grid-distance between each pair of solutions on the same front is computed (Lines 7 to 13). Afterwards, for each solution, the Grid-based crowding distance (Gr$_{dec}$) value is computed based on equations 1 and 2 (Lines 14 to 18).

In order to normalize the Grid-based crowding distance values (for a subsequent combination with other distance metrics), a max-min normalization is used to put the results into the same range (0,1] (Line 19). To avoid obtaining zero values for the solutions with the minimum value after applying the normalization method, the min value in the equation is changed to a very small value (e.g. 0.001) as follows:

$$norm(V_i) = \frac{V_i - (min(V) - 0.001)}{max(V) - (min(V) - 0.001)} \tag{3}$$

where $V = (V_1, ..., V_N)$ is a set of values (the Gr$_{dec}$ values in this work) and $norm(V_i)$ is the i_{th} normalized value.

Based on this method, the solutions with lowest Gr$_{dec}$ values are selected and transferred to the next generation. These lowest values represent solutions that are located in the neighbourhood of sparse areas.

In our proposed approach called NSGA-II-Gr, we replace the crowding distance method in the NSGA-II by the above Grid-based crowding distance in decision space (Gr_{dec}).

Algorithm 1: Grid-based crowding distance in decision space (Gr$_{dec}$) approach.

Input: Number of decision variables: D,
List S of solutions of current front (with Grid index Gr_{Ind} values for each dimension)
Output: List S with the extra property Grid-based crowding distance in decision space (Gr$_{dec}$) for each solution

```
1  . for i ∈ {1,..,|S| − 1} do
2        S[i].Gr_dec = 0;
3        for j ∈ {1,..,|S|} do
4            GD(i, j) = 0;
5        end
6  end
7  for i ∈ {1,..,|S| − 1} do
8        for j ∈ {i + 1,..,|S|} do
9            for k ∈ {1,..,D} do
10               GD(i, j) + = S[i].Gr_Ind[k] − S[j].Gr_Ind[k] ;
11               GD(j, i) + = S[j].Gr_Ind[k] − S[i].Gr_Ind[k] ;
12           end
13       end
14 end
15 for i ∈ {1,..,|S|} do
16       for j ∈ {1,..,|S|} do
17           S[i].Gr_dec + = max(D − GD(i, j), 0);
18       end
19 end
20 S = norm_Gr_dec(S) ;    // normalization of Gr_dec
21 return S
```

Although the proposed method (NSGA-II-Gr) is able to approximate the density of solutions, there are still some cases where this method fails to highlight the solutions that are located in crowded areas. Since the Gr$_{dec}$ value of solutions is highly dependent on the grid resolution, in some cases the Gr$_{dec}$ values of some solutions can be the same, even if these solutions are not located in equally crowded areas. To compensate such shortcoming, the proposed method NSGA-II-Gr is furthermore improved by dividing the Gr$_{dec}$ with the crowding distance technique applied to decision space (CD_{dec}) proposed in Omni-optimizer algorithm [5]. We call this variation NSGA-II-Gr-CDdec approach.

The crowding distance value in decision space [5], [7] is calculated as follows: first, the solutions are ranked based on each of the decision variables. Then, for each solution, the final crowding distance value is computed based on the summation of the normalized distance between the two adjacent neighbours on each dimension. For boundary points in any dimension, the distance between the solution and its nearest neighbor is multiplied by 2.

In order to demonstrate the importance of this combination, an example is provided in Figure 1. As shown in this example, the Gr$_{dec}$ values of the solutions represent if they are located sparse areas. In this example, solutions D and E have the same Gr$_{dec}$ values, but different values of CD_{dec}. By dividing the obtained Gr$_{dec}$ values with the crowding distance CD_{dec}

values, solution E gets a smaller Gr_{dec} value than D. This combination reveals that solution E is located in a less crowded area than D.

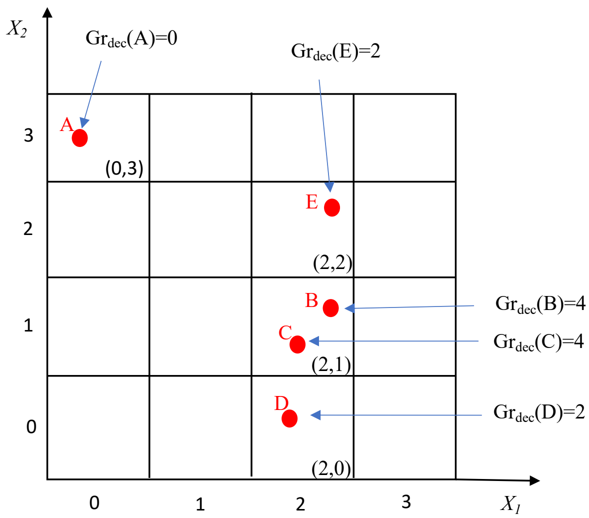


Fig. 1: An instance to demonstrate the importance of combining the Grid-based crowding distance method with the crowding distance in decision space.

In the following, we propose two more variations. In NSGA-II-Gr-CDobj, the proposed Gr_{dec} method is combined with the usual crowding distance approach used in NSGA-II (CD_{obj}) [2]. In this way, we aim to keep the diversity in the objective space.

In NSGA-II-Gr-CDws, we aim to take the advantages of using crowding distance approach in both decision and objective spaces. Therefore, we combine the presented Gr_{dec} approach with the weighted sum Crowding Distance (CD_{ws}) [9], which leads to the increase of the diversification in the decision space while not deteriorating the distribution of the obtained optimal solutions in objective space.

In addition, we examine the following. Since lower Gr_{dec} values and higher CD_{dec}, CD_{obj}, and CD_{ws} values are preferable, these combined algorithms are proposed as previously pointed out by dividing the obtained Gr_{dec} values with the respective crowding distance. Then, this final distance values are sorted in ascending order, and the lowest values (representing the solutions located in the neighborhood of sparser areas) are transferred to the next generation.

All these proposed methods are focused on the preservation of solutions by increasing the selection pressure towards more diverse solutions in decision space. On the other hand, it is important to enhance the exploration of the search space so as to locate more diverse solutions as well as to get rid of getting trapped into local optima. To achieve this goal, the NBM operator proposed by [9], [10] was used in all of the proposed methods. This operator is a modified version of polynomial mutation, one of the most well-known operators for MOEAs [13]. NBM operator works as follows: at first, the Euclidean distance between each solution and the rest of solutions is calculated. For each solution, the solution itself and its K nearest neighbors (where K is the neighborhood size) locate into a cluster, and then the polynomial mutation operator is applied to each of the clusters. This operator increases the probability of mutation for those solutions that are located in denser areas, helping to explore a higher range of the search space and escape the trap of local optima.

IV. Experimental Preliminaries

In this study, the subsequent parameter settings are considered by all of the proposed algorithms as well as by the state-of-the-art algorithms used for comparison (NSGA-II, MO-Ring-PSO-SCD, NSGA-II-CDdec and NSGA-II-WSCD).

Except for MO-Ring-PSO-SCD, Simulated Binary Crossover (SBX) and Neighborhood Polynomial Mutation (NBM) [10] were used as recombination and mutation operators, respectively. However, for NSGA-II, the common polynomial mutation operator is considered. The probability of doing crossover and mutation are respectively set to $p_c = 1$ and $p_m = 1/D$, being D the number of decision variables. The distribution indexes for these operators are $\eta_c = 20$ and $\eta_m = 20$. The neighborhood size for the NBM operator is set to 20. For all the competitors, the population size was set to 100, and the termination criterion consisted on 10000 evaluations.

The weight factor in NSGA-II-WDCD follows the original literature, so it has the same value for both the decision and the objective spaces. The parameter settings for MO-Ring-PSO-SCD is taken from the original publication [7], that are respectively set to $C_1 = C_2 = 2.05$ and $W = 0.7298$.

Furthermore, to set the proper grid size, some experiments were performed for all of the proposed algorithms on every test problem with different grid sizes considering 1, 5, 10, 15, 20, 25, 30, 35 or 40 grids on each dimension. The obtained results showed that increasing the grid size from 1 to 20 improves the performance of all the proposed algorithms. After grid size 20, the curve of improvement keeps its steady state with a bit of fluctuation, due to the stochastic nature of evolutionary algorithms. Therefore, we considered a grid size of 20 for all the experiments.

We used the Matlab-based platform PlatEMO [14] to implement our proposed algorithms and state-of-the-art algorithms.

The test problems used in these experimental studies are taken from the CEC2019 competition on Multimodal Multi-objective optimization (denoted as MMF1z, MMF1-MMF9) [15], [16]. The considered test problems are bi-dimensional, both in terms of decision variables and objective functions. The different characteristics of these test problems are presented in Table I.

To challenge the functionality of the proposed algorithms, the difficulty level of the test problems varies based on the number of PSs, the shape of the optimal solutions (e.g., symmetry), and the overlap of the PSs on each dimension. For example, MMF1z is considered a complex test problem with two non-symmetric PSs, so the complexity for searching each of these PS is different (MMF1z could be similar to a real-world problem).

TABLE I: Properties of the test problems from CEC2019 competition on Multimodal Multi-objective Optimization

Test Problem	Number of PSs	PS Geometry	PF Geometry
MMF1	2	non-linear & symmetric	concave
MMF1z	2	non-linear & non-symmetric	concave
MMF3	2	non-linear & symmetric	concave
MMF3	2	non-linear & symmetric	concave
MMF4	4	non-linear & symmetric	concave
MMF5	4	non-linear & symmetric	convex
MMF6	4	non-linear & symmetric	convex
MMF7	2	non-linear & symmetric	convex
MMF8	4	non-linear & symmetric	convex
MMF9	2	linear & symmetric	convex

To assess the performance of the competitive algorithms over multiple runs, we use three different performance indicators: Inverted Generational Distance in decision space (IGDx), Pareto Set Proximity (PSP), and Inverted Generational Distance in objective space (IGD). The first two are adopted to measure the effectiveness of the proposed algorithms over the rest based on the diversity and convergence of solutions in decision space. Whereas IGD indicator is employed to reflect the quality of the obtained optimal solutions according to the diversity and convergence in objective space.

IGDx is computed as the average Euclidean distance between the obtained solutions and the PS. This metric provides a comprehensive quantification of both diversity and convergence of the obtained optimal solutions in decision space. Let P^* be a sample of the PS of the problem, and R the set of solutions (their values in decision space) obtained by an algorithm, then the IGDx indicator can be defined as:

$$IGDx(P^*, R) = \frac{\sum_{v \in P^*} \|R - v\|_2}{|P^*|} \tag{4}$$

where $\|R - v\|_2$ is the minimum Euclidean distance between the sampled point v and any point in R. Lower IGDx values are preferable.

PSP measures the convergence and the degree of similarity between the obtained optimal solutions and the PS. This indicator is formulated as:

$$PSP = CR/IGDx \tag{5}$$

where CR is the maximum spread of the obtained solutions in decision space. Higher PSP values are preferred to lower ones.

IGD is calculated with the same formula as IGDx but in objective space. This metric indicator measures both the diversity and convergence of the obtained optimal PF. All of these performance indicators require a reference sample of solutions in decision and objective space, which is uniformly distributed over the PS and the PF.

V. Results and Discussions

In this section, we compared our proposed algorithms: NSGA-II-Gr, NSGA-II-Gr-CDdec, NSGA-II-Gr-CDobj and NSGA-II-Gr-CDws, with other MMOEAs: NSGA-II-WSCD, MO-Ring-PSO-SCD and NSGA-II-CDdec algorithms, as well as NSGA-II (as the baseline algorithm).

The Mann-Whitney-U statistical test is employed to determine whether there is a statistical significance difference between the best-performed algorithm from the others on each test problem, that is, $h = 1$ demonstrates a rejection of the null hypothesis, and $h = 0$ demonstrates a failure to reject the null hypothesis at the 5% significance level.

The experimental results for each algorithm on each test problem were obtained out of 31 independent runs. The median value and interquartile range (IQR) for the corresponding IGDx, IGD, and PSP performance indicators are provided in Table II. For each test problem, the outperforming results are shown in bold. An asterisk is used to represent if the algorithm with the best performance is significantly better than the other competitors.

As can be seen from the analysis of the results in Table II, in all the cases the combination of the Grid-based approach with crowding distance in both decision (NSGA-II-Gr-CDdec) and objective (NSGA-II-Gr-CDobj) spaces leads to an improvement in the quality of the obtained optimal solutions in decision space. According to these results, there is no statistical difference between these two algorithms. It is interesting to observe how both of the proposed algorithms performed the best in every test problem over the rest of MMOEAs. As can be seen from the pairwise comparison between NSGA-II-Gr-CDdec and NSGA-II-CDdec, the first one preformed better than the second one in all of the test problems in terms of IGDx and PSP, demonstrating the role of Gr_{dec} approach to improve the distribution of solutions in decision space.

Furthermore, the obtained IGDx and PSP value of NSGA-II-Gr outperformed the ones obtained by MO-Ring-PSO-SCD (as an state-of-the-art algorithm) on approximating the PS for all of the test problems. The results of PSP values also prove the superiority of the two proposed NSGA-II-Gr-CDdec and NSGA-II-Gr-CDws in every test problem on representing both the diversity and convergence of the obtained optimal solutions in the decision space.

It is known for multimodal algorithms that manipulating the diversity of solutions in decision space may leads to deteriorating the diversity in objective space. The goal of our proposed method is to made a trade-off between these two conflicting objectives in two spaces. As can be seen in Table II, in general, the improvement of the IGDX value is much higher than the deterioration of the IGD value for the proposed methods. As an example, the improvement of IGDX value for the MMF5 and MMF8 test problems on NSGA-II-Gr-CDws over NSGA-II-CDws (as an state-of-the-art algorithm) is about 1.2e-02 and 1.05e-02. On the other hand, the IGD value is worsened by 8.36e-04 and 2.97e-04. These results proved that the proposed algorithms were successful in making

TABLE II: Median and IQR for the IGDX, PSP and IGD values of the compared algorithms in different test problems. An asterisk (*) indicates statistical significance compared to the respective best performing algorithm

		NSGA-II-Gr	NSGA-II-Gr-CDDec	NSGA-II-Gr-CDobj	NSGAII-Gr-CDws	NSGA-II-CDDec	NSGA-II-WSCD	Mo-Ring-PSO-SCD	NSGA-II
MMF1	IGDx	0.067124(0.008875)*	0.061202(0.003417)	0.062595(0.002853)*	**0.060603**(0.003125)	0.061958(0.002577)*	0.062318(0.002726)*	0.073865(0.005652)*	0.11003(0.02105)*
	PSP	14.8073(1.9046)*	16.2504(0.87617)	15.8527(0.72453)*	**16.4489**(0.78003)	15.9796(0.68082)*	15.9255(0.73988)*	13.2792(0.98137)*	8.8227(1.6945)*
	IGD	0.007599(0.00073)*	0.006187(0.000424)*	**0.005055**(0.000386)	0.00584(0.000639)*	0.005754(0.000336)*	0.005418(0.00055)*	0.006533(0.000504)*	0.005331(0.000233)*
MMF1z	IGDx	0.052395(0.003493)*	**0.044782**(0.00213)	0.047838(0.002698)*	0.045687(0.002583)	0.045896(0.001208)*	0.046543(0.002298)*	0.055018(0.004518)*	0.12034(0.025452)*
	PSP	19.0266(1.2466)*	**22.2021**(1.1178)	20.8208(1.3242)*	21.8624(1.3449)	21.6625(0.5915)*	21.3178(1.1579)*	17.9275(1.5125)*	8.0487(1.837)*
	IGD	0.007038(0.000836)*	0.005919(0.000341)*	**0.004838**(0.000185)	0.005703(0.000548)*	0.005782(0.000281)*	0.005298(0.000462)*	0.006554(0.00057)*	0.005145(0.000222)*
MMF2	IGDx	0.018418(0.004607)	0.018879(0.003991)	0.018996(0.004189)	**0.017765**(0.003369)	0.019397(0.004749)	0.018523(0.003559)	0.031923(0.012669)*	0.10784(0.087009)*
	PSP	54.1538(14.3124)	52.9243(13.3026)	52.6005(10.6054)	**55.1813**(9.7281)	51.541(12.4749)	53.9273(9.5183)	29.1782(10.7133)*	7.9555(6.7582)*
	IGD	**0.014274**(0.003097)	0.015225(0.00409)	0.015057(0.003806)	0.015032(0.002328)	0.014505(0.002001)	0.014852(0.002164)	0.019609(0.006034)*	0.020704(0.019092)*
MMF3	IGDx	0.015085(0.002805)	**0.014636**(0.001038)	0.016876(0.00313)*	0.014908(0.002532)	0.015515(0.00159)*	0.015936(0.002998)*	0.024832(0.008584)*	0.066651(0.031745)*
	PSP	66.0964(11.5562)	**67.9741**(4.8889)	59.2371(11.6847)*	66.7617(10.7485)	64.4272(6.5669)*	62.7239(11.1474)*	38.3825(12.4391)*	14.5135(7.267)*
	IGD	0.011869(0.002062)	0.011685(0.002171)	0.011666(0.002669)	0.012051(0.002231)	**0.011566**(0.001405)	0.012024(0.002053)	0.015091(0.003205)*	0.014244(0.006389)*
MMF4	IGDx	0.045234(0.003421)*	0.038038(0.002652)	0.038727(0.002337)	**0.037983**(0.002413)	0.039865(0.00547)*	0.041913(0.003713)*	0.04535(0.003056)*	0.10256(0.047135)*
	PSP	21.8348(1.7856)*	26.1288(1.9195)	25.6534(1.579)	**26.1896**(1.6784)	24.941(3.2196)*	23.7223(2.1679)*	21.5821(1.454)*	9.5321(4.0232)*
	IGD	0.008182(0.00144)*	0.006185(0.00058)*	**0.005068**(0.000225)	0.00619(0.000593)*	0.005954(0.000284)*	0.00545(0.00056)*	0.006793(0.000809)*	0.005221(0.000259)*
MMF5	IGDx	0.11035(0.005561)*	0.1034(0.003865)	0.10417(0.006102)	**0.10222**(0.005236)	0.11292(0.005694)*	0.11425(0.010062)*	0.12693(0.011658)*	0.20553(0.048088)*
	PSP	9.0458(0.46834)*	9.6169(0.33912)	9.5596(0.57918)	**9.7424**(0.51451)	8.8265(0.42334)*	8.7221(0.69202)*	7.7577(0.71897)*	4.7613(1.1202)*
	IGD	0.007493(0.000737)*	0.006361(0.000506)*	**0.005031**(0.000332)	0.006111(0.000515)*	0.005691(0.000347)*	0.005275(0.000651)	0.006285(0.000481)*	0.00537(0.000226)*
MMF6	IGDx	0.095281(0.008412)*	**0.089501**(0.00341)	0.091343(0.002659)	0.091507(0.003796)	0.098542(0.006039)*	0.09933(0.007392)*	0.10804(0.011067)*	0.1892(0.073865)*
	PSP	10.467(0.91076)*	**11.0811**(0.46192)	10.9065(0.33621)	10.8866(0.48719)	9.9693(0.62399)*	10.0447(0.69786)*	9.1072(1.0346)*	5.1003(2.3573)*
	IGD	0.007036(0.001198)*	0.006074(0.000735)*	**0.004996**(0.000283)	0.005967(0.00066)*	0.005702(0.000356)*	0.005445(0.000493)*	0.006374(0.000755)*	0.005198(0.000334)*
MMF7	IGDx	0.044455(0.003232)*	0.038889(0.002492)*	**0.036447**(0.002983)	0.03801(0.002741)*	0.03704(0.002444)	0.036946(0.002157)	0.043546(0.003618)*	0.067719(0.021536)*
	PSP	22.4605(1.4481)*	25.351(1.9344)*	**27.1832**(2.1236)	26.1666(1.7906)*	26.7337(1.678)	26.7984(1.5704)	22.674(2.0256)*	14.1855(4.2449)*
	IGD	0.008725(0.00095)*	0.007058(0.00083)*	0.005098(0.000304)	0.006122(0.000588)*	0.006702(0.000767)*	0.005271(0.000542)*	0.007676(0.001388)*	**0.005016**(0.000264)
MMF8	IGDx	0.088645(0.005978)*	0.0791(0.006105)	0.088348(0.007404)*	**0.076404**(0.004284)	0.083033(0.010748)*	0.086917(0.007588)*	0.10699(0.011313)*	0.8061(0.63768)*
	PSP	11.1843(0.69393)*	12.576(0.97508)	11.2055(0.9299)*	**12.9898**(0.6339)	12.0055(1.6526)*	11.4086(0.88867)*	9.2193(1.0427)*	0.99354(0.51834)*
	IGD	0.007482(0.000784)*	0.006989(0.000668)*	**0.005144**(0.000178)	0.00552(0.000312)*	0.006857(0.00046)*	0.005223(0.000338)*	0.007582(0.000663)*	0.005289(0.000292)*
MMF9	IGDx	0.012333(0.000966)*	**0.010691**(0.000806)	0.011113(0.001116)*	0.010808(0.000977)	0.011338(0.001384)*	0.013275(0.001564)*	0.013383(0.002343)*	0.24982(0.20045)*
	PSP	81.0639(6.131)*	**93.4914**(6.7324)	89.9869(9.4493)	92.5107(8.4157)	87.9327(10.8428)*	75.2867(9.1232)*	74.2549(12.4973)*	0.71175(19.3066)*
	IGD	0.033322(0.005984)*	0.027648(0.002816)*	0.019675(0.00109)*	0.01997(0.001073)*	0.02757(0.003082)*	**0.019284**(0.000882)	0.028642(0.003486)*	0.020262(0.001267)*

a reasonable trade-off between the diversity of decision and objective spaces.

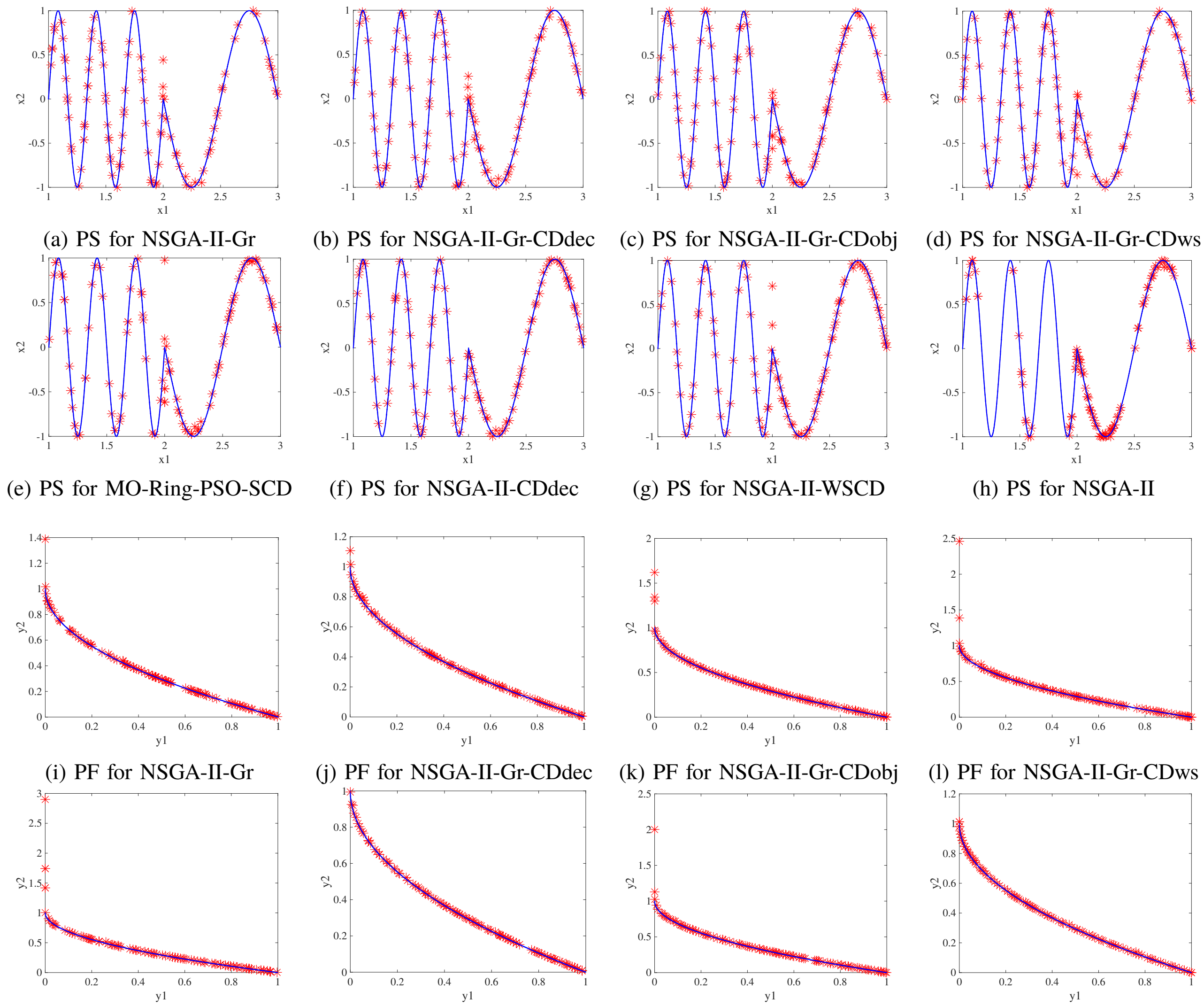
A closer inspection of Table II shows that for the most complex test problem (i.e. MMF1z with the asymmetrical shape of the PS, that is similar to a real-world problem), the grid-based approach contributes in the distribution of solutions in the decision space.

To further evaluate the performance of the proposed algorithms on the approximation of the PF, IGD values were studied. These results, as shown in Table II, indicate that NSGA-II-Gr-CDobj algorithm significantly outperforms the original NSGA-II algorithm in nine out of ten test cases. The obvious reason for this observation is that the focus of this algorithm is on providing better-distributed solutions in objective space while ignoring the decision space. On the other hand, the NBM operator enhances the exploration of the search space, leading to a better locating of Pareto-optimal solutions during the search process.

As a general notice, it can be seen that the obtained IGDx and PSP values for both NSGA-II-Gr-CDdec and NSGA-II-Gr-CDws performed better than the previous algorithms not considering the Gr_{dec} approach, while not deteriorating that much the obtained IGD values.

The overall reason for the better performance of both of the proposed NSGA-II-Gr-CDdec and NSGA-II-Gr-CDws over the rest of compared MMOEAs is that for the early generations where solutions may be randomly distributed over the search space, Grid-based crowding distance value in decision space is equal to zero for some cases, and after applying the normalization it is turned into a non-zero value. Afterwards, by diving the obtained values with the crowing distance in decision space, or weighted sum of crowding distance in both of the spaces, the solutions are mostly selected according to their crowding distance values, that compensates the absence in the obtained grid-based crowding distance values. In the later generations, the solutions converge toward the PS. In these generations, the solutions are closer to each other, therefore, they have different Grid-based crowding distance values. So the combination with the crowding distance outperforms this distance by taking into account a broader neighborhood of solutions compared to the two-nearest-neighborhood considered by crowding distance. This combination leads to the selection of more diverse solutions along the PS, as it increases the selection pressure on decision space compared to NSGA-II-CDdec.

When Gr_{dec} is not considered, as in NSGA-II-CDdec, the later generations only considering the crowding distance in decision space tend to have a lower selection pressure on



(a) PS for NSGA-II-Gr (b) PS for NSGA-II-Gr-CDdec (c) PS for NSGA-II-Gr-CDobj (d) PS for NSGA-II-Gr-CDws

(e) PS for MO-Ring-PSO-SCD (f) PS for NSGA-II-CDdec (g) PS for NSGA-II-WSCD (h) PS for NSGA-II

(i) PF for NSGA-II-Gr (j) PF for NSGA-II-Gr-CDdec (k) PF for NSGA-II-Gr-CDobj (l) PF for NSGA-II-Gr-CDws

(m) PF for MO-Ring-PSO-SCD (n) PF for NSGA-II-CDdec (o) PF for NSGA-II-WSCD (p) PF for NSGA-II

Fig. 2: Obtained solutions in both decision and objective spaces for MMF1z test problem

decision space, because it focus on the two nearest neighbors of the solutions, while Gr_{dec} takes into account a broader neighborhood of solutions. In order to make a better understating of the similarities between the obtained solutions in both decision and objective space, some results are graphically represented for the median run of the IGDx performance indicator in Figure 2. The obtained solutions for all the competitors are shown in both decision and objective spaces. The solid blue lines represent the true PS and PF, while the obtains solutions are represented with red markers. As can be seen from Figure 2, for instance, the obtained solutions in both NSGA-II-Gr-CDdec (Figure 2b) and NSGA-II-Gr-CDws (Figure 2d) are more evenly distributed over the PS and cover a bigger area for the two PSs of the problem, which are located in the ranges $x_1 \in [1, 2]$ (left side) and $x_1 \in [2, 3]$ (right side), respectively. However, in NSGA-II-WSCD (Figure 2g), as an state-of-the-art algorithm, the solutions are unevenly distributed over the PS, i.e. the density of the optimal obtained solutions is higher on the right side rather than on the left side.

Moreover, for NSGA-II (Figure 2h), the optimal solutions have covered more points on the right side of the PS and these points are located close to each other. In this specific problem, because of the more straightforward shape of the right side, both NSGA-II and NSGA-II-WSCD, could find more Pareto-optimal solutions mainly in this area. Whereas, as it is visible, the proposed Grid-based methods can cover more points in both of the PSs and the solutions are more uniformly distributed along both of them.

VI. Conclusion and future works

In this work, we introduced a novel Grid-based crowding distance method for the decision space to solve MMOPs. By combining this method with different crowding distance approaches in both decision and objective spaces, we designed four algorithms: NSGA-II-Gr, NSGA-II-Gr-CDdec, NSGA-II-Gr-CDobj and NSGA-II-Gr-CDws. We tested our proposed algorithms on 10 different test problems, and compared them against several MMOEAs and NSGA-II. Through the experimental assessments, we demonstrated the significant improvement of our proposed algorithms (for the approximation of PS in terms of diversity and convergence) in comparison with the other competitors on many of the test problems.

As a future research topic, an analysis of the Gr$_{dec}$ method on test functions of higher dimensions in both decision and objective spaces, as well as on real world problems, will be considered. Moreover, our work might be involved with the further investigation on the adaptive tuning of the grid size parameter for each generation, which could be applicable for different dimensions in both decision and objective spaces.

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