Variance of particle location in the stochastic model of PSO with inertia weight

1st Krzysztof Wójcik
Cardinal Stefan Wyszyński University in Warsaw
Warsaw, Poland
ORCID: 0000-0003-2759-8738

2nd Tomasz Kulpa
Cardinal Stefan Wyszyński University in Warsaw
Warsaw, Poland
ORCID: 0000-0002-7847-7955

3rd Krzysztof Trojanowski
Cardinal Stefan Wyszyński University in Warsaw
Warsaw, Poland
ORCID: 0000-0001-9009-049X

***Abstract*—In the Particle Swarm Optimization (PSO) method, the behavior of particles depends on movement parameters. Effective application of the PSO method in real-world problems requires stable behavior of particles in the swarm. In the stochastic particle stability analysis, recurrent formulas of expected value and variance of particle location are used. An explicit formula for the variance can also be obtained, however, it cannot be applied in practice due to its complexity. In our research, we propose assumptions guaranteeing a simple explicit formula for location variance. For the formula and given assumptions, we show stability areas in the particle configuration space, which guarantee order-2* stability of particles also for probability distributions of movement parameters other than uniform. The areas are verified in simulations.**

***Index Terms*—Particle Swarm Optimization, PSO with inertia weight, order-2 stability, stagnation**

I. Introduction

Particle Swarm Optimization belongs to a family of modern heuristic techniques of optimization. It is a population-based stochastic method developed in [1] and originally addressed to the problems of numerical optimization. For the last twenty years, PSO has been successfully applied to solve numerous real-world optimization problems. The growing popularity of the method started to be accompanied by the appearance of alterations proposed to PSO algorithms aimed to improve its performance and control. One of the versions, namely PSO with inertia weight (IPSO), became a subject of particular interest, both in the domain of applications and theoretical analysis of its properties and behavior.

The PSO behavior depends on values assigned to its control parameters. Early works about PSO reported that the system was prone to entering a state of explosion, where incorrect parameter settings caused velocities and thus particle positions to increase rapidly, approaching infinity. The convergent behavior of particles has a significant impact on algorithm performance. For example, in [2], for a wide range of test cases, the authors experimentally showed that swarms with unstable parameter configurations perform worse than random search. Therefore, analysis of swarm and particle convergence, stagnation, and stability conditions having regard to the values of the parameters remains vital.

In the presented research, we focus mainly on particle convergence in the stochastic model of the IPSO particles. Specifically, we study the particle location's variance. Under some reasonable assumptions, we obtain a simple explicit formula for the second moment of particle's location in the stochastic model. Then we directly apply the formula to investigate and visualize stability areas in the particle's configuration space. Finally, we empirically confirm the theoretical findings of this paper in simulations.

The text consists of 7 sections. Section II presents a brief review of selected areas of PSO theoretical analysis concerning stability in the PSO method with inertia weight (IPSO). In Section III, a stochastic model of the particle movement is recalled, and an analysis of the model is conducted. In Section IV, under some assumptions, an explicit formula for particle location's variance is obtained. In section V the formula is used to derive and visualize stability areas with respect to inertia weight and acceleration coefficients. In Section VI, an experiment is set up, and simulations are conducted in reference to the theoretical findings. Section VII summarizes the presented research.

II. Related Work

Particles are the primary elements of a swarm, each representing a certain proposition of a solution, intending to find an optimal solution. Their movement can be generally described by the equation

$$x_i(t+1) = x_i(t) + v_i(t+1), \tag{1}$$

where x_i is the location of the particle, v_i — velocity of the particle.

The movements of a particle depend only on its velocity and so-called attractors. The attractors represent locations where good solutions have already been found by the particle itself or other particles of the swarm. This is an analogy to, e.g., a bird flock, where individuals make their decisions based on cognitive aspects (modeled by the influence of particle attractors) and social aspects (modeled by the influence of the swarm attractor). Each particle keeps track of the coordinates in the search space, which are associated with the best solution it has found so far. Another value that is tracked by each particle is the best value obtained so far by the swarm.

At each iteration, the velocity of each particle is changed towards the two attractors mentioned above: personal and global

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best locations. Also, some random component is incorporated into the velocity update. In the case where the attractors remain unchanged for a sufficiently long time, we talk about the so-called stagnation. The swarm becomes stagnant when none of the particles can find a better position anymore. As a direct result, the swarm is not capable of further optimization.

When a particle no longer deviates from a particular solution, it reaches a so-called stable behavior, and thus we talk about the stability of the particle. When the above situation happens for every particle, then we talk about the stability of the swarm.

Particle velocity is an essential factor to consider in terms of swarm optimization. The velocity sets the size of consecutive steps, thus influences the stability of particles and the swarm. The decrease of the velocity is mandatory for a particle, and thus a swarm, to stabilize. Lack of decrease and especially uncontrolled velocity increase makes the particle unstable, or in some cases, even explode.

The first stability analysis publications dates back to years 2002-2006, when the authors usually did not assume the random behavior of the particles, thus agreeing on the deterministic model. They defined stability as a simple convergence of particle locations $\mathbf{x}_t$ to a certain vector of constants $\mathbf{y}$:

$$\lim_{t\to\infty} \mathbf{x}_t = \mathbf{y}. \tag{2}$$

In 2009, Poli [3] used the expected value of particle locations and presented a new stability definition

$$\lim_{t\to\infty} E[\mathbf{x}_t] = \mathbf{y}, \tag{3}$$

which he named order-1 stability, or first-order stability.

Both Jiang [4] and Poli [3] agreed in their works that the order-1 stability alone is not sufficient to ensure the convergence of the particles' locations. They correctly observed that the variances of particles' locations have to be convergent as well. In 2007 Jiang came up with a following condition:

$$\lim_{t\to\infty} E[\mathbf{x}_t - \mathbf{y}]^2 = 0, \tag{4}$$

where $\mathbf{y} = \lim_{t\to\infty} E[\mathbf{x}_t]$. In 2009 Poli proposed another condition for the convergence of variance:

$$\lim_{t\to\infty} E[\mathbf{x}_t]^2 = \beta_0, \;\; \lim_{t\to\infty} E[\mathbf{x}_t\mathbf{x}_{t-1}]^2 = \beta_1, \tag{5}$$

where β_0, β_1 are vectors of constant values. Poli was also the first to use the terminology of order-2 stability, or second-order stability

The third proposition by Cleghorn, maybe the most intuitive one, was proposed in [5]. It says that the variance of locations of particles should be convergent, that is

$$\lim_{t\to\infty} Var[\mathbf{x}_t] = \beta, \tag{6}$$

where β is a vector of constant values. It is easy to imagine the proposition above — particles converge to a point in space and reach the stability only when they do not stray away from that point too much anymore.

In [6], second-order stability is defined as:

$$\lim_{t\to\infty} Var[\mathbf{x}_t] = \mathbf{0}. \tag{7}$$

In contrary to the previous propositions, this one is much stronger — convergence of the variance to zero forces the particles to entirely seize their oscillating movement.

The first analysis of the variance convergence to zero was carried out in [3]. It was shown that the only way of reaching such state would be to associate the global attractor $\mathbf{y}^*$ with an individually found best solutions vector $\mathbf{y}$, or simply when $\mathbf{y}^* = \mathbf{y}$ (so called "weak stagnation assumption"). More general results for second order stability, including Bare Bones PSO, were obtained in [7]. Another argument showing that the first-order and the second-order stability do not imply the convergence of variance to zero comes from [8]. The term of **order-2* stability** or **second-order stability with a star**, to which we will relate in this paper, was also used in [8].

A. Particle swarm optimization with inertia method

In the IPSO (*Inertia Particle Swarm Optimization*) method movement of particles is described by the set of equalities

$$\begin{cases} \mathbf{v}_{t+1} = w \cdot \mathbf{v}_t + \varphi_{t,1} \otimes (\mathbf{y}_t - \mathbf{x}_t) + \varphi_{t,2} \otimes (\mathbf{y}^*_t - \mathbf{x}_t) \\ \mathbf{x}_{t+1} = \mathbf{x}_t + \mathbf{v}_{t+1}, \end{cases} \tag{8}$$

where $\mathbf{v}_t$ is a vector of particle velocities, $\mathbf{x}_t$ is a vector of particle locations, $\mathbf{y}_t$ is the best location that the particle has found so far, $\mathbf{y}^*_t$ - the best location found by particles overall, w is an inertia weight, $\varphi_{t,1}$ and $\varphi_{t,2}$ are random variables, $\otimes$ is a Hadamard product.

Furthermore, we assume $\varphi_{t,1} = R_{t,1}c_1$, $\varphi_{t,2} = R_{t,2}c_2$, where c_1, c_2 are acceleration coefficients, $R_{t,1}$ and $R_{t,2}$ are vectors of numbers generated from a uniform distribution over an interval of $[0, 1]$.

B. IPSO stability areas

The primary purpose of the inertia weight w is to help control the velocities of the particles and to prevent the swarm from exploding. Nevertheless, the coefficient itself is not sufficient — an analysis of the behavior of the swarm concerning all the particle parameters is required. In other words, we need to establish a domain of parameters guaranteeing a stable behavior of particles. In the further text, we relate to such a domain as to the stability area.

1) Assumptions: In theoretical analysis of particle swarm optimization the following assumptions are used:

- **a)** Deterministic assumption [9]: $\varphi_{t,1} = R_{t,1}c_1 = R_1c_1 = \varphi_1$, $\varphi_{t,2} = R_{t,2}c_2 = R_2c_2 = \varphi_2$ for all t,
- **b)** Stagnation assumption [9]: $\mathbf{y}_t = \mathbf{y}$, $\mathbf{y}^*_t = \mathbf{y}^*$ for sufficiently large t,

c) Weak chaotic assumption [10]: $\mathbf{y}_t$, $\mathbf{y}^*_t$ will occupy an arbitrarily large finite number of unique positions,

d) Weak stagnation assumption [11]: $y_{it} = y^*_{it}$ for large enough t, where i is an index of the particle, which has found the best location of all particles,

e) Stagnant distribution assumption [6]: y_{it} and y^*_{it} are random variables sampled from a fixed distribution and have well-defined expected value and variance.

Notice that the objective behind the assumptions listed above is to simplify PSO in order for the mathematical analysis of its properties to become more approachable.

2) Stability under the deterministic assumptions: Definition (2) of stability in IPSO appeared in [12]–[14]. Under the deterministic assumption each of authors consequently reached a similar stability area:

$$0 < c_1 + c_2 < 2(1+w), \ \ -1 < w < 1 \tag{9}$$

in [12], [13] and

$$0 < c_1 + c_2 < 2(1+w), \ 0 < w < 1 \tag{10}$$

in [14].

3) Order-1 stability: In 2009, Poli [3] used expected value of particle locations to assess stability in IPSO:

$$\lim_{t\to\infty} E[\mathbf{x}_t] = \mathbf{y}. \tag{11}$$

He updated the stability area as well:

$$0 < E[\varphi_{t,1}] + E[\varphi_{t,2}] = \frac{c_1}{2} + \frac{c_2}{2} < 2(1+w), \ \ -1 < w < 1, \tag{12}$$

or, equivalently,

$$0 < c_1 + c_2 < 4(1+w), \ \ -1 < w < 1. \tag{13}$$

Recall that $\frac{c_1}{2}$, $\frac{c_2}{2}$ are expected values of $\varphi_{t,1}, \varphi_{t,2}$ respectively.

4) Order-2 stability: In 2007, Jiang presented condition (4) together with the following stability area:

$$\frac{5c - \sqrt{25c^2 - 336c + 576}}{24} < w < \frac{5c + \sqrt{25c^2 - 336c + 576}}{24}, \tag{14}$$

where $c = c_1 = c_2$. In 2009, Poli used condition (5), while assuming $c = c_1 = c_2$, to reach another area

$$c < \frac{12(1-w^2)}{7-5w}. \tag{15}$$

As shown in [11], (14) and (15) are in fact equivalent.

5) Order-2 stability:* In [3] the following equation for variance of each particle was derived:

$$Var[x_i] = \frac{c(w+1)}{4c(5w-7) - 48w^2 + 48}(y^*_i - y_i)^2 \tag{16}$$

where $c = c_1 = c_2$. It follows that the limit of each particle variance would be zero if and only if every particle should find a global optimum (i.e. $\mathbf{y} = \mathbf{y}^*$). Similar conclusions were made in [8].

III. IPSO Stochastic Model Stability Analysis

Effective analysis of the IPSO method requires additional assumptions imposing constraints on the variability of attractors. The majority of theoretical research is based on several variants of stagnation. In the analysis given below, we also assume that personal and global attractors are held constant.

A. IPSO stochastic model

Let's assume stagnation in IPSO method as defined in subsection II-B1, point **b)**. In this case the velocity equation is as follows:

$$\mathbf{v}_{t+1} = w \cdot \mathbf{v}_t + \varphi_{t,1} \otimes (\mathbf{y} - \mathbf{x}_t) + \varphi_{t,2} \otimes (\mathbf{y}^* - \mathbf{x}_t). \tag{17}$$

Substituting $\mathbf{v}_{t+1} = \mathbf{x}_{t+1} - \mathbf{x}_t$:

$$\mathbf{x}_{t+1} - \mathbf{x}_t = w \cdot (\mathbf{x}_t - \mathbf{x}_{t-1}) + \varphi_{t,1} \otimes (\mathbf{y} - \mathbf{x}_t) + \varphi_{t,2} \otimes (\mathbf{y}^* - \mathbf{x}_t), \tag{18}$$

$$\mathbf{x}_{t+1} = \mathbf{x}_t + w \cdot \mathbf{x}_t - (\varphi_{t,1} + \varphi_{t,2}) \otimes \mathbf{x}_t - w \cdot \mathbf{x}_{t-1} + \varphi_{t,1} \otimes \mathbf{y} + \varphi_{t,2} \otimes \mathbf{y}^*. \tag{19}$$

Furthermore, limiting (19) to a single dimension allows us to rewrite it as

$$x_{t+1} = (1 + w - \varphi_{t,1} - \varphi_{t,2})x_t - wx_{t-1} + \varphi_{t,1}y + \varphi_{t,2}y^*. \tag{20}$$

In [14], it is shown that the equilibrium point for a particle is a weighed average of y, y^* and is equal to $\frac{\varphi_1 y + \varphi_2 y^*}{\varphi_{t,1} + \varphi_{t,2}}$. Still, without loss of generality, we can assume $y = y^*$ and reformulate (20) as

$$x_{t+1} = (1 + w - \phi_t)x_t - wx_{t-1} + \phi_t y, \tag{21}$$

where $\phi_t = \varphi_{t,1} + \varphi_{t,2}$. From now on we will relate to (21) as the **stochastic model** where x_t is a random variable representing particle location in time t.

B. Model analysis

Recall that $w \in (-1, 1)$ (inertia weight) is a constant. We also assume that random variables ϕ_t, for $t = 1, 2, 3...$, are independent and identically distributed.

Let $e_t = E[x_t]$, $m_t = E[x_t^2]$, $h_t = E[x_t x_{t-1}]$, $f = E[\phi_t]$, $g = E[\phi_t^2]$. According to theory of probability, if X, Y are independent random variables, then $E[X \cdot Y] = E[X] \cdot E[Y]$. We should exercise this fact when applying the expected value operator to both sides of equation (21):

$$e_{t+1} = (1 + w - f)e_t - we_{t-1} + fy. \tag{22}$$

Let's then raise both sides of (21) to the second power:

$$\begin{aligned} x_{t+1}^2 = &(1 + w + \phi_t)^2 x_t^2 + w^2 x_{t-1}^2 + \phi_t^2 y^2 \\ &- 2(1 + w - \phi_t)wx_t x_{t-1} \\ &- 2wy\phi_t x_{t-1} + 2y\phi_t(1 + w - \phi_t)x_t \end{aligned} \tag{23}$$

and again apply the expected value operator to both sides:

$$\begin{aligned} m_{t+1} = &m_t((1+w)^2 - 2(1+w)f + g) + m_{t-1}w^2 \\ &- h_t 2w(1 + w - f) + e_t 2y(f(1+w) - g) \\ &- e_{t-1}2wyf + y^2 g \end{aligned} \tag{24}$$

Ultimately, let's multiply both sides of equation (21) by x_t

$$x_{t+1}x_t = (1+w+\phi_t)x_t^2 - wx_tx_{t-1} + \phi_t y x_t \tag{25}$$

and apply the expected value operator to both sides for the last time:

$$h_{t+1} = (1+w-f)m_t - wh_t + fye_t. \tag{26}$$

Let $\mathbf{z}_t = (e_t, e_{t-1}, m_t, m_{t-1}, h_t)^T$. Equations (22), (24), (26) can be collected and presented as

$$\mathbf{z}_{t+1} = M\mathbf{z}_t + \mathbf{b}, \tag{27}$$

where

$$M = \begin{bmatrix} m_{1,1} & -w & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & w^2 & m_{3,5} \\ 0 & 0 & 1 & 0 & 0 \\ fy & 0 & m_{5,3} & 0 & -w \end{bmatrix}, \tag{28}$$

is a square matrix with terms of

$m_{1,1} = 1+w-f,$
$m_{3,1} = 2y(f(1+w)-g),$
$m_{3,2} = -2wyf,$
$m_{3,3} = (1+w)^2 - 2(1+w)f + g,$
$m_{3,5} = -2w(1+w-f),$
$m_{5,3} = 1+w-f$

and $\mathbf{b}$ is a vector of free expressions:

$$\mathbf{b} = (fy, 0, y^2g, 0, 0)^T. \tag{29}$$

We will now show that without any loss of generality we can assume $y=0$. Notice that:

$$x_t = x_t - y + y = z_t + y$$

for some $z_t \in \mathbb{R}$. Substituting into (21) we get

$$\begin{aligned} z_{t+1} + y &= (1+w-\phi_t)(z_t+y) - w(z_{t-1}+y) + \phi_t y, \\ z_{t+1} + y &= (1+w-\phi_t)z_t + y + wy - \phi_t y + wz_{t-1} \\ &\quad - wy + \phi_t y, \end{aligned} \tag{30}$$

which evaluates to

$$z_{t+1} = (1+w-\phi_t)z_t + wz_{t-1}. \tag{31}$$

A simple translation of x_t by vector y can be always done, hence the models described by (21) and (31) are equivalent. Let's set $y=0$. It follows that $\mathbf{b} = \mathbf{0}$ and

$$M = \begin{bmatrix} m_{1,1} & -w & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{3,3} & w^2 & m_{3,5} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & m_{5,3} & 0 & -w \end{bmatrix}, \tag{32}$$

where

$m_{1,1} = 1+w-f,$
$m_{3,3} = (1+w)^2 - 2(1+w)f + g,$
$m_{3,5} = -2w(1+w-f),$
$m_{5,3} = 1+w-f.$

Observe that M is a block matrix. Let

$$M_1 = \begin{bmatrix} m_{1,1} & -w \\ 1 & 0 \end{bmatrix}, \; M_2 = \begin{bmatrix} m_{3,3} & w^2 & m_{3,5} \\ 1 & 0 & 0 \\ m_{5,3} & 0 & -w \end{bmatrix}. \tag{33}$$

The analysis of eigenvalues of matrix M_1 is relatively simple and was done in [15]. On the other hand, the analysis of eigenvalues of matrix M_2 is very complicated. Therefore, it is justified to search for ways of making the analysis more viable. One way is to make appropriate assumptions, and we should choose this path in this paper.

IV. An Explicit Formula

The level of complexity of calculation of eigenvalues of the matrix M_2 is very high. Therefore, we look for assumptions or any kind of relations that could simplify the analysis. Observe the following elements of the matrix M_2: $m_{1,1}, m_{3,5}$ and $m_{5,3}$ have the same factor of $1+w-f$ in common. Let's consider a relation $f = w+1$. As $w \in (-1,1)$ and thus $f \in (0,2)$, it implies that we do not stray away from the values of the order-1 stable region: $f = E[\phi] < 2(w+1) = 2f$.

Let's assume $f = w+1$. Eventually, we get $m_{1,1} = m_{3,5} = m_{5,3} = 0$ and $m_{3,3} = g - f^2$. We can also notice that $Var[\phi_t] = g - f^2$. Let's set $\gamma = Var[\phi_t]$. Now:

$$M_2 = \begin{bmatrix} \gamma & w^2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -w \end{bmatrix}. \tag{34}$$

Current state of the matrix M_2 allows us to proceed with the analysis of its eigenvalues:

$$\begin{aligned} \lambda_1 &= -w, \\ \lambda_2 &= \frac{1}{2}(\gamma - \alpha), \\ \lambda_3 &= \frac{1}{2}(\gamma + \alpha). \end{aligned}$$

where $\alpha = \sqrt{\gamma^2 + 4w^2}$. In result we can easily calculate

$$(M_2)^t = P\Delta^t P^{-1} = \tag{35}$$

$$\begin{bmatrix} 0 & \lambda_2 & \lambda_3 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^t \begin{bmatrix} 0 & 0 & 1 \\ \frac{-1}{\alpha} & \frac{\lambda_3}{\alpha} & 0 \\ \frac{1}{\alpha} & \frac{-\lambda_2}{\alpha} & 0 \end{bmatrix}. \tag{36}$$

After brief calculations we get

$$(M_2)^t = \begin{bmatrix} \frac{\lambda_3^t - \lambda_2^t}{\alpha} & \frac{\lambda_3\lambda_2^t - \lambda_2\lambda_3^t}{\alpha} & 0 \\ \frac{\lambda_3^{t-1} - \lambda_2^{t-1}}{\alpha} & \frac{\lambda_3\lambda_2^{t-1} - \lambda_2\lambda_3^{t-1}}{\alpha} & 0 \\ 0 & 0 & \lambda_1^t \end{bmatrix}. \tag{37}$$

Using the fact that

$$\begin{bmatrix} m_t \\ m_{t-1} \\ h_t \end{bmatrix} = (M_2)^t \begin{bmatrix} m_1 \\ m_0 \\ h_1 \end{bmatrix} \tag{38}$$

we are able to obtain an explicit formula for m_t

$$m_t = A\lambda_2^t + B\lambda_3^t \quad (39)$$

where

$$A = \frac{1}{\alpha}(\lambda_3 m_0 - m_1),\ B = \frac{1}{\alpha}(m_1 - \lambda_2 m_0). \quad (40)$$

V. Analysis of Stability Areas

The explicit formula for m_t is key to further investigation of the stability of the stochastic model. Until now, the analysis of variance was too complex to perform. Previous papers concerning this area of research usually restricted their analysis to the expected value only, which, according to the order-2 and order-2* stability forms, is not sufficient. The explicit formula for m_t also allows us to calculate the areas of stability directly.

A. Stability areas

Before proceeding to analysis of the stability areas, let's observe that the first-order stability condition is always satisfied:

$$1 + w = f = E[\phi_t] = E[\varphi_{t,1}] + E[\varphi_{t,2}]. \quad (41)$$

Substituting into inequality (12) describing the area of order-1 stability we get

$$0 < E[\varphi_{t,1}] + E[\varphi_{t,2}] = 1 + w < 2(1 + w), \quad (42)$$

which always holds true as $0 < 1 + w < 2$.

Another observation is the fact that the order-2* stability implies order-2 stability. Notice that:

$$Var[x_t] = E[x_t^2] - (E[x_t])^2 = m_t - e_t^2. \quad (43)$$

Remembering $\lim_{t\to\infty} e_t = y = 0$ we obtain

$$\lim_{t\to\infty} Var[x_t] = \lim_{t\to\infty} m_t - e_t^2 = \lim_{t\to\infty} m_t. \quad (44)$$

We can rewrite conditions (6) and (7) in relation to the stochastic model:

$$\text{order-2:}\ \lim_{t\to\infty} Var[x_t] = \lim_{t\to\infty} m_t = \beta, \quad (45)$$

$$\text{order-2*:}\ \lim_{t\to\infty} Var[x_t] = \lim_{t\to\infty} m_t = 0. \quad (46)$$

By setting $\beta = 0$ in (45) we obtain (46), therefore (46) implies (45). Thus, it suffices to show only the stronger condition. Remembering that

$$E[x_t^2] = m_t = A\lambda_2^t + B\lambda_3^t,$$

the order-2* stability holds if and only if

$$\lim_{t\to\infty} A\lambda_2^t + B\lambda_3^t = 0.$$

Observe that the above equation is true only in the case of $\lim_{t\to\infty} \lambda_2^t = 0$ and $\lim_{t\to\infty} \lambda_3^t = 0$, or equivalently $|\lambda_2| < 1$ and $|\lambda_3| < 1$. Let's review the priorly found eigenvalues.

Knowing that $v > 0$ and $-1 < w < 1$, we get

$$|\lambda_1| = |-w| = |w| < 1,$$
$$|\lambda_2| = \left|\frac{1}{2}(\gamma - \alpha)\right| < \left|\frac{1}{2}(\gamma + \alpha)\right| = |\lambda_3|,$$
$$|\lambda_3| = \left|\frac{1}{2}(\gamma + \alpha)\right| < 1 \iff \gamma \in (0, 1 - w^2).$$

Hence we conclude that the second-order stability with a star holds if and only if

$$\gamma \in (0, 1 - w^2). \quad (47)$$

Notice that we also obtain the convergence area, within which all of the stability types hold true. We can present the area in several ways. Substituting $g - f^2$ for γ and keeping in mind that $\gamma > 0$, we get

$$\gamma = g - f^2 = g - (w + 1)^2 > 0 \implies g > (w + 1)^2,$$

$$|\lambda_3| = \left|\frac{1}{2}\left((g - (w+1)^2) + \sqrt{(g - (w+1)^2)^2 + 4w^2}\right)\right| < 1$$
$$\implies g \in (0, 2(w + 1)),$$

which results in

$$g \in \left((w + 1)^2, 2(w + 1)\right), \quad (48)$$

as $0 < w + 1 < 2$.

B. Stability areas visualized

1) Stability areas with respect to inertia weight: The analysis of the stability areas resulted in conditions (47) and (48). As $w \in (-1, 1)$, we get the following stability areas: with respect to w and γ (Fig. 1)

$$\{-1 < w < 1 \wedge 0 < \gamma < 1 - w^2\} \quad (49)$$

and with respect to w and g (Fig. 2):

$$\{-1 < w < 1 \wedge (w + 1)^2 < g < 2(w + 1)\}. \quad (50)$$

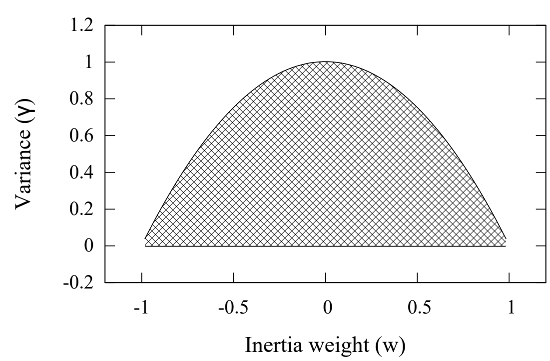


Fig. 1. Stability area with respect to w and γ given by ineq. (49)

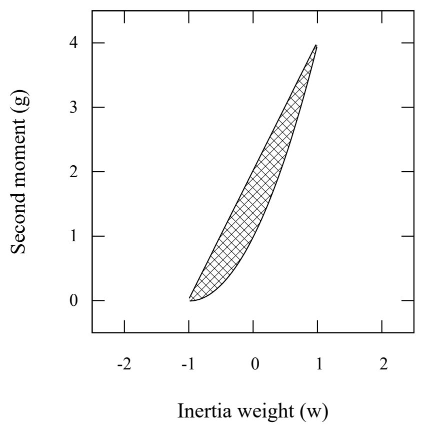


Fig. 2. Stability area with respect to w and g given by ineq. (50)

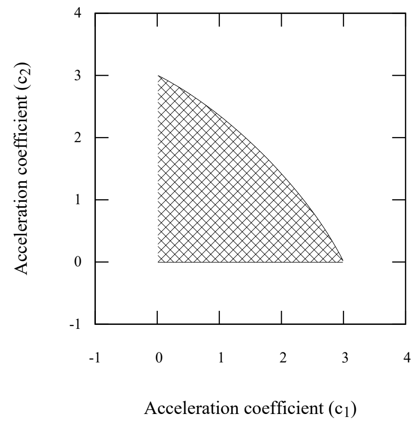


Fig. 3. Stability area with respect to c_1 and c_2 given by ineq. (51)

2) Stability area with respect to acceleration coefficients — uniform distribution: Let's recall that the IPSO method assumes $\varphi_{t,1}$ and $\varphi_{t,2}$ are random variables of uniform distribution over $[0, c_1]$ and $[0, c_2]$ respectively. Thus $E[\varphi_{t,1}] = \frac{c_1}{2}$ and $E[\varphi_{t,2}] = \frac{c_2}{2}$, as well as $Var[\varphi_{t,1}] = \frac{c_1^2}{12}$ and $Var[\varphi_{t,2}] = \frac{c_2^2}{12}$. Remembering the independence of the variables we calculated the distribution of ϕ_t:

$$f = E[\phi_t] = E[\varphi_{t,1}] + E[\varphi_{t,2}] = \frac{c_1 + c_2}{2},$$
$$g - f^2 = \gamma = Var[\phi_t] = Var[\varphi_{t,1}] + Var[\varphi_{t,2}] = \frac{c_1^2 + c_2^2}{12},$$
$$w = f - 1 = \frac{c_1 + c_2}{2} - 1$$

Observe the impact on the stability area:

$$-1 < w = \frac{c_1 + c_2}{2} - 1 < 1 \implies 0 < c_1 + c_2 < 4,$$

$$\gamma = \frac{c_1^2 + c_2^2}{12} < 1 - \left(\frac{c_1 + c_2}{2} - 1\right)^2 = 1 - w^2$$
$$\implies 2c_1^2 + 3c_1(c_2 - 2) + 2(c_2 - 3)c_2 < 0$$

Recollecting the above inequalities and given that $c_1 > 0$ and $c_2 > 0$ we get the following stability area (Fig. 3)

$$\{0 < c_1 + c_2 < 4 \wedge 2c_1^2 + 3c_1(c_2 - 2) + 2(c_2 - 3)c_2 < 0 \wedge c_1 > 0 \wedge c_2 > 0\} \quad (51)$$

3) Stability area with respect to acceleration coefficients — joint uniform distribution: Another common assumption made during exploration of the IPSO method is for $\phi_t = \varphi_{t,1} + \varphi_{t,2}$ to have a uniform distribution over $[0, c_1 + c_2]$. Let's set $c = c_1 + c_2$ and calculate:

$$E[\phi_t] = \frac{c}{2},$$
$$Var[\phi_t] = Var[\varphi_{t,1} + \varphi_{t,2}] = \frac{c^2}{12}.$$

Observe the impact on the stability area:

$$-1 < w = \frac{c}{2} - 1 < 1 \implies 0 < c < 4,$$
$$\gamma = \frac{c^2}{12} < 1 - \left(\frac{c}{2} - 1\right)^2 = 1 - w^2 \implies 0 < c < 3$$

Recollecting the above inequalities and given that $c_1 > 0$ and $c_2 > 0$ we get the following stability area (Fig. 4)

$$\{0 < c_1 + c_2 < 3 \wedge c_1 > 0 \wedge c_2 > 0\} \quad (52)$$

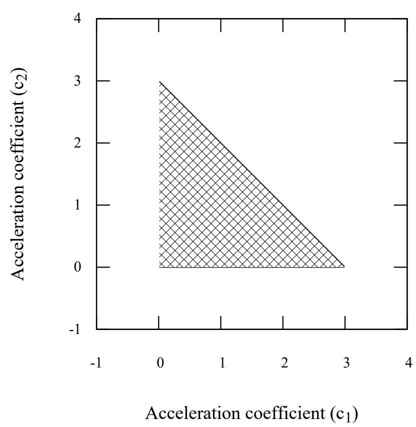


Fig. 4. Stability area with respect to c_1 and c_2 given by ineq. (52)

VI. Simulations

The experimental part of the research verified graphs of stability areas obtained from theoretical analysis. In this part, we did simulations for a set of particles where the stagnation condition is satisfied. Due to stagnation, no communication between particles arise. Thus we can regard this experiment as a series of tests for a sample of $N = 1000$ particles. For all particles, their initial locations z_1 are uniformly distributed in $[-10, 10]$. The second locations equal the first ones, that is, initial velocities equal zero for all particles. The particles change their locations in subsequent steps according to the eq. (31) and for each step, the variance of the location in the sample is calculated. Algorithm 1 presents the details of this procedure.

Algorithm 1 particle location variance evaluation procedure

1: for each particle in a set P of size $N = 1000$, initialize its two successive distances from the attractor: z_1^i and z_2^i where $i \in [1, N]$. $T_{\max} = 150$.
2: $t = 2$
3: **repeat**
4:     **for all** $i \in \{1, 2 \ldots N\}$ **do**
5:         $z_{t+1}^i = (1 + w - \phi_t) z_t^i + w z_{t-1}^i$.
6:     **end for**
7:     $t = t + 1$
8: **until** $t \leq T_{\max}$
9: **for all** $t \in \{1, 2 \ldots, T_{\max}\}$ **do**
10:     $m_t = \frac{1}{N} \sum_{i=1}^{N} (z_t^i)$ ▷ m_t: mean of z_t
11:     $v_t = \frac{1}{N} \sum_{i=1}^{N} (z_t^i - m_t)^2$ ▷ v_t: variance of z_t
12: **end for**
13: **return** v ▷ v: vector of variances of z

We executed the procedure for a grid of 100×100 nodes representing particle configurations. For every node, the experiment returned a series of location variances v for subsequent time steps. When the variance values were higher than 10^6, we regarded the configurations as representing the unstable region and clamped the variance to 10^6. With maps of variances obtained for subsequent time steps, we observed the process of clarification of stable and unstable configuration areas and identified their final shape.

Due to the fact that for the uniform distribution over an interval of $[0, c]$ all f, g and γ are dependent on w (see Sec. V-B2, V-B3), for simulations we decided to use Gauss distribution (see eq. (53), (54)). The results related to Fig. 1 and Fig. 2 can be viewed in Fig. 5 and Fig. 6 respectively.

The ϕ_t coefficient in the line 5 of Algorithm 1 is defined depending on the tested stability area parameters. For the stability area with respect to w and $\gamma = Var[\phi_t]$ (Fig. 1), ϕ_t equals:

$$\phi_t = N(1 + w, \gamma) \tag{53}$$

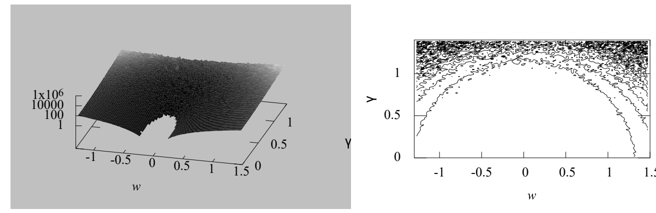
where $N(\cdot, \cdot)$ represents Gauss distribution. The variances are generated for a grid of configurations (w, γ) starting from $[w = -1.272, \gamma = 0.014]$ and changing with step 0.028 for w and with step 0.014 for γ. Fig. 5 depicts the final shape of stable and unstable configuration areas obtained in simulations.

For the stability area with respect to w and $g = E[\phi_t^2]$ (Fig. 2), ϕ_t equals:

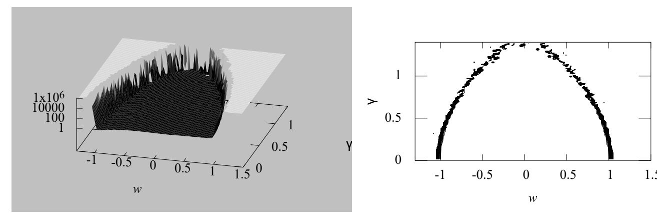
$$\phi_t = N(1 + w, g - (1 + w)^2) \tag{54}$$

The variances are generated for a grid of configurations (w, g) starting from $[w = -1.272, g = 0.05]$ and changing with step 0.028 for w and with step 0.05 for g. Fig. 6 presents the results of simulations.

The results of the simulations confirming validity of calculations leading to Fig. 3 and Fig. 4 can be viewed in Fig. 7 and Fig. 8 respectively. In these cases we are testing the impact

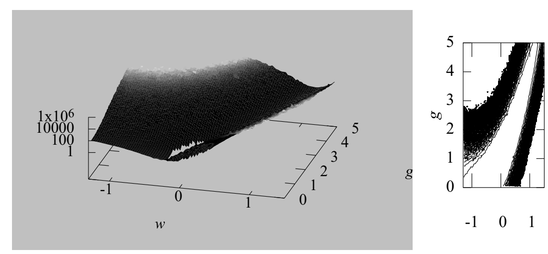


(a) $t = 5$

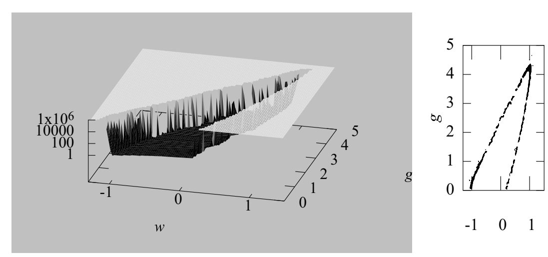


(b) $t = 150$

Fig. 5. Stability area with respect to w and γ obtained in simulations; snapshots for $t = 5$ (the top figures) and $t = 150$ (the bottom figures)



(a) $t = 5$



(b) $t = 150$

Fig. 6. Stability area with respect to w and g obtained in simulations; snapshots for $t = 5$ (the top figures) and $t = 150$ (the bottom figures)

of acceleration coefficients c_1 and c_2, therefore we choose the continuous uniform probability distribution and ϕ_t equals:

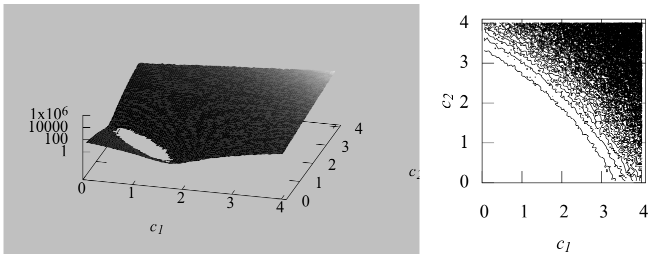
$$\phi_t = U(0, c_1) + U(0, c_2) \qquad \text{for ineq. (51)} \tag{55}$$

$$\phi_t = U(0, c_1 + c_2) \qquad \text{for ineq. (52)} \tag{56}$$

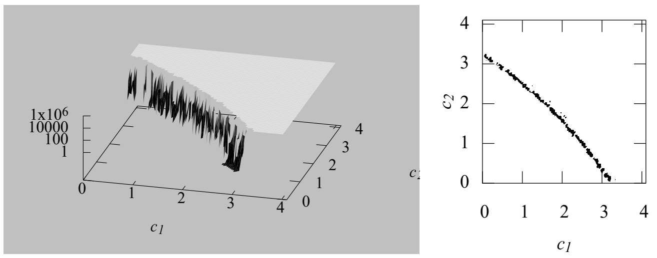
where $U(\cdot, \cdot)$ represents uniform distribution. The variances are generated for a grid of configurations (c_1, c_2) starting from $[c_1 = 0.05, c_2 = 0.05]$ and changing with step 0.04 for both c_1 and c_2. Fig. 7 and 8 present the results of simulations.

VII. Summary and Conclusions

In this paper, we investigated the method of particle swarm optimization with inertia weight. We made a brief overview of the stability definition and presented three types of stability

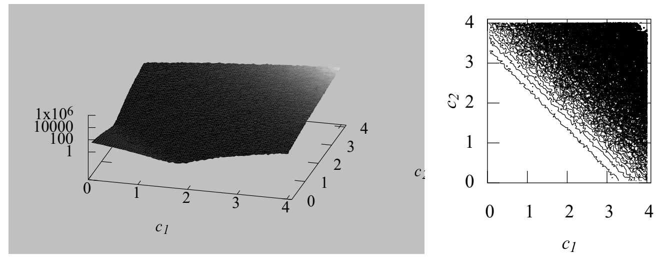


(a) $t = 5$

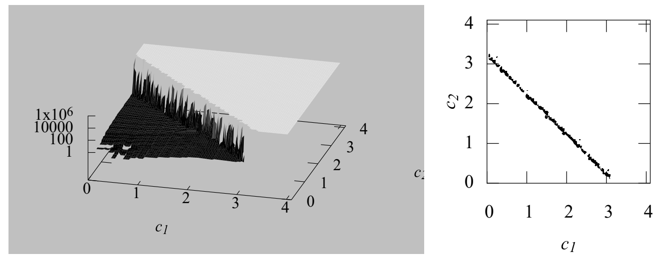


(b) $t = 150$

Fig. 7. Stability area with respect to c_1 and c_2 obtained in simulations; snapshots for $t = 5$ (the top figures) and $t = 150$ (the bottom figures)



(a) $t = 5$



(b) $t = 150$

Fig. 8. Stability area with respect to c_1 and c_2 obtained in simulations; snapshots for $t = 5$ (the top figures) and $t = 150$ (the bottom figures)

together with the related research. Next, we showed the main principles of the stochastic model of the IPSO method. Then, we indicated the area of parameters, within which analytical formulas for the variance of the location of a particle are simple enough for further analysis. This led us to establish an explicit formula for the second moment in the model. Finally, using the formerly found formula, the stability areas were calculated and visualized with respect to the inertia weight, as well as some selected distributions of random variables that parametrize acceleration coefficients of the particle. The findings were confirmed empirically in simulations.

The main novelty of research conducted in this paper lies in establishing a simple explicit formula for the second moment of particle location under some sensible assumptions. In previous works, despite the first moment analysis being thoroughly conducted, the second moment analysis was often referred to as too complicated to perform. This paper shows that there exist assumptions for which the analysis becomes viable.

References

[1] J. Kennedy and R. C. Eberhart, "Particle swarm optimization," in *Proc. of the IEEE Int. Conf. on Neural Networks*. Piscataway, NJ: IEEE, 1995, pp. 1942–1948.

[2] C. W. Cleghorn and A. P. Engelbrecht, "Particle swarm optimizer: The impact of unstable particles on performance," in *2016 IEEE Symposium Series on Computational Intelligence (SSCI)*, 2016, pp. 1–7.

[3] R. Poli, "Mean and variance of the sampling distribution of particle swarm optimizers during stagnation," *IEEE T. Evolut. Comput.*, vol. 13, no. 4, pp. 712–721, 2009.

[5] C. W. Cleghorn and A. P. Engelbrecht, "Particle swarm stability: a theoretical extension using the non-stagnate distribution assumption," *Swarm Intelligence*, vol. 12, no. 1, pp. 1–22, 2018.

[4] M. Jiang, Y. P. Luo, and S. Y. Yang, "Stochastic convergence analysis and parameter selection of the standard particle swarm optimization algorithm," *Information Processing Letters*, vol. 102, no. 1, pp. 8–16, Apr. 2007.

[6] M. R. Bonyadi and Z. Michalewicz, "Stability analysis of the particle swarm optimization without stagnation assumption," *IEEE Trans. Evolutionary Computation*, vol. 20, no. 5, pp. 814–819, 2016.

[7] T. Blackwell, "A study of collapse in bare bones particle swarm optimization," *IEEE Transactions on Evolutionary Computation*, vol. 16, no. 3, pp. 354–372, 2012.

[8] C. W. Cleghorn, "Particle swarm optimization: Understanding order-2 stability guarantees," in *Applications of Evolutionary Computation - 22nd International Conference, EvoApplications 2019, Held as Part of EvoStar 2019, Leipzig, Germany, April 24-26, 2019, Proceedings*, 2019, pp. 535–549. [Online]. Available: https://doi.org/10.1007/978-3-030-16692-2_36

[9] E. Ozcan and C. K. Mohan, *Analysis of a simple particle swarm optimization system*. ASME Press, St. Louis, 1998, vol. 8, pp. 253–258.

[10] C. W. Cleghorn and A. P. Engelbrecht, "A generalized theoretical deterministic particle swarm model," *Swarm Intelligence*, vol. 8, no. 1, pp. 35–59, 2014.

[11] Q. Liu, "Order-2 stability analysis of particle swarm optimization," *Evol. Comput.*, vol. 23, no. 2, pp. 187–216, 2015.

[12] M. Clerc and J. Kennedy, "The particle swarm-explosion, stability, and convergence in a multidimensional complex space," *IEEE Transactions on Evolutionary Computation*, vol. 6, no. 1, pp. 58–73, 2002.

[13] I. C. Trelea, "The particle swarm optimization algorithm: convergence analysis and parameter selection," *Inform. Process. Lett.*, vol. 85, no. 6, pp. 317 – 325, 2003.

[14] F. van den Bergh and A. P. Engelbrecht, "A study of particle swarm optimization particle trajectories," *Inform. Sciences*, vol. 176, no. 8, pp. 937–971, Apr. 2006.

[15] R. Poli and D. Broomhead, "Exact analysis of the sampling distribution for the canonical particle swarm optimiser and its convergence during stagnation," in *GECCO '07: Proceedings of the 9th annual conference on Genetic and evolutionary computation*, vol. 1. ACM Press, 7-11 Jul. 2007, pp. 134–141.