A New Method for Generating and Indexing Reference Points in Many Objective Optimisation

Kai Eivind Wu
Department of ACSE
University of Sheffield
Sheffield, UK
kewu1@sheffield.ac.uk

George Panoutsos
Department of ACSE
University of Sheffield
Sheffield, UK
g.panoutsos@sheffield.ac.uk

***Abstract*— A new method for the creation of reference points is proposed, for use in reference point-based Many objective Optimisation Problems. Unlike the current common practice of generating reference points on a unit simplex plane by Dan and Dennis method, which leads to uneven distribution of reference points on actual Pareto Front (PF) that the generated reference points are projected onto, the reference points are more evenly generated on an m dimensional B-norm surface which is created adaptively by tracking the true PF. The method is thus named as B-norm based PF tracking method (Bn-PFt). To ease the work of algorithmic development, an indexing system of reference points is also proposed. Systematic numerical studies performed on B-norm surfaces of various B values show that reference points created by the proposed Bn-PFt method are more evenly distributed on the surfaces than those of the projected onto the same surfaces generated using Dan and Dennis method. Simulation results using NSGA III on benchmark problems MaF3 and MaF5 with 3, 5, and 7 objective functions, show that the candidate solution sets obtained, via adopting reference points with the proposed method, have better diversity.**

Keywords—Many Objective Optimisation Problem (MaOP), Pareto-optimal solutions, Unit Simplex Plane, Reference Points, indexing of reference points

I. Introduction

Solving a multi-objective optimisation (MOP) problem is about finding decision variables of objective functions at which Trade-offs among the objective functions are attained, which are often referred to as Pareto optimal solutions, where none of the objective functions can be improved in value without worsening one or more of the other objective functions [1]. The methodology can be used in analysis of all kinds of work processes where the process parameters influencing the final outputs are evaluated and optimised in terms of finding trade-offs of the conflicting objective functions. Big savings can be obtained on operations of various kinds, based on recommendations reached through such analysis. The vector set of objective functions together, corresponding to all Pareto optimal solutions, make the Pareto Optimal Front (PF). Different Pareto optimal solutions represent different possibilities for the choice of a Decision Maker (DM) in decision making in an operation. Without further choice made by a human DM, all Pareto optimal solutions are equally good since they represent different operational options [1].

When the number of objective functions to be optimised simultaneously exceeds three, the optimisation problem is considered as Many Objective Optimisation (MaOP) problem [1]. The approximated sets to PF of a MOP or a MaOP solution shall ideally be fully converged to a PF, uniformly distributed along the surface of PF and spread to all peripheries of the front. Uniformity and spread properties of the PF is often referred to as its diversity property. MOP problems have been successfully solved based on Pareto-based or dominance based strategy, where non-dominated solutions are chosen as parents to generate new children candidates in further iterations in evolutionary algorithms. This is often combined with the use of a secondary diversity-related strategy, such as crowding distance-based diversity maintenance, to distribute and to spread the solutions to the whole of the PF. Non-dominated solutions which are more distant apart to their neighbours are preferred to be selected in the choice of candidate solutions for next iteration process. This strategy is frequently used in many evolutionary multi-objective optimisation algorithms (EMO), such as NSGA-II [2], and SPEA2 [3]. However, this strategy is less efficient on MaOP problems due to the fact that the number of non-dominated solutions increases dramatically as the number of objectives increases. Hence the number of non-dominated solutions overflows the pre-set storage space of the archive, which makes the algorithm unable to efficiently identify new candidates for further iterations.

To deal with the challenge, several other solution strategies have been implemented in the literature, often in combinations, in solving MaOP problems, such as, dominance enhancement-based, grid-based, knee point-based, DM's preference-based, indicator-based, two archive-based, objective function space reduction-based, decomposition-based, reference vector-based or reference points-based approaches [4].

Algorithms as ε-dominance [5], [6], L-optimality [7], and fuzzy dominance [8] are some representatives of dominance enhancement-based methods. Algorithm as Grid-based Evolutionary Algorithm [9] is a typical grid-based one. Knee Point-driven Evolutionary Algorithm [10] is a knee points based method. Preference-Inspired Coevolutionary Algorithms [11] is the most frequently used algorithm of DM's preference based method. [4]. When indicator-based methodology is concerned, Indicator Based Evolutionary Algorithm [12], S-metric selection based evolutionary multi-objective algorithm [13], and the fast hypervolume based evolutionary algorithm [14], can be

978-1-7281-6929-3/20/$31.00 ©2020 IEEE

mentioned. Two-Archive Algorithm [15] and Two-Archive Algorithm II [16] use two separate archives in MaOP algorithm, one for convergence criterion and one for diversity criterion. In algorithms based on objective function space reduction, number of objective functions is reduced where less important objective functions are removed from the optimisation process, based on eigen value analysis or correlation analysis of objective functions [17].

Decomposition-based algorithms transform MaOP into single objective functions by using aggregation function with the help of the weight vectors, which are optimised simultaneously. Multiobjective Evolutionary Algorithm based on Decomposition (MOEA/D) [18] is the first algorithm of the kind. Pareto Sampling (MSOPS) [19], later improved version MSOPS-II [20], and MOEA/D-M2M [21], MOEA/DD [22] are some of the representative algorithms of the category. The weight vectors are established with the aid of reference points.

In reference vector-based or reference points-based algorithms, the objective space is covered by a given number of reference vectors or reference points which guide the iteration process towards the final goal, both in terms of diversity and convergence. Non-dominate Sorting Genetic Algorithm III (NSGA-III) [23], Reference Vector-guided Evolutionary Algorithm [24] are typical algorithms of the kind [25], [26]. Reference vectors or points are formed a priori to algorithm start up.

Li et al. [4] have performed a comprehensive and systematic comparable study among 13 MaOP algorithms, formed based on different strategies of MaOPs, by using them to analyse three major groups of test functions. The conclusion is that none of the algorithms outperforms the others on all types of problems. However, decomposition-based and reference vector/reference points-based algorithms of the approaches are competitive on a large number of test problems. Methodology of generating equidistant reference points is essential in these MaOP algorithms.

Reference vectors are also frequently used as basic tools in forming pure diversity indicators in MaOP research, where the included angles between candidate solutions and reference vectors are adopted as indicative measure to diversity [2], [26].

Ideally, reference points should be created equidistantly on the searched true PF, but this is not possible since PF is not known a priori. A feasible way out of the dilemma is to create reference points adaptively as the search for the true PF progresses.

Reference points are most commonly created by using Das and Dennis's [27] boundary intersection approach, where uniformly spaced reference points are generated on a unit simplex plane. Since the reference points generated in this way are equidistant on the unit simplex plane only, it is most efficient for the use on searching for PFs which have unit simplex plane as Pareto Front or at best in its vicinity. It is common to vector-wise project the obtained reference points to PF surfaces of other form [28]. The major drawback of the practice is that the formed reference points are not equidistant on the surface these are projected onto. Depending on whether the surface of PF is in convex or concave form, the distributions of the points vary differently. With larger distances in mid-area in PF of concave type and decreases towards the peripheral. For PF of convex shape, on the contrary, the reference points are closer to each other in the mid-area and are more distant apart towards the edges.

The proposed research work introduces a new method for the creation of reference points. The proposed methodology aims to generate more evenly distributed reference points on the surface in the vicinity of the true PF. It is shown that the reference point creation method of Dan and Dennis is a special case of the proposed method – hence the proposed is the general case. The main contributions of this paper to the field of study are:

- Establishment of an indexing system for reference points so that each reference point can be readily identified in m dimensional space.
- Introduction of a new creation method for more equally spaced reference points on curved surfaces based on equidistant points along each two dimensional edge of the approximated PF. An initial candidate solution set is first obtained by using reference points created using Dan and Dennis method. The candidate solutions are then used to determine the parameter B in B-norm curve. Equidistant reference points are first generated on a line of edge of approximated PF. The coordinate values of all intermediate points are put to be equal to their respective amounts on the edge based on equal index.
- Establishment of an adaptation procedure for adaptive creation of reference points in MaOP algorithm.

This work is organised as follows. Section II gives an overview of methods of creation of reference points. Section III covers reasoning and formulation of an indexing system for reference points. Section IV provides a detailed description on formulation of the proposed reference point system. Section V is devoted to comparable studies of the proposed reference points with the existing ones, where improvements of the introduced new reference point creation method compared with the existing ones are highlighted. Some discussions about future work of the newly introduced system of reference points can be found in Section VI. Finally, conclusions are drawn in Section VII.

II. EXISTING CREATION METHODS OF REFERENCE VECTORS

The dominating method for generating reference points is Dan and Dennis method [27]. The reference points generated are widely used both as a tool in measuring diversity and as guidance for candidate selection for next iteration in multi-objective evolutionary algorithms [18], [23].

The method of Dan and Dennis creates uniformly spaced reference points on a normalised hyper-plane – a (m − 1)-dimensional unit simplex plane to all objective axes which have an intercept of one on each axis. Reference points are generated as: [27]

$$\lambda^i = (\lambda_1^i, \lambda_2^i, \dots, \lambda_m^i) \tag{1}$$

in which

$$\lambda_j^i \in \left\{\frac{0}{p}, \frac{1}{p}, \dots, \frac{p}{p}\right\} \ and \ \sum_{j=1}^{m} \lambda_j^i = 1$$

where p is the number of divisions along each objective axis and λ_j^i is normalised coordinate value, where i is the ith reference point and j is the jth objective functions.

The number of reference points generated, H, is given by:

$$\mathrm{H} = \binom{\mathrm{m}+\mathrm{p}-1}{\mathrm{p}} \qquad (2)$$

where p is number of divisions along each axis, and m is number of axes or objective functions.

The number of reference points to be created may be determined by the resolution requirement on PF, which again should be decided by a DM. Normally, this number is put to that of closest to and slightly bigger than the number of candidate solutions [23].

The reference points on the unit simplex are equidistant. But when they are projected to surfaces of an arbitrary form of a true PF, the distance between the points are not equal. See Fig. 2 for an illustration. As any of the equidistant points on the simplex plane are projected onto a convex curve along the line linking the point and the origin, the projected point will be closer to the center than it is a the point at the edge, resulting in unevenly distributed points along the convex curve. This means that the reference points used in a MaOP algorithm to start with are not uniformly distributed. When used to find candidates solutions, the approximation sets cannot be evenly distributed, resulting in a deterioration of uniformity.

The number of reference points increases exponentially with increasing p and m. Already for p=m=8, it generates 6,435 reference points which requires considerable amount of computational power. To address this challenge, Deb and Jain [23] suggest using two layers of reference points with each of smaller value of p, p1 for boundary layer and p2 for inside layer so that the total number of reference lines are dramatically reduced to a manageable level. See an illustrative drawing depicted in Fig. 1. But the consequence of this is further reduction of even distribution of reference points on the unit simplex plane and gaps of non-existing reference points are created between the two layers.

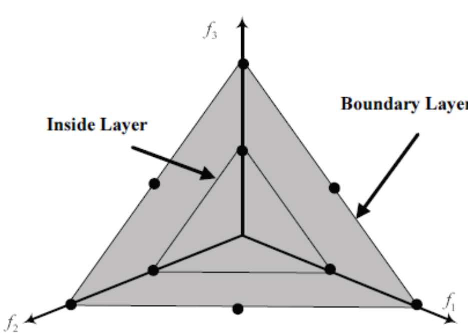


Fig. 1. A schematic view on the principle of Deb and Jain's multi-layer method, where p1=2, p2=1. [18]

He et al. [29] proposed a novel sampling method of generating reference points, by considering the actual shape and location of PF, which is described by equation of B norm. A point $\boldsymbol{F}'=(f_1', f_2', \dots, f_i', \dots, f_m')$ is on a regular surface in m dimensional space if it satisfies Eq. (3).

$$\left(\sum_{i=1}^{m} f_i'^{B}\right)^{1/B} = 1 \qquad (3)$$

in which the value of parameter B determines the shape of the surface:

B < 1, convex surface
B = 1, simplex plane
B > 1, concave surface

Any point $\boldsymbol{F}=(f_1, f_2, \dots, f_i, \dots f_m)$ that is not lying on this surface can be projected to it by using normalisation strategy with B-norm:

$$\boldsymbol{F}' = \left(\frac{f_1}{\|\boldsymbol{F}\|^B}, \frac{f_2}{\|\boldsymbol{F}\|^B}, \dots, \frac{f_m}{\|F\|^B}\right) \qquad (4)$$

in which

$$\|\boldsymbol{F}\|_B = \left(\sum_{i=1}^{m} |f_1|^B\right)^{1/B} \qquad (5)$$

Fig. 2 shows the principle of the sampling method. Starting with the centroid of unit simplex plane, reference points are successively sampled through subdivisions. Reference points are sampled both as vertices of subregions and as centroid of the subregions.

When using B-norm of 2.0, the sampling method can generate uniformly distributed reference points on the reference surface of a hypersphere in 1st octant.

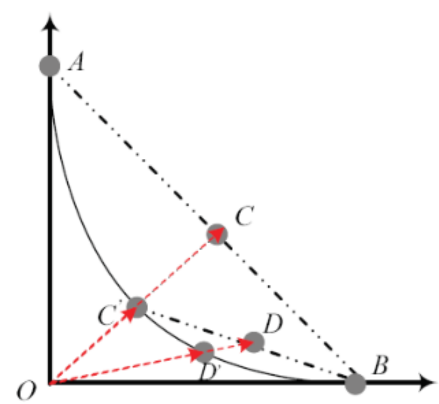


Fig. 2. The principle of sampling method [30], where a point C on unit simplex plane is projected on surface of B norm C' after the first division, and point D' is sampled after further division.

A weakness of the method is that it is unable to create equidistant points on surfaces of B-norm except on surface of B-norm of B=2.0. It fails especially on creation of evenly distributed reference points on surfaces of convex form. See Fig. 2 for further explanation. Point D is created as mid- point of B and C'. As it is projected to D' onto the convex curve BD'C' along OD, the point D' is closer to C' than it is to point B, resulting unevenly distributed points of B, D' and C' along the convex curve of BD'C'. When the parameter B of B-norm surface equals to 2.0, the surface becomes a perfect hyper cube, the method can create equidistant points on the concave curve of B=2.0.

Tian et al. [28] suggest a method of generating reference points on known true PFs starting with the points created by Das and Dennis method on unit simplex plane and projecting them to the actual Pareto fronts. The final distribution of the reference points generated in this way is unfortunately not uniform. As any of equidistant points on unit simplex plane is projected onto a convex curve along the line linking the point and the origin, the projected point will be closer to the center than it is to point of the outskirt, resulting unevenly distributed points along the convex curve. Contrarily, when the point is projected onto a concave curve along the line linking the point and the origin, the projected point will be more far away from the center than it is to a point on the edge, resulting in unevenly distributed points along the concave curve as well.

III. Proposed Indexing System for Reference points

An efficient indexing system to reference points can enhance clarity of formulation, save computer power when searching for a specific reference point and ease work of algorithmic development of new methods. Finding neighbouring points to a given reference point are made easier. A reference point can be easily turned on/off when needed, and new reference points can be generated between existing ones if required. Arbitrary number of reference points can thus be created on boundary layer and on inside layers.

Another feature is that instead of handling coordinate value of reference points in real number, one can identify the reference points with indexes of integers, which can speed up computation and save data storage space.

The description of the indexing system is first done with an example in creating reference points on unit simplex plane. Fig. 3 displays reference points on unit simplex plane of 3 objectives and 5 equal divisions along each objective axis.

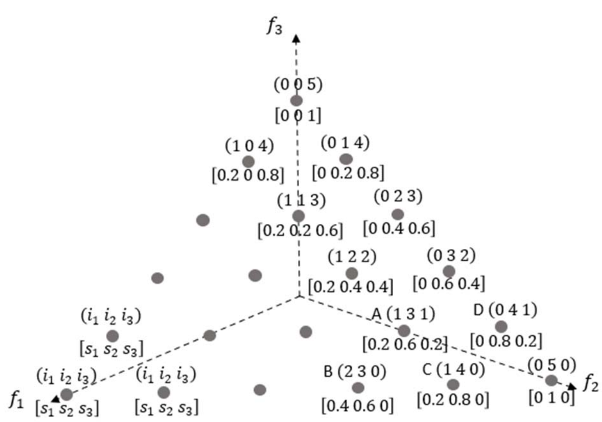


Fig. 3. Indexing of reference points created using Dan and Dennis method. Numbers in parentheses are indexes while numbers in brackets are coordinate values of a reference point.

All reference points in Fig. 3 can be created by:

$$[s_1\ \ s_2\ \ s_3] = \begin{bmatrix} \frac{i_1}{p} & \frac{i_2}{p} & \frac{i_3}{p} \end{bmatrix} \tag{6}$$

where s_1, s_2, and s_3 are coordinate values of a reference point. $(i_1\ i_2\ i_3)$ are their indexes along f_1, f_2, and f_3 axis respectively. We have:

$$s_1 + s_2 + s_3 = 1 \tag{7}$$

$$i_1 + i_2 + i_3 = p \tag{8}$$

in which p is number of divisions on each objective axis.

All reference points are readily found by systematically varying $i_j, j \in (1, 2, \ldots, m)$ under the assumption that Eq. 8 is satisfied. Another interesting feature is that an index and its corresponding coordinate value are directly linked, meaning that for two arbitrary reference points, that indexes from each of the two are equal, results to the same coordinate value. For example, see Fig. 3, for points A, B, C and D:

$$\begin{gathered} i_1^A = i_1^C = i_3^D = 1 \rightarrow f_1^A = f_1^C = f_3^D = 0.2 \\ i_2^A = i_2^B = 3 \rightarrow f_2^A = f_2^B = 0.6 \\ i_3^A = i_3^D = i_1^C = 1 \rightarrow f_3^A = f_3^D = f_1^C = 0.2 \end{gathered} \tag{9}$$

Eq. 9 indicates that coordinates are the same for points that have the same index. It can be interpreted as that coordinates of an arbitrary point can be found by comparing its indexes with those of points along a line of edge (2D hyper-line) and are put equal to the coordinate values of same index.

In general, the coordinates of a reference point in m objective space are given as:

$$(s_1\ \ s_2\ \ \ldots\ s_m) = \left(\frac{i_1}{p}\ \ \frac{i_2}{p}\ \ \ldots\ \ \frac{i_m}{p}\right) \tag{10}$$

and

$$s_1 + s_2 + \cdots + s_m = 1 \tag{11}$$

$$i_1 + i_2 + \cdots + i_m = p \tag{12}$$

The principle illustrated in Eq. 9 can be expanded for determination of coordinates of an arbitrary reference point in *m* objective function space by equating the value to the corresponding values to a two-dimensional line with same index:

$$i_j = i_1^{k,2D} = k \rightarrow f_j = f_1^{k,2D}, j \in (1, 2, \ldots, m) \tag{13}$$

where m is the number of objective functions. i_j, $j \in (1, \ldots, m)$, is the index of jth component of the searched reference point. f_j, $j \in (1, \ldots, m)$, is the value of jth component of the searched reference point. $i_1^{k,2D}$ and $f_1^{k,2D}$ are the index and the value of the first coordinate of reference point with $i_1^{k,2D}$ =k along a two-dimensional line of edge on the simplex plane respectively.

Eq. 13 implies that the coordinate values of an arbitrary reference point in m dimensional space can be put equal to corresponding coordinate values of points on a two- dimensional line of edge on the simplex plane, by contrasting indexes of them. This property of Eq. 13 can be utilised to develop new system of reference points. See section IV for details.

IV. Propsed Method for Generating Refernce Points

The proposed method for creation of reference points is done on a (m-1) dimensional B-norm surface which is generated based on curve fitting of an approximation set of PF. To start with, an approximation is created using existing MaOP algorithm and with reference points created by, for instance, Dan and Dennis method. The candidate solution set obtained is then used to determine the parameter B in B-norm curve. Projection of B-norm curve on a 2D plane, for instance $f_1 - f_2$ plane, can be attained, which is an 2D edge or boundary of the estimate of PF. Equidistant reference points are first generated on the edge. The coordinate values of all internal points are put to be equal to their respective amounts on the edge, which have the same index.

Algorithm 1 Procedure for the proposed algorithm **(Bn-PFt)**

Input: X (Solution set)
Output: Z (New refernce line)

1: $X \leftarrow$ Apply non-dominent sorting (X)
2: $X_N \leftarrow$ Normalise the solution (X)
3: $B \leftarrow$ a single optimisation process is carried out to obtain parameter **B** (X_N)
4: $f_N^D \leftarrow$ Division of B-norm edge in plane formed by a pair of objective axis (B)
5: Z $\leftarrow$ Determination of coordinate values of reference points (f_N^D)

The reference point creation method is done on surface of a B-norm, which adaptively trace the true PF, which is thus named as B-norm based PF tracing method (Bn-PFt).

A. *Approximate PF with B-norm equation*

After the first converged solution set has been obtained using reference point generating method of Dan and Dennis, the solution set is curve-fitted to take a form of B-norm equation. The solution set can be used to estimate parameter B as:

$$F(f_1, f_2, \dots, f_m) = (f_1^B + f_2^B + \dots + f_m^B)^{\frac{1}{B}} = 1 \quad (14)$$

$$h(B) = 1 - (f_1^B + f_2^B + \dots + f_m^B)^{\frac{1}{B}} \quad (15)$$

$$B = argmin_B\left(\sum_{k=1}^{N} h_k(B)\right) \quad (16)$$

where N is the number of obtained candidate solutions.

B. *Division of B-norm edge in plane formed by a pair of objective axis*

Due to symmetry, the curve fitted B-norm surface has exactly the same projection on planes formed by any pairwise objective functions. Equidistant reference points need first to be generated on such a projection. See Fig. 4, illustration of a division of p=7 equal parts.

Due to symmetry, only the locations of reference points on one half of the $f_1 \sim f_2$ curve need to be first determined.

1) Midpoint M's coordinates: due to symmetry

$$f_1^* = f_2^* \text{ and } ((f_1^*)^B + (f_2^*)^B)^{\frac{1}{B}} = 1 \quad (17)$$

which leads to $f_1^* = f_2^* = 2^{-\frac{1}{B}}$

2) The arc length l_{aM}:

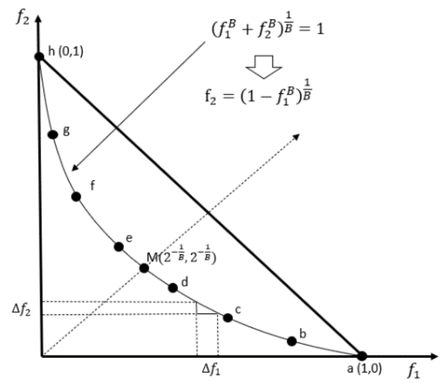


Fig. 4. Determination of part's length for p=7

$$l_{aM} = \sum_{j=1}^{n} \sqrt{(\Delta f_1)^2 + (\Delta f_2)^2} \quad (18)$$

where n is number of divisions between point f_1^a and that of midpoint $2^{-\frac{1}{B}}$.

$$\Delta f_1 = \frac{1 - 2^{-\frac{1}{B}}}{n}$$

$$f_{1,j} = 2^{-\frac{1}{B}} + \Delta f_1 \cdot j \text{ and } f_{1,0} = 2^{-\frac{1}{B}}, f_{1,n} = 1,$$
$$j \in (0, 1, 2, \dots, n)$$

By Eq. 17:

$$f_{2,j} = \left(1 - (f_{1,j})^B\right)^{\frac{1}{B}}$$
$$\Delta f_2 = f_{2,j} - f_{2,j-1}, \; j \in (0, 1, 2, \dots, n)$$

n determines the accuracy of the calculated arc length l_{aM}, and a large number is chosen since the calculation is done only once per tracking operation on the PF (in this work, it is chosen to be n=1000).

3) The arch length:

$$l_{ab} = 2l_{aM} \quad (19)$$

4) The length per division:

$$l_{pd} = \sum_{j=1}^{n_1} \sqrt{(\Delta f_1)^2 + (\Delta f_2)^2} = \frac{l_{ab}}{p} \quad (20)$$

where n_1 is number of summation points from f_1^b to f_1^a to obtain the length per division l_d. Eq. 20 gives n_1.

5) The coordinate values of point b:

$$f_1^b = f_1^a - n_1 \cdot \Delta f_1 \quad (21)$$

$$f_2^b = \left(1 - (f_1^b)^B\right)^{\frac{1}{B}} \quad (22)$$

6) The coordinate of next point c:

$$\sum_{j=1}^{n_2} \sqrt{(\Delta f_1)^2 + (\Delta f_2)^2} = \frac{l_{ab}}{p} \quad (23)$$

where n_2 is number of summation points from f_1^c to f_1^b.

Eq. 21 and 22 are again used to determine (f_1^c, f_2^c):

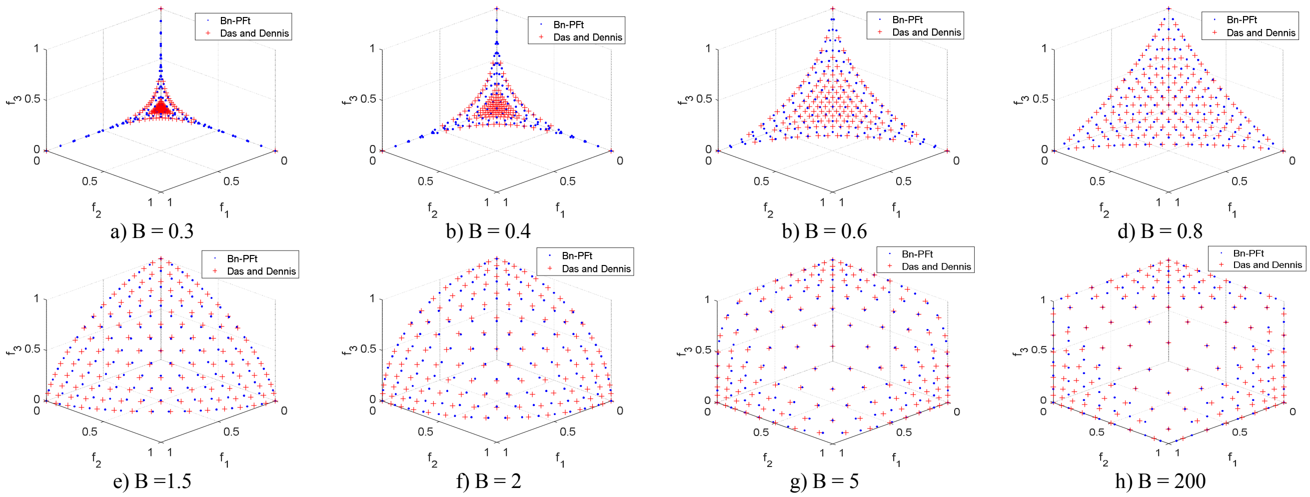


Fig. 5. Comparisons between reference points created by Das and Dennis method and the proposed Bn-PFt method for various B values.

$$f_1^c = f_1^b - n_2 \cdot \Delta f_1 \quad (24)$$

$$f_2^c = (1-(f_1^c)^B)^{\frac{1}{B}} \quad (25)$$

The procedure is repeated until coordinates of all points of divisions are obtained.

C. Determination of coordinate values of reference points

For an arbitrary reference point, $S[f_1\ f_2\ \ ...\ f_m]$ with indexes $(i_1\ \ i_2\ \ ...\ \ i_m)$, its coordinate values can be found directly by picking up corresponding values on 2D boundary of the attained B-norm curve

$$[(f_1^1, f_2^1), (f_1^2, f_2^2), \dots, (f_1^k, f_2^k), \dots\ (f_1^{p+1}, f_2^{p+1})]$$

$$k \in (1, 2, \dots, p+1)$$

which are organized as given by Eq. 13:

$$i_j = i_1^{k,2D} = k \rightarrow f_j = f_1^{k,2D}, j \in (1, 2, \dots, m)$$

V. Numerical Stdies on B-norm Surfaces and on Benchmark Problems with the Aid of Proposed Creation Method of Reference Points

Eq. 4 is used to project reference points generated by the Das and Dennis method on the actual B-norm surface as the common practice up to now, and the location of projected reference points are contrasted with those created by Bn-PFt method.

The first part of testing is done in displaying reference points created by Dan and Dennis and Bn-PFt methods onto surface of B-norm with various B values with 3 objective functions.

In the second part of testing, Non-dominated Sorting Genetic Algorithm III (NSGA III) [23] is used to analyse Benchmark problem of MaF3 and MaF5 [30] with 3, 5 and 7 objective functions, with use of reference point creation of Dan and Dennis method and of Bn-PFt method. A list of parameters used in the study is given in Table I. The outcomes are compared so that the efficiency and efficacy of the proposed creation method of reference points are highlighted.

TABLE I. Parameter used in NSGA III calculations

Benchmark problem	Number of objectives	Number of decision variables	Number of evaluations
MaF 3 MaF 5	3	12	100000
	5	14	200000
	7	16	500000

Hypervolume [31] and Spacing metric [32] are used to check quantitively the evenness of reference points created by Bn-PFt method compared with that of Das and Dennis method. Hypervolume (HV) metric calculates the space enclosed by the candidate solutions and a reference point (Nadir point is often chosen.). HV is a Pareto compliant metric which means that as long as a solution set A dominates a solution set B, it leads to that HV of A will be greater than that of B.

Spacing metric is defined as:

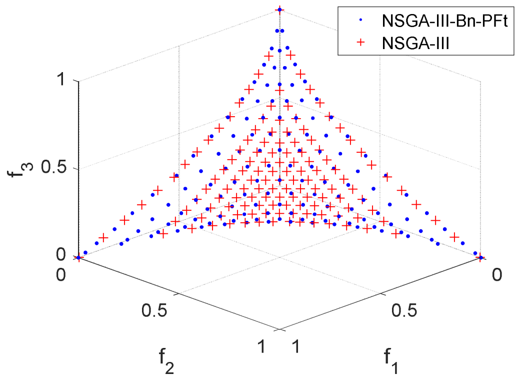
$$SP(S) = \sqrt{\frac{1}{|S|-1}\sum_{i=1}^{|S|}\left(d_i - \bar{d}\right)^2} \quad (26)$$

in which $d_i = min_{(s_i,s_j)\in S, s_i \neq s_j}\left\|F(s_i) - F(s_j)\right\|_1$, which is the $l_1$distance between point $s_i \in S$ and its closest point in S other than s_i, and $\bar{d}$ is the mean value of d_i.

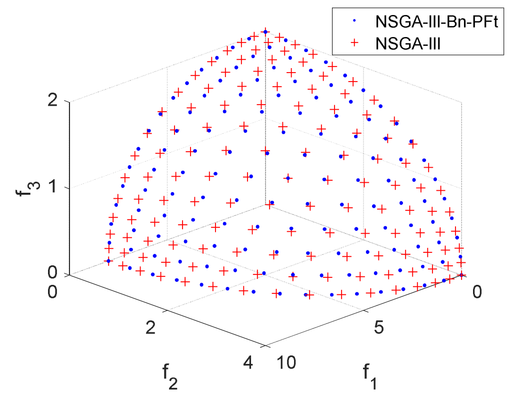
Spacing metric is expressed in averaged sum of variance of distance between a point and its closest neighbor. It is improper to be used to evaluate an approximation set which has holes or strong local clusters in its domain [33]. The solution sets being studied here do not have holes and local clusters in their data sets, which is the reason for why Spacing metric is used in this study. The lower the value of Spacing metric the better the distribution of a solution set.

A. Reference points created on various B-norm surfaces

Fig. 5 shows comparisons between reference points created by Das and Dennis method and the proposed Bn-PFt method for various B values. As can be observed from the figures, reference points generated by Bn-PFt method have more even distributions than those of Das and Dennis. On surfaces of convex form, the reference points of Bn-PFt method spread out and cover the surface while those of Das and Dennis method



a) MaF3 M3



b) MaF5 M3

Fig. 6. Distribution of candidate solution set created using NSGA III and reference points of Das and Dennis method and reference points of Bn-PFt method on Benchmark test of MaF3 and MaF5, p=15, m=3

TABLE II. HV AND SPACING VALUES (MEAN AND SD) OF ANALYSIS ON MAF3 AND MAF5 IN 3, 5 AND 7 OBJECTIVE FUNCTIONS USING NSGA III WITH REFERENCE POINTS GENERATED BY DAN AND DENNIS METHOD AND BY BN-PFT METHOD.

N obj	Benchmark	NSGAIII		NSGAIII-Bn-PFt	
		HV(↑)	Spacing (↓)	HV(↑)	Spacing (↓)
3 objective functions	MaF3	0.96167(0.00045)	2.8404e-2(0.0556e-2)	**0.96545(0.00015)**	**2.7856e-2 (0.1482e-2)**
	MaF5	0.54603(0.06918)	0.17162(0.04474)	**0.57058(0.00006)**	**0.14159(0.00046)**
5 objective functions	MaF3	0.99927(0.00021)	6.4105e-2(0.4386e-2)	**0.99961(0.00006)**	**4.8076e-2 (1.0589e-2)**
	MaF5	0.81241(0.00042)	1.0664(0.0045)	**0.81660(0.00043)**	**0.94334(0.00466)**
7 objective functions	MaF3	0.999982(0.000005)	9.3940e-2(0.062e-2)	**0.999988(0.0000126)**	**8.4815e-2 (0.0548e-2)**
	MaF5	0.87410(0.00037)	6.9930(0.0361)	**0.88266(0.00082)**	**6.3094(0.4461)**

gather more in the mid areas. On surfaces of concave form, the opposite occurs, the reference points of Bn-PFt method gather more densely in the mid area of the surface while those of Das and Dennis spread more towards peripherals.

B. Bn-PFt method applied in Benchmark testing when implemented into NSGA III

Fig. 6 shows typical calculated solution sets of Benchmark problems of MaF3 and MaF5, p=15, m=3, based on NSGA-III algorithm with reference points generated by Dan and Dennis and by Bn-PFt. Table II depicts quantitatively the HV and Spacing values of analysis on MaF3 and MaF5 with 3, 5 and 7 objective functions, and averaged after approximations of 20 times.

As can be seen in Fig. 6 as well as quantitively in Table II, the candidate solution set obtained using reference points of Bn-PFt method has better diversity, as evaluated using HV and Spacing metrics, than that of adopting reference points of using the standard Dan and Dennis method.

VI. DISCUSSION

Reference points created by Bn-PFt method with B=1 (unit simplex plane) have exactly the same location and distribution as those of Dan and Dennis method. In other words, Das and Dennis method is a special case of the proposed Bn-PFt method, which is the general case.

B-norm surface curve used in this study is symmetric in m dimensional objective space which is most suitable to be used to trace PF of approximately symmetric forms. For strong non symmetric PFs, the reference points created by Bn-PFt method will be much out of the surface of the true PF and the evenness of their projections on the true PF is hampered when these are projected on the true PF, although they are still more suitable to be used to guide search for PF than those of Dan and Dennis method due to the fact that they locate much nearer to the true PF than the later ones.

The efficacy of Bn-PFt method shall be further studied by comparisons using more different existing MaOP algorithms and Benchmark functions, and also by exploring its performance in higher dimensions, and with more decision variables. Bn-PFt method using non-symmetric B-norm surface curve will be a subject of study in future work, combined with development of MaOP Benchmarks of non-symmetric PFs.

VII. CONCLUSION

An new indexing system for generating reference points is introduced, for many objective optimisation, which can enhance clarity of formulation, save computing resources when searching for a specific reference point as well as help facilitate work on algorithmic development of new methods.

A new method of generation of reference points is proposed, B-norm based PF tracking method (Bn-PFt). The reference points are more evenly created on an m dimensional B-norm surface which is created adaptively by tracking the true PF. Numerical studies performed on B-norm surfaces of various B values show that reference points created by Bn-PFt method are more evenly distributed on PF than those of the projected onto the same surfaces generated using Dan and Dennis method. Simulation results, using NSGA III to the Benchmark problems of MaF3 and MaF5 with 3, 5, and 7 objective functions, show that the candidate solution sets obtained using reference points of Bn-PFt method have better diversity than those when using the popular Dan and Dennis method. Crucially, the proposed method, generates better results in terms of diversity.

Reference points created using the proposed Bn-PFt method with B=1 (unit simplex plane) have exactly the same location and distribution as those of Dan and Dennis method. In other words, Das and Dennis method is a special case of the proposed Bn-PFt method, which entails the general case.

The efficacy of Bn-PFt method shall be further studied by comparisons using more existing MaOP algorithms and Benchmark functions, and also by exploring its performance in MaOPs of higher dimensions, more decision variables and of more complex PF. Application on non-symmetric B-norm surface curves will also need to be studied in future work.

REFERENCES

[1] A. Zhou, B.-Y. Qu, H. Li, S.-Z. Zhao, P. N. Suganthan, and Q. Zhang, "Multiobjective evolutionary algorithm: A survey of the state of art," *Swarm Evol. Comput.*, vol. 1, no. 1, pp. 32–49, 2011.

[2] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan, "A fast and elitist multiobjective genetic algorithm: NSGA-II," *IEEE Trans. Evol. Comput.*, vol. 6, no. 2, pp. 182–197, 2002.

[3] E. Zitzler, M. Laumanns, and L. Thiele, "SPEA2: Improving the Strength Pareto Evolutionary Algorithm," *TIK-Report*, vol. 103, Jul. 2001.

[4] K. Li, R. Wang, T. Zhang, and H. Ishibuchi, "Evolutionary Many-Objective Optimization: A Comparative Study of the State-of-the-Art," *IEEE Access*, vol. 6, pp. 26194–26214, 2018.

[5] M. Laumanns, L. Thiele, K. Deb, and E. Zitzler, "Combining Convergence and Diversity in Evolutionary Multiobjective Optimization," *Evol. Comput.*, vol. 10, no. 3, pp. 263–282, 2002.

[6] D. Hadka and P. Reed, "Borg: An Auto-Adaptive Many-Objective Evolutionary Computing Framework," *Evol. Comput.*, vol. 21, no. 2, pp. 231–259, Mar. 2012.

[7] X. Zou, Y. Chen, M. Liu, and L. Kang, "A New Evolutionary Algorithm for Solving Many-Objective Optimization Problems," *IEEE Trans. Syst. Man, Cybern. Part B*, vol. 38, no. 5, pp. 1402–1412, 2008.

[8] G. Wang and H. Jiang, "Fuzzy-Dominance and its Application in Evolutionary Many Objective Optimization," in *International Conference on Computational Intelligence and Security Workshops*, 2007, pp. 195–198.

[9] S. Yang, M. Li, X. Liu, and J. Zheng, "A Grid-Based Evolutionary Algorithm for Many-Objective Optimization," *IEEE Trans. Evol. Comput.*, vol. 17, no. 5, pp. 721–736, 2013.

[10] X. Zhang, Y. Tian, Y. Jin, and S. Member, "A Knee Point-Driven Evolutionary Algorithm for Many-Objective Optimization," *IEEE Trans. Evol. Comput.*, vol. 19, no. 6, pp. 761–776, 2015.

[11] R. Wang, R. C. Purshouse, and P. J. Fleming, "Preference-Inspired Coevolutionary Algorithms for Many-Objective Optimization," *IEEE Trans. Evol. Comput.*, vol. 17, no. 4, pp. 474–494, 2013.

[12] E. Zitzler and S. Künzli, "Indicator-Based Selection in Multiobjective Search BT - Parallel Problem Solving from Nature - PPSN VIII," 2004, pp. 832–842.

[13] N. Beume, B. Naujoks, and M. Emmerich, "SMS-EMOA: Multiobjective selection based on dominated hypervolume," *Eur. J. Oper. Res.*, vol. 181, no. 3, pp. 1653–1669, 2007.

[14] J. Bader and E. Zitzler, "HypE: An Algorithm for Fast Hypervolume-Based Many-Objective Optimization," *Evol. Comput.*, vol. 19, no. 1, pp. 45–76, 2011.

[15] K. Praditwong and X. Yao, "A New Multi-objective Evolutionary Optimisation Algorithm: The Two-Archive Algorithm," in *2006 International Conference on Computational Intelligence and Security*, 2006, vol. 1, pp. 286–291.

[16] H. Wang, L. Jiao, and X. Yao, "Two_Arch2: An Improved Two-Archive Algorithm for Many-Objective Optimization," *IEEE Trans. Evol. Comput.*, vol. 19, no. 4, pp. 524–541, 2015.

[17] N. Pholdee, S. Bureerat, and A. Yildiz, "Hybrid real-code population-based incremental learning and differential evolution for many-objective optimisation of an automotive floor-frame," *Int. J. Veh. Des.*, vol. 73, pp. 20–53, Jan. 2017.

[18] Q. Zhang and H. Li, "MOEA/D: A Multiobjective Evolutionary Algorithm Based on Decomposition," *IEEE Trans. Evol. Comput.*, vol. 11, no. 6, pp. 712–731, 2007.

[19] E. J. Hughes, "Multiple single objective Pareto sampling," in *2003 IEEE Congress on Evolutionary Computation*, 2003, vol. 4, pp. 2678-2684 Vol.4.

[20] E. J. Hughes, "MSOPS-II: A general-purpose Many-Objective optimiser," in *2007 IEEE Congress on Evolutionary Computation*, 2007, pp. 3944–3951.

[21] H. Liu, F. Gu, and Q. Zhang, "Decomposition of a Multiobjective Optimization Problem Into a Number of Simple Multiobjective Subproblems," *IEEE Trans. Evol. Comput.*, vol. 18, no. 3, pp. 450–455, 2014.

[22] K. Li, K. Deb, Q. Zhang, and S. Kwong, "An Evolutionary Many-Objective Optimization Algorithm Based on Dominance and Decomposition," *IEEE Trans. Evol. Comput.*, vol. 19, no. 5, pp. 694–716, 2015.

[23] K. Deb and H. Jain, "An Evolutionary Many-Objective Optimization Algorithm Using Reference-Point-Based Nondominated Sorting Approach, Part I: Solving Problems With Box Constraints," *IEEE Trans. Evol. Comput.*, vol. 18, no. 4, pp. 577–601, 2014.

[24] R. Cheng, Y. Jin, M. Olhofer, and B. Sendhoff, "A Reference Vector Guided Evolutionary Algorithm for Many-Objective Optimization," *IEEE Trans. Evol. Comput.*, vol. 20, no. 5, pp. 773–791, 2016.

[25] G. Chiandussi, M. Codegone, S. Ferrero, and F. E. Varesio, "Comparison of multi-objective optimization methodologies for engineering applications," *Comput. Math. with Appl.*, vol. 63, no. 5, pp. 912–942, 2012.

[26] G. O. Odu, "Review of Multi-criteria Optimization Methods – Theory and Applications," vol. 3, no. 10, pp. 1–14, 2013.

[27] J. E. Dennis, "NORMAL-BOUNDARY INTERSECTION : A NEW METHOD FOR GENERATING THE PARETO SURFACE IN NONLINEAR," *Soc. Ind. Appl. Mathmatics*, vol. 8, no. 3, pp. 631–657, 1998.

[28] Y. Tian, X. Xiang, X. Zhang, R. Cheng, and Y. Jin, "Sampling Reference Points on the Pareto Fronts of Benchmark Multi-Objective Optimization Problems," in *2018 IEEE Congress on Evolutionary Computation (CEC)*, 2018, pp. 1–6.

[29] C. He, L. Pan, H. Xu, Y. Tian, and X. Zhang, "An improved reference point sampling method on Pareto optimal front," in *2016 IEEE Congress on Evolutionary Computation (CEC)*, 2016, pp. 5230–5237.

[30] R. Cheng *et al.*, "A benchmark test suite for evolutionary many-objective optimization," *Complex Intell. Syst.*, vol. 3, no. 1, pp. 67–81, 2017.

[31] E. Zitzler and L. Thiele, "Multiobjective evolutionary algorithms: a comparative case study and the strength Pareto approach," *IEEE Trans. Evol. Comput.*, vol. 3, no. 4, pp. 257–271, 1999.

[32] J. R. Schott, "Fault Tolerant Design Using Single and Multicriteria Genetic Algorithm Optimization," Massachusetts Institute of Technology, 1995.

[33] N. Riquelme, C. Von Lücken, and B. Baran, "Performance metrics in multi-objective optimization," in *2015 Latin American Computing Conference (CLEI)*, 2015, pp. 1–11.