Comparing Intervals Using Type Reduction

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***Abstract*—Many decision making processes are based on choosing options with maximum utility. Often utility assessments are associated with uncertainty, which may be mathematically modeled by intervals of utilities. Intervals of utilities may be mapped to single utility values by so–called type reduction methods which have been originally developed in the context of interval type–2 defuzzification: the method by Nie and Tan (NT), consistent linear type reduction (CLTR), consistent quadratic type reduction (CQTR), and the uncertainty weight method (UW). This paper considers the problem of comparing pairs of utility intervals using type reduction methods. Three different possible relations between pairs of intervals (disjoint, overlapping, and inclusive) are distinguished in an extensive experimental study, which yields recommendations for the choice of type reduction methods with respect to the level of risk that the decision maker is willing to take. If the focus is on mean utility, then we recommend the Nie–Tan method. For more cautious decision making, when very low utilities should be avoided, we recommend consistent linear type reduction with a high value of the cautiousness parameter or consistent quadratic type reduction. For more risky decision making with a strong focus on very high utilities we recommend consistent linear type reduction with a low value of the cautiousness parameter.**

I. Introduction

In many decision making processes the best option is chosen based on qualitative assessments of the utility of decision options, such as movies, project proposals, machine tools, or waste disposal sites. Such assessments are often associated with uncertainty. One way of modeling this uncertainty is to specify the utility of each option as an interval of membership values [1], [2], which leads to interval type–2 fuzzy sets [3], [4], [5]. Another way of modeling this uncertainty leads to general type–2 fuzzy sets that may be constructed from intervals [6] and may then be processed in an efficient algorithmic scheme [7]. Here, for simplicity, we restrict to the case of interval type–2 fuzzy sets, and we are planning to extend this to the case of general type–2 fuzzy sets in a forthcoming publication.

The problem of comparing options with different intervals of utilities is related to the problems of comparing numerical intervals [8] and ordering fuzzy subsets of the unit interval [9]. Comparing intervals of utilities is at the core of decision making with interval type–2 fuzzy sets [10], [11], [12], [13].

In this paper we specifically consider comparing pairs of intervals with so–called type reduction methods [14], [15], [16] that have been developed in the context of interval type–2 defuzzification [17], [18], in order to map interval type–2 fuzzy sets to type–1 fuzzy sets. In the literature, various different type reduction methods have been proposed: the method by Nie and Tan (NT) [19], consistent linear type reduction (CLTR) [20], consistent quadratic type reduction (CQTR) [20], and the uncertainty weight method (UW) [21],

The idea to apply such type reduction methods to decision making has first been discussed in [22], where four simple examples have been considered for illustration. This paper is a substantial extension of the study at [22]. We distinguish the three different possible relations between pairs of intervals (disjoint, overlapping, and inclusive), and for each of these relations we perform extensive statistical experiments evaluating the resulting utilities obtained by the four considered type reduction methods (NT, CLTR, CQTR, and UW), leading to specific recommendations for selecting appropriate type reduction methods with respect to the level of risk that the decision maker is willing to take.

This paper is structured as follows: Section II briefly reviews the different type reduction methods considered in this paper. Section III describes the setup of our experimental study. Sections IV–VI present the results of the experiments with disjoint, overlapping, and inclusive pairs of intervals. Section VII finally summarizes our conclusions.

II. Type Reduction Methods

Type reduction maps each membership interval $[\underline{u}, \overline{u}]$, $\underline{u}, \overline{u} \in [0, 1]$, $\underline{u} \leq \overline{u}$, to a membership value $u \in [0, 1]$. We call a type reduction method *convex*, if and only if for all $\underline{u}, \overline{u} \in [0, 1]$ it yields $u \in [\underline{u}, \overline{u}]$.

In this paper we consider the following four type reduction methods:

1) the Nie–Tan method (NT) [19]

$$u_{\text{NT}} = \frac{\underline{u} + \overline{u}}{2} \tag{1}$$

2) consistent linear type reduction (CLTR) [20]

$$u_{\text{CLTR}} = a \cdot \underline{u} + (1 - a) \cdot \overline{u} \tag{2}$$

with the parameter $a \in [0, 1]$ that quantifies the degree of caution in the decision making process; for $a = 0$ we call this the *risky* method

$$u_{\text{risky}} = \overline{u} \tag{3}$$

*This paper is dedicated to the memory of Professor Robert I. John.

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for $a = 1$ we call this the *cautious* method

$$u_{\text{cautious}} = \underline{u} \quad (4)$$

and for $a = 0.5$ we obtain the Nie–Tan method (1).

3) consistent quadratic type reduction (CQTR) [20]

$$u_{\text{CQTR}} = a \cdot \underline{u} + (1-a) \cdot \overline{u} - (1-a) \cdot (\overline{u} - \underline{u})^2 \quad (5)$$

with the parameter $a \in [0, 1]$

4) the uncertainty weight method (UW) [21]

$$u_{\text{UW}} = \frac{1}{2}(\underline{u} + \overline{u}) \cdot (1 + \underline{u}(x) - \overline{u}(x))^{\alpha} \quad (6)$$

with the parameter $\alpha \geq 0$

For $a = 1$, CLTR and CQTR are both equivalent to the cautious method, and for $\alpha = 0$, NT and UW are equivalent. The first three methods (NT, CLTR, CQTR) are convex, but the uncertainty weight method (UW) is not convex. To see this, consider for example the case $\underline{u} = 0.8$, $\overline{u} = 1$, $\alpha = 1$, for which (6) yields

$$u_{\text{UW}} = \frac{1}{2}(0.8 + 1) \cdot (1 + 0.8 - 1) = 0.72 < 0.8 = \underline{u} \quad (7)$$

so in this example $u_{\text{UW}} = 0.72$ is outside the interval $[\underline{u}, \overline{u}] = [0.8, 1]$.

III. Interval Comparison

In our experiments we consider the following decision making scenario based on comparing intervals using type reduction: Assume we have two decision options 1 and 2 associated with uncertain utilities quantified by the utility intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_2, \overline{u}_2]$, respectively. Based on these two utility intervals we decide for either option 1 or option 2. If the decision is option 1, then our decision utility u^* will be randomly drawn from $[\underline{u}_1, \overline{u}_1]$, and if the decision is option 2, then our decision utility u^* will be randomly drawn from $[\underline{u}_2, \overline{u}_2]$, with uniform distributions. The goal is to maximize the utility u^*.

Each of the utility intervals $[\underline{u}, \overline{u}]$ is generated randomly using the following process: We randomly choose a mean value

$$u \in [0, 1] \quad (8)$$

and a value Δu, so that

$$\underline{u} = u - \Delta u, \quad \overline{u} = u + \Delta u \quad (9)$$

To make sure that the membership values are normalized, $\underline{u}, \overline{u} \in [0, 1]$, the half spread Δx of each interval has to be randomly chosen in the interval

$$\Delta x \in [0, \min\{u, 1 - u, \}] \quad (10)$$

as illustrated by the grey area in Fig. 1. So for normalized intervals the membership (mean) and uncertainty (spread) are not independent, but the maximum possible uncertainty is maximal ($\max \Delta u = 0.5$), for membership $u = 0.5$, and the maximum possible uncertainty decreases for lower or higher memberships, until for memberships zero and one the uncertainty can only be zero.

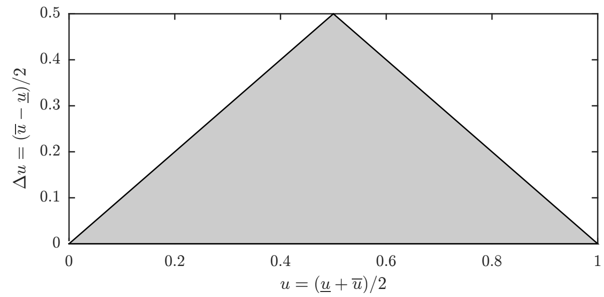

Fig. 1. Possible means (horizontal) and spreads (vertical) of the randomly generated intervals.

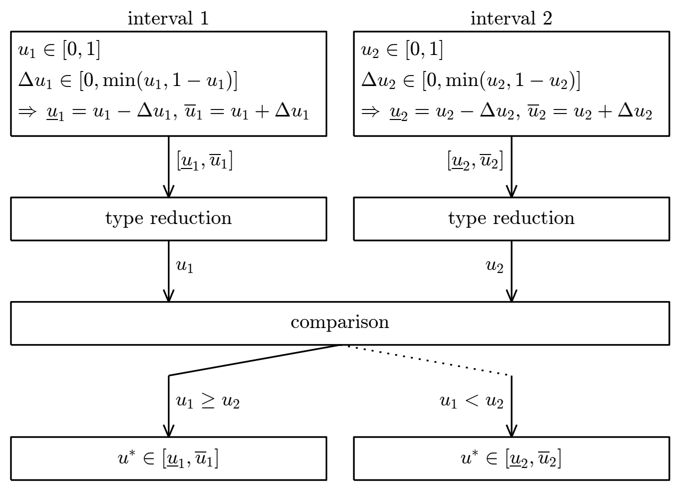

Fig. 2. Experimental setup for comparing random pairs of intervals.

Here, for decision making we use one of the four type reduction methods (NT, CLTR, CQTR, and UW) to map the utility intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_2, \overline{u}_2]$ of the two decision options to the single utility values u_1 and u_2. Then we decide for the option with larger utility: option 1 if $u_1 \geq u_2$, and option 2 if $u_1 < u_2$. The resulting overall experimental setup is illustrated in Fig. 2.

For pairs of intervals $\{[\underline{u}_1, \overline{u}_1], [\underline{u}_2, \overline{u}_2]\}$ without loss of generality we require $\overline{u}_1 > \overline{u}_2$. We distinguish three different relations, as shown by the examples in Fig. 3:

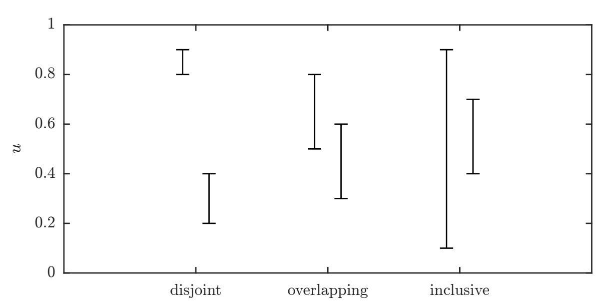

Fig. 3. Each pair of intervals has one of these three different types of relations.

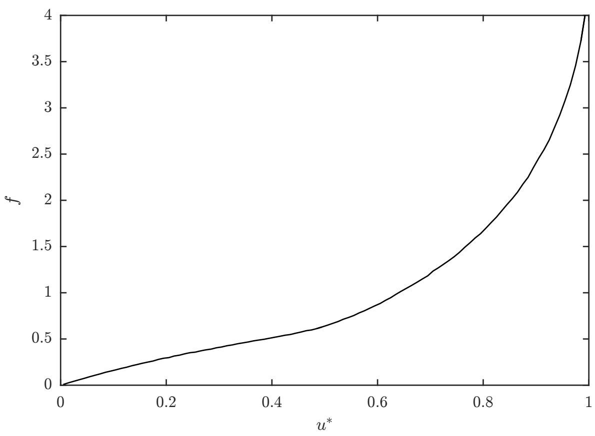

Fig. 4. Utility distribution for disjoint intervals for the convex methods.

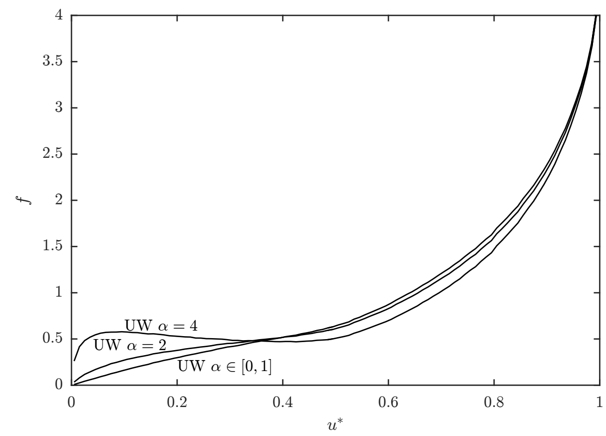

Fig. 5. Utility distribution for disjoint intervals for the uncertainty weight method (UW).

1) disjoint

$$\underline{u}_1 > \overline{u}_2 \tag{11}$$

2) overlapping

$$\underline{u}_2 \leq \underline{u}_1 \leq \overline{u}_2 \tag{12}$$

3) inclusive

$$\underline{u}_1 < \underline{u}_2 \tag{13}$$

The reader may easily verify that any possible pair of intervals will have one and only one of these relations. If we randomly generate pairs of intervals using the method presented above, then we obtain the following distribution of the three types of relations: 55.9% disjoint, 23.7% overlapping, and 20.4% inclusive.

For each of the three relations (disjoint, overlapping, inclusive) we run $10,000,000$ experiments (where the randomly generated intervals are discarded if they do not match the desired relation). In each of the experiments we use each of the four type reduction methods (NT, CLTR, CQTR, and UW) with various values of the parameters a and α to decide for one of the two options, and for each case we report the average utility u^*.

IV. Comparing Disjoint Intervals

We begin our experiments with the case of disjoint intervals as shown in the left of Fig. 3. For disjoint intervals, we have $\underline{u}_1 > \overline{u}_2$, so the convex methods will always pick the upper interval $[\overline{u}_1, \underline{u}_1]$ and therefore all convex methods (NT, CLTR, CQTR) yield the same results. The histogram (normalized to area one) of the ground truth utilities obtained by the convex methods is shown in Fig. 4. Here, the relative frequency of the utilities is monotonically increasing, with the highest frequency for utility one and the lowest frequency (almost zero) for utility zero. The average utility obtained in this experiment is about 0.74.

Fig. 5 shows the results for the same experiment with the uncertainty weight method (UW) for $\alpha \in [0, 1]$, $\alpha = 2$, and $\alpha = 4$. These parameters have been chosen, so that example curves are obtained that can be easily distinguished visually. For $\alpha \in [0, 1]$ we obtain almost the same distribution as for the convex methods (Fig. 4). For $\alpha = 2$ the distribution is similar, but with slightly higher frequencies for $u^* < 0.35$ and slightly lower frequencies for $u^* > 0.35$. For $\alpha = 4$ we obtain even higher frequencies for $u^* < 0.35$ and lower frequencies for $u^* > 0.35$, and for $u^* < 0.5$ the highest frequency is at $u^* \approx 0.1$.

TABLE I

Utilities obtained for disjoint intervals.

method	mean u^*	$u^* \leq 0.1$	$u^* \geq 0.9$
convex	$\mathbf{0.73724}$	$\mathbf{0.00823\%}$	$\mathbf{0.30961\%}$
UW $\alpha = 2$	0.72114	0.01687%	0.30523%
UW $\alpha = 4$	0.68281	0.05084%	0.29866%

Table I shows the mean utilities and the precentages of very low utilities ($u^* \leq 0.1$) and of very high utilities ($u^* \geq 0.9$) obtained by the different methods for disjoint intervals. In this case, the best results (shown in bold) are obtained with the convex methods, and with UW for $\alpha \in [0, 1]$, which yields almost the same results. UW with higher values of α yields lower mean utilities, more very low utilities ($u^* \leq 0.1$), and less very high utilities ($u^* \geq 0.9$).

V. Comparing Overlapping Intervals

In our next set of experiments we consider overlapping intervals as shown in the middle of Fig. 3. Also here we observe that all considered convex methods yield exactly the same utility distribution that is shown in Fig. 6. Notice however that we may construct convex type reduction methods for which this property does not hold. Compared with disjoint intervals (Fig. 4), we obtain more medium and almost no very high utility values for overlapping intervals, which reflects the fact that intervals with medium membership are more likely to overlap than intervals with high membership.

Fig. 7 shows the results for the same experiment with UW for $\alpha \in [0, 0.2]$, $\alpha = 1$, and $\alpha = 2$. Again, these parameters

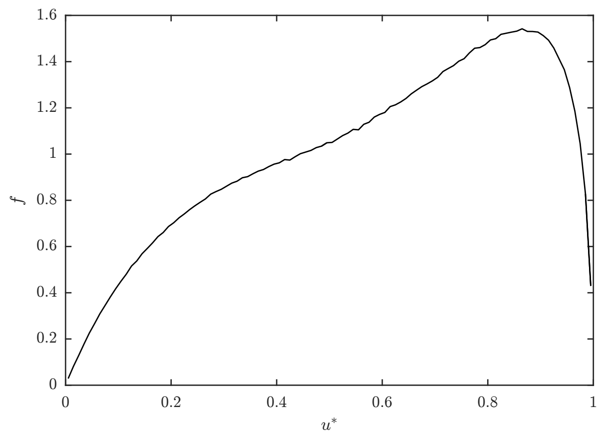

Fig. 6. Utility distribution for overlapping intervals for the convex methods.

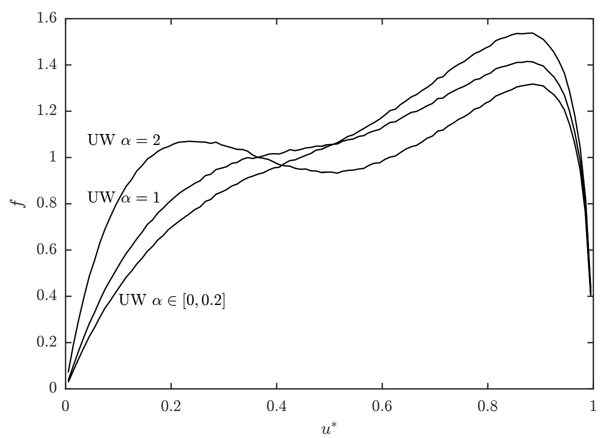

Fig. 7. Utility distribution for overlapping intervals for the uncertainty weight method (UW).

have been chosen to obtain example curves that can be easily distinguished. For $\alpha \in [0, 0.2]$ we obtain almost the same distribution as for the convex methods (Fig. 6). For $\alpha = 1$ we obtain higher frequencies for $u^* < 0.5$ and lower frequencies for $u^* > 0.5$. For $\alpha = 2$ lower utilities are even more frequent, and higher utilities even less, and we can see a clear local maximum at about $u^* \approx 0.2$.

Table II shows the mean utilities and percentages of very low and very high utilities for the different methods for overlapping intervals. Again, the best results are obtained with the convex methods, and with UW for $\alpha \in [0, 0.2]$, which

TABLE II
UTILITIES OBTAINED FOR OVERLAPPING INTERVALS.

method	mean u^*	$u^* \leq 0.1$	$u^* \geq 0.9$
convex	**0.59834**	**0.02364%**	**0.12017%**
UW $\alpha = 1$	0.57580	0.02938%	0.11191%
UW $\alpha = 2$	0.54080	0.04865%	0.10628%

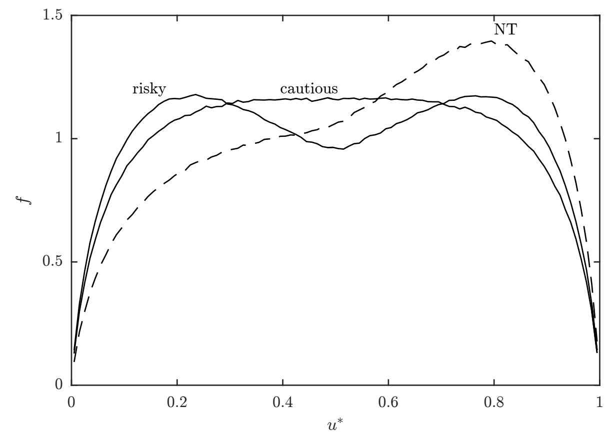

Fig. 8. Utility distribution for inclusive intervals for the risky, cautious, and Nie–Tan (NT) methods.

yields almost the same results. UW with higher values of α yields lower mean, more very low and less very high utilities.

VI. Comparing Inclusive Intervals

Our third set of experiments uses inclusive intervals as shown in the right of Fig. 3. Here, the considered methods yield significantly different utility distributions. The distributions for the risky, cautious, and Nie–Tan (NT, dashed) methods are shown in Fig. 8. The risky and cautious methods yield distributions that are approximately symmetric with respect to the axis $u^* = 0.5$. The distribution for the risky method has two maxima at $u^* \approx 0.2$ and $u^* \approx 0.8$, and the distribution for the cautious method is quite flat in $u^* \in [0.3, 0.7]$. The NT method (dashed) yields significantly more higher and less lower utilities, and the distribution has its maximum at $u^* \approx 0.8$.

The risky, cautious, and NT methods are all instances of the consistent linear type reduction (CLTR) for $a = 0$, $a = 1$, and $a = 0.5$, respectively. Fig. 9 shows the distributions obtained by two more instances of CLTR for $a = 0.3$ and $a = 0.7$, about half way between the special cases $a = 0$, 0.5, and 1, in comparison with NT (dashed). For $a = 0.3$ we obtain slightly more very high utilities at the cost of of slightly more very low utilities, and for $a = 0.7$ we obtain slightly less very low utilities at the cost of of slightly less very high utilities.

Fig. 10 shows the distributions obtained by the consistent quadratic type reduction (CQTR) for $a \in \{0, 0.5, 1\}$. In all three cases, CQTR yields less utilities > 0.5 and more utilities < 0.5 than NT, similar to the risky method in Fig. 8. However, for $a = 0.5$ CQTR yields slightly less very low utilities ($u^* \leq 0.1$) than NT.

Fig. 11 shows the result of the uncertainty weight method (UW) with $\alpha = 0.2$, which out of several tested choices for α visually yields the lowest utilities for low u^* and the highest utilities for high u^*. Compared with NT, UW yields slightly less very low utilities at the cost of slightly less very high utilities.

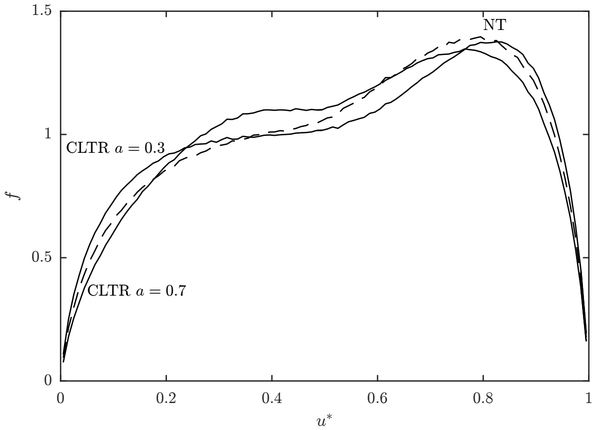

Fig. 9. Utility distribution for inclusive intervals for consistent linear type reduction (CLTR).

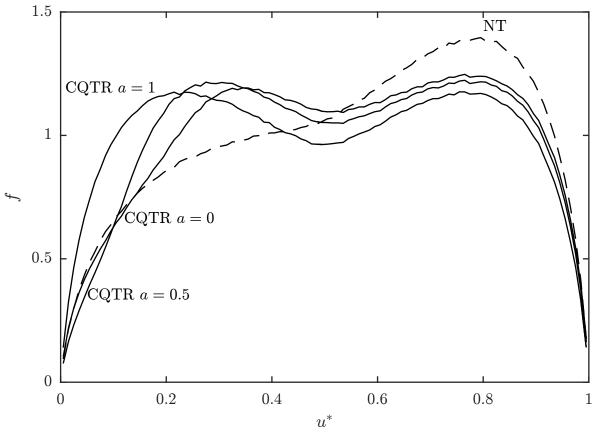

Fig. 10. Utility distribution for inclusive intervals for consistent quadratic type reduction (CQTR).

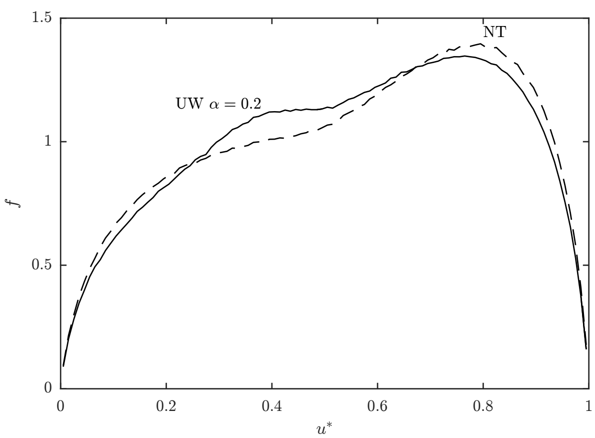

Fig. 11. Utility distribution for inclusive intervals for the uncertainty weight method (UW).

TABLE III
UTILITIES OBTAINED FOR INCLUSIVE INTERVALS.

method	mean u^*	$u^* \leq 0.1$	$u^* \geq 0.9$
risky	0.50009	6.4701%	6.4699%
cautious	0.49997	5.7576%	5.7518%
NT	**0.55250**	4.2540%	7.9728%
CLTR $a = 0.3$	0.54653	4.8214%	**8.4823%**
CLTR $a = 0.7$	0.54652	**3.7435%**	7.4106%
CQTR $a = 0.5$	0.52453	**3.5894%**	7.0388%
UW $\alpha = 0.2$	0.54780	**3.9357%**	7.3379%

Table II shows the mean and the percentages of very low and very high utilities for the different methods for inclusive intervals. The risky and cautious methods yield the lowest mean utility (mean $u^* \approx 0.5$) of all methods considered here, and the NT method yields the highest mean utility (mean $u^* \approx 0.55$). However, the methods listed in the bottom four rows of this table yield very similar mean utilities as NT, but CLTR $a = 0.3$ yields more very high utilities ($u^* \geq 0.9$), and CLTR $a = 0.7$, CQTR $a = 0.5$, and UW $\alpha = 0.2$ yields less very low utilities ($u^* \leq 0.1$) than NT.

VII. Conclusions

In this paper we have experimentally studied the use of different type reduction methods (Nie–Tan, consistent linear type reduction, consistent quadratic type reduction, and the uncertainty weight method) for comparing interval utilities in decision making processes. We have distinguished three relations between pairs of intervals: disjoint, overlapping, and inclusive. In our comparisons we have considered the distribution of the resulting (ground truth) utilities, the mean utilities, and the relative percentages of very low (≤ 0.1) and very high utilities (≥ 0.9). Our experiments show that for disjoint or overlapping pairs of intervals, all considered convex methods yield the same results and outperform the nonconvex uncertainty weight method. However, the different methods yield very different results for inclusive pairs of intervals. Here, the best mean utility is obtained by the Nie–Tan method, the lowest percentages of very low utilities are obtained by consistent linear type reduction with $a = 0.7$, consistent quadratic type reduction with $a = 0.5$, and the uncertainty weight method with $\alpha = 0.2$, and the highest percentage of very high utilities is obtained by consistent linear type reduction with $a = 0.3$. Therefore, our recommendation for comparing intervals using type reduction is as follows: If the focus of the decision making process is on mean utility, then we recommend the Nie–Tan method. For more cautious decision making, when very low utilities should be avoided, we recommend consistent linear type reduction with $a = 0.7$ or consistent quadratic type reduction with $a = 0.5$. For more risky decision making with a strong focus on very high utilities we recommend consistent linear type reduction with $a = 0.3$.

In this paper we considered the case of comparing *pairs* of intervals, i.e. we have two decision alternatives. As a future work we are planning to extend this study to a larger number of decision alternatives. In addition, we are planning to support this experimental study with theoretical considerations.

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