Quantitative Study of Fuzzy Logics

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***Abstract*—In this paper, we focus on two main 3-valued logics used by the fuzzy logic community. The Gödel-Dummett logic and the Łukasiewicz one. Both are based on the same language of implication and negation. In both, we consider fragments consisting of formulas formed with one variable. We investigate the proportion of the number of true (or satisfiable) formulas of a certain length n to the number of all formulas of such length. We are especially interested in the asymptotic behavior of this fraction when length n tends to infinity. If the limit exists it is represented by a real number between 0 and 1 which is called the density of truth or the density of SAT. Using the powerful theory of analytic combinatorics, we state several results comparing the density of truth and the density of satisfiable formulas for both Gödel-Dummett and Łukasiewicz logics.**

***Index Terms*—Fuzzy logic, Gödel-Dummett logic, Łukasiewicz logic, analytic combinatorics, generating functions, asymptotic densities, density of truth**

I. Introduction

Gödel-Dummett and Łukasiewicz 3-valued logics play a crucial role in the mathematics of fuzzy logic. Nice overview and the draught of the history of fuzzy logic is well presented in classical Petr Hájek [9] paper and in the other paper at the same volume by Vilém Novák [18]. Fuzzy logic is used to express facts in which we are able to describe the vagueness phenomenon and the notion of uncertainty treated as degrees of truth. As it is mentioned in Vilém Novák's paper

> *Fuzzy logic is a special many-valued logic addressing the vagueness phenomenon and developing tools for its modeling via truth degrees taken from an ordered scale. It is expected to preserve as many properties of classical logic as possible.*

In this paper we investigate only density problems for 3-valued logics in the simplest case of one variable formulas. We know that formulas expressing even simple fuzzy properties within 3-valued logic may be extremely long. Hence the truth or satisfiability of these formulas is difficult to estimate. Using technics developed in this paper, we may calculate for long formulas the likelihood of being satisfiable or tautology before their actual evaluation.

This paper is a continuation of quantitative research in logic and computability. For a formal logical system equipped with predicate variables and logical connectives we denote by $Form$ the set of all formulas. The set $Form$ is naturally equipped with the standard notion of length of formulas which for a formula $\alpha \in Form$ will be denoted by $l(\alpha)$. One may see that for every $n \in \mathbb{N}$ the set of formulas of length n is finite. This leads to the following definition. For any set of formulas $A \subset Form$ we define its ***asymptotic density***, denoted by $\mu(A)$, as follows:

$$\mu(A) = \lim_{n\to\infty} \frac{|\{\alpha \in A : l(\alpha) = n\}|}{|\{\alpha \in Form : l(\alpha) = n\}|}, \tag{1}$$

where $|B|$ means the cardinality of the finite set B. Note that the density may not exist for some sets of formulas. The number $\mu(A)$ if it exists is an asymptotic probability of finding a formula from the class A among all formulas from $Form$ or it can be interpreted as the asymptotic density of the set A in the set of formulas $Form$. It can be seen immediately that the density μ is finitely additive, so if A and B are disjoint classes of formulas such that $\mu(A)$ and $\mu(B)$ exist then $\mu(A\cup B)$ also exists and $\mu(A \cup B) = \mu(A) + \mu(B)$. It is straightforward to observe that for any finite set A, the density $\mu(A)$ exists and is 0. Conversely for co-finite sets the density also exists and is always 1. The density μ is not countably additive. A good counterexample is to take the denumerably family $\{A_i\}_{i\in\mathbb{N}}$ of singletons of formulas from our language under any natural order of formulas. Obviously $\mu(A_i) = 0$ but $\mu(\bigcup_{i=0}^{\infty} A_i) = 1$. If A is the set of tautologies of a given logic, then $\mu(A)$ is called the ***density of truth*** of this logic.

There are numerous results on the density of truth and other asymptotic properties of logics in literature. In the first place, the density was computed for various fragments of classical propositional logic (see [16], [20], [1] and [7]). The problem of asymptotic equality between density of truth of classical and intuitionistic logic with implication was raised by Moczurad, Tyszkiewicz and Zaionc in [17] and continued by Kostrzycka in [11]. In Kostrzycka paper it was shown that the implicational fragments of classical and intuitionistic logic with two propositional variables are asymptotically different. Later on it was shown by Gardy, Fournier, Genitrini and Zaionc in [5] that the asymptotic difference between classical and intuitionistic logics of implication over the same language with the finite numer of propositional variables tends to zero when the number of variables tends to infinity. A similar research was performed in [14] comparing classical and intuitionistic logics of one variable in the language equipped with implication and negation. Obviously, intuitionistic logic

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INT is a subset of classical one CL. The appropriate densities of truth were obtained analytically. It occurred that numerically $\mu(CL) \approx 0.4232$ whereas $\mu(INT) \approx 0.3953$. So, for the first time we get relative density result stating that for a long implicational-negational classical tautology of one variable there are about 93% chances to have the intuitionistic proof. For more variables the difference between classical and intuitionistic logics of implication and negation is getting smaller as it is proved in [5]. As it was proved in [14] the implicational- negational fragment with one propositional variable of intuitionistic logic is equal to the appropriate fragment of the Gödel-Dummett logic. Furthermore, one may easily prove that the $\rightarrow, \neg$ fragment with one variable of the Gödel-Dummett logic is, in fact, the appropriate fragment of 3-valued linear logic, see [13]. In this paper we shall denote this fragment as G. Then, without getting into details, in [13] we have obtained the density of the set of tautologies of Gödel-Dummett logic and $\mu(E(G)) \approx 0.3953$. And this is our starting point for a new research described in this paper.

II. Novelty of this work

The main novelty of this work is the proof of existence and the exact analytically obtained value of the density of tautologies which is approximately $0,3824$ and the density of satisfiable formulas of about $0,8378$ for fuzzy Łukasiewicz's logic of one variable. We prove that randomly chosen huge formula of Łukasiewicz's logic has quite good chances, almost 84% to be satisfiable. This way we may also compute the relative density of Łukasiewicz logic being a fragment of the classical logic answering the **intriguing question** how big is the fuzzy logic fragment of the classical one. The class consisting of classical tautologies of one variable has a density of approximately 0.4232 obtained in papers [21] and [14]. The density of our class of Łukasiewicz tautologies being a proper subset of classical is approximately 0.3824. So finally, we obtain that the random long formula chosen from the set of classical tautologies has pretty good chances, more than 90% to be provable in Łukasiewicz fuzzy logic.

III. Gödel-Dummett and the Łukasiewicz logics

The Gödel-Dummett and the Łukasiewicz logics are different propositional logics, both being a part of classical logic. They are based on different axiom systems. And they have two different semantics as well. However, as many-valued logics they are both treated as important parts of the so-called **fuzzy logic** technology. The Gödel-Dummett and the Łukasiewicz logics have the Hilbert-style axiomatizations which are based on the axiomatization for the Monoidal T-norm based Logic (or shortly MTL), see [2]. Another important fuzzy logic is the Basic fuzzy Logic (or shortly BL), the logic of continuous t-norms, see [8]. In our research we focus on three valued implicational-negational fragments of the Göodel-Dummett logic and the Łukasiewicz one. Moreover, our approach in counting tautologies will be purely semantic. Let us recall the matrixes of these logics presented at Table I and Table II. We use the symbols $\rightarrow_G$ and $\neg_G$ for the Göodel-Dummett implication and negation while $\rightarrow_L$ and $\neg_L$ is reserved for the appropriate Łukasiewicz connectives. We denote fragments of the considered logics as $G^{\rightarrow,\neg}$ and $Ł^{\rightarrow,\neg}$. The implicational-negational fragment of 3-valued Göodel-Dummett logic is characterized by the matrix $\mathbb{G} = \langle\{0, \frac{1}{2}, 1\}, \rightarrow_G, \neg_G, \{1\}\rangle$, where the operations $\{\rightarrow_G, \neg_G\}$ are defined as in Table I. The fragment of 3-valued Łukasiewicz logic is characterized by the matrix $\mathbb{L} = \langle\{0, \frac{1}{2}, 1\}, \rightarrow_L, \neg_L, \{1\}\rangle$, presented in Table II. One may see that the matrixes for implications differ in exactly one place: $\frac{1}{2} \rightarrow_G 0 = 0$, while $\frac{1}{2} \rightarrow_L 0 = \frac{1}{2}$.

TABLE I
Truth table of 3-valued Göodel-Dummett logic

$\rightarrow_G$	0	$\frac{1}{2}$	1	$\neg_G$
0	1	1	1	1
$\frac{1}{2}$	0	1	1	0
1	0	$\frac{1}{2}$	1	0

TABLE II
Truth table of 3-valued Łukasiewicz logic

$\rightarrow_L$	0	$\frac{1}{2}$	1	$\neg_L$
0	1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1	0

This difference implies the difference for negations: $\neg_G(\frac{1}{2}) = 0$ and $\neg_L(\frac{1}{2}) = \frac{1}{2}$ since negation $\neg_G p$ is originally defined in fuzzy logic as $p \rightarrow_G \bot$, an analogous definition exists for $\neg_L$. Other standard functors of conjunction and disjunction are defined in the same way: $p \wedge q = \min\{p, q\}$ and $p \vee q = \max\{p, q\}$ (we omit here the subscripts). The considered logics treated as parts of the fuzzy logic may be enriched with other functors i.e. strong conjunction. However, in our paper we shall focus on the sets of implicational-negational formulas. It is denoted as $Form^{\rightarrow,\neg}$.

Example 1. *Gödel-Dummett and Łukasiewicz logics are not identical. For example the formula* $(\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p)$ *is a tautology of* $Ł^{\rightarrow,\neg}$*, but is not a tautology of* $G^{\rightarrow,\neg}$ *and on the contrary the formula* $\neg p \rightarrow \neg(\neg p \rightarrow p)$ *is a tautology of* $G^{\rightarrow,\neg}$ *but is not a tautology of* $Ł^{\rightarrow,\neg}$.

IV. Fragment of Łukasiewicz's logic with one variable

The density of truth for the implicational-negational fragment with one variable of the 3-valued Gödel-Dummett logic has been already determined in [13]. We know analytically this quantity. Numerically, it is approximately 0.3953. Below, we present the semantic method of classifying formulas to determine generating functions and to count the density of truth for the Łukasiewicz 3-valued logic. The language of implicational - negational formulas $Form_p^{\rightarrow,\neg}$ of one propositional variable p consists of formulas from $Form^{\rightarrow,\neg}$ built from the single variable p by means of negation and implication only. One variable fragments of 3-valued Gödel-Dummett logic $G^{\to,\neg}$ and Łukasiewicz logic $Ł^{\to,\neg}$ will be denoted respectively by $G_p^{\to,\neg}$ and $Ł_p^{\to,\neg}$. In the set $Form_p^{\to,\neg}$ we introduce an equivalence relation $\equiv$ in the standard way:

Definition 2. *For formulas $\varphi, \psi \in Form_p^{\to,\neg}$ we write $\varphi \equiv \psi$ if both $\varphi \to \psi$ and $\psi \to \varphi$ are tautologies of the of 3-valued Łukasiewicz logic.*

Since the relation $\equiv$ is an equivalence relation it divides the set $Form_p^{\to,\neg}$ into some equivalence classes. Such a quotient algebra of formulas is known as the Tarski-Lindenbaum algebra. Every formula from our language $Form_p^{\to,\neg}$ falls into one of the 12 equivalence classes. Below, we lists the equivalence classes represented by their simplest representatives.

$$
\begin{aligned}
I =& [p]_{\equiv}, \\
II =& [\neg p]_{\equiv}, \\
III =& [\neg p \to p]_{\equiv}, \\
IV =& [p \to \neg p]_{\equiv}, \\
V =& [(\neg p \to p) \to p]_{\equiv}, \\
VI =& [\neg(\neg p \to p)]_{\equiv}, \\
VII =& [\neg(p \to \neg p)]_{\equiv}, \\
VIII =& [(\neg p \to p) \to \neg(p \to \neg p)]_{\equiv}, \\
IX =& [\neg((\neg p \to p) \to p)]_{\equiv}, \\
X =& [((\neg p \to p) \to p) \to \neg((\neg p \to p) \to p)]_{\equiv} \\
\mathbf{T} =& [p \to p]_{\equiv}, \\
\mathbf{0} =& [\neg(p \to p)]_{\equiv}.
\end{aligned}
$$

One may see that the class **T** is the class of tautologies of $Ł_p^{\to,\neg}$ whereas class denoted as **0** is the class of contra tautologies. We may observe that all classes except IX and obviously **O** are satisfiable meaning that all formulas from those classes are satisfiable. We define semantic operations $\{\to, \neg\}$ on these classes by $[\alpha]_{\equiv} \to [\beta]_{\equiv} = [\alpha \to \beta]_{\equiv}$ and $\neg[\alpha]_{\equiv} = [\neg\alpha]_{\equiv}$. We may displayed these operations in the 12×12 truth table presented in Table III. The order on classes of equivalence is defined as $[\alpha]_{\equiv} \leq [\beta]_{\equiv}$ iff $[\alpha \to \beta]_{\equiv} = \mathbf{T}$. It forms the following diagram of a distributive lattice with the class of tautologies **T** being on the top (see Fig. 1.) This lattice is denoted as $\mathfrak{L}_{12}$. From the finiteness of the Tarski-Lindenbaum algebra presented above, we see that the logic $Ł_p^{\to,\neg}$ is locally finite (locally tabular). Therefore we are able to use the proof technique developed in [12] which is summarized in Theorem 7, Corollary 8. Using this in Section VIII we shall prove the existence of the asymptotic density for the set of tautologies and satisfiable formulas for the Łukasiewicz fuzzy logic of implication and negation with one variable.

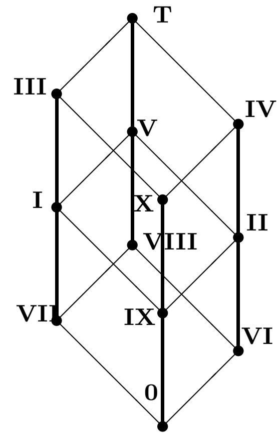

Fig. 1. The lattice $\mathfrak{L}_{12}$ of equivalence classes.

TABLE III
TARSKI-LINDENBAUM 12×12 TRUTH TABLE ALGEBRA.

$\to$	**0**	I	II	III	IV	V	VI	VII	VIII	IX	X	**T**	$\neg$
0	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**T**
I	II	**T**	IV	**T**	IV	**T**	II	V	V	IV	IV	**T**	II
II	I	III	**T**	III	**T**	**T**	V	I	V	III	III	**T**	I
III	VI	V	II	**T**	IV	V	VI	VIII	VIII	II	IV	**T**	VI
IV	VII	I	V	III	**T**	V	VIII	VII	VIII	I	III	**T**	VII
V	IX	III	IV	III	IV	**T**	II	I	V	X	X	**T**	IX
VI	III	III	**T**	III	**T**	**T**	**T**	III	**T**	III	III	**T**	III
VII	IV	**T**	IV	**T**	IV	**T**	IV	**T**	**T**	IV	IV	**T**	IV
VIII	X	III	IV	III	IV	**T**	IV	III	**T**	X	X	**T**	X
IX	V	**T**	**T**	**T**	**T**	**T**	V	V	V	**T**	**T**	**T**	V
X	VIII	V	V	**T**	**T**	V	VIII	VIII	VIII	V	**T**	**T**	VIII
T	**0**	I	II	III	IV	V	VI	VII	VIII	IX	X	**T**	**0**

V. GENERATING FUNCTIONS

We shall investigate the ratio of formulas that are tautologies or satisfiable among all formulas of size n. Our interest lays in finding the limit of this fraction when n grows to infinity which will be called density. For this purpose analytic combinatorics has developed an exceptionally efficient formal apparatus, in the form of analytic generating functions. A nice exposition of the method can be found in Wilf [19], or in Flajolet and Sedgewick [3]; see also Gardy [6, Section 5.2] for a systematic application of these techniques to the computation of probability distributions for Boolean functions.

Let $(a_0, a_1, a_2, \dots)$ be a sequence of real numbers. The *ordinary generating series* for the sequence (a_n) is the formal power series

$$\sum_{n=0}^{\infty} a_n z^n \ .$$

Obviously, formal power series are in one-to-one correspondence to sequences. However, considering z as a complex variable, this series, as it is known from the theory of analytic functions, converges uniformly to a function $f(z)$ in some open disc $\{z \in \mathbb{C} : |z| < R\}$ of maximal diameter, and $R > 0$ is called its radius of convergence. So, when $R > 0$, we can associate with the sequence (a_n) a complex function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, called the *ordinary generating function* for (a_n), defined in a neighbourhood of 0. In the other way, as is well known from the theory of analytic functions, the expansion of a complex function $f(z)$, analytic in a neighbourhood of z_0, into a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is unique. For a function $g(z)$ analytic in a neighbourhood of 0, we shall denote by $[z^n]g$ the coefficient of z^n in the series expansion of g in 0.

Many questions concerning the asymptotic behaviour of the sequence (a_n) can be precisely determined and resolved by analyzing the behaviour of $\sum a_n z^n$ at the complex circle $|z| = R$. The approach we take to determine the size of asymptotic fraction of tautologies (or satisfiable formulas or other classes of formulas) among all formulas of the fuzzy logic under consideration.

VI. Counting formulas and the basic generating function

The length of a formula from the set $Form_p^{\to,\neg}$ is defined as follows:

Definition 3.

$$l(p) = 1, \;\; l(\neg\phi) = l(\phi) + 1, \;\; l(\phi \to \psi) = l(\phi) + l(\psi) + 1$$

In fact, this definition reflects the size of the binary-unary Motzkin tree, which is the number of internal nodes and leafs of this tree. We may notice that for any positive integer n the number of formulas $\phi \in Form_p^{\to,\neg}$ such that $l(\phi) = n$ is finite. By F_n we mean the finite set of formulas from $Form_p^{\to,\neg}$ of the length $n-1$ and, by $|F_n|$ we mean the cardinality of F_n. The sequence $|F_n|$ enumerates the Motzkin trees. We will also consider several subclasses of F_n. For any $B \subset Form_p^{\to,\neg}$, we take $B_n = B \cap F_n$ and by $|B_n|$, we denote the cardinality of the class B_n.

Lemma 4. *The generating function f for the sequence $|F_n|$ is the following:*

$$f(z) = \frac{1-z}{2} - \frac{\sqrt{(z+1)(1-3z)}}{2}. \qquad (2)$$

Proof. The numbers $|F_n|$ of formulas from F_n are given by the recursion:

$$|F_0| = 0, |F_1| = 0, |F_2| = 1, \qquad (3)$$

$$|F_n| = |F_{n-1}| + \sum_{i=1}^{n-1} |F_i||F_{n-i}|. \qquad (4)$$

Let us emphasize that the number of formulas of length $n-1$ is $|F_n|$. Any formula of length $n-1$ for $n > 2$ is either a negation of some formula of length $n-2$ for which the fragment $|F_{n-1}|$ is responsible, or is the implication between some pair of formulas of lengths $i-1$ and $n-i-1$, respectively. The length of any of such implicational formulas must be $(i-1)+(n-i-1)+1$ which is exactly $n-1$. Therefore the total number of such formulas is $\sum_{i=1}^{n-1} |F_i||F_{n-i}|$. The recurrence $|F_n| = |F_{n-1}| + \sum_{i=1}^{n-2} |F_i||F_{n-i}|$ becomes the equality

$$f(z) = zf(z) + f^2(z) + z^2$$

since the recursion fragment $\sum_{i=1}^{n-2} |F_i||F_{n-i}|$ exactly corresponds to the multiplication of power series. The term $|F_{n-1}|$ corresponds to the function $zf(z)$. The quadratic term z^2 corresponds to the first non-zero coefficient in the power series of $f(z)$. After solving this quadratic equation we get two solutions:

$$f(z) = \frac{1-z}{2} + \frac{\sqrt{(z+1)(1-3z)}}{2}$$
$$f(z) = \frac{1-z}{2} - \frac{\sqrt{(z+1)(1-3z)}}{2}.$$

With the boundary condition: $f(0) = 0$ we obtain (2). □

The main singularity of the function f is $\rho = \frac{1}{3}$. The expansion of f around ρ is the following:

$$f(z) = \frac{1}{3} - \frac{1}{\sqrt{3}}\sqrt{1-3z} + O(1-3z). \qquad (5)$$

VII. Solving polynomial systems

In our paper we deal with sequences of numbers of formulas. We consider different classes of formulas (e.g. class of tautologies) and count number of formulas with the established length. To determine limits of such sequences we use generating functions. An comprehensive, presentation of this method can be found, for instance, in [3] and [19]. The following result is known as Drmota-Lalley-Woods theorem; see [3], Thm. 8.13, p.71. It is the key result to compute the densities of the classes of formulas being represented by the generating functions f_j.

Theorem 5. *Consider a nonlinear polynomial system, defined by a set of equations. Unknown functions $f_1, \ldots, f_m$ are mutually dependent on all other functions in the system and the $i-th$ dependency is named Φ_i .*

$$\{f_i = \Phi_i(z, f_1, ..., f_m)\}, \quad 1 \le i \le m$$

which is a-proper, a-positive and a-irreducible. Then

1) *All component solutions f_i for $1 \le i \le m$ have the same radius of convergence $\rho < \infty$.*
2) *There exist functions h_j analytic at the origin such that*

$$f_j(z) = h_{j0} + h_{j1}(\sqrt{1-z/\rho}) + h_{j2}(1-z/\rho) + h_{j3}(\sqrt{1-z/\rho})^3 + \cdots,$$

   where $h_{j1} \neq 0$ and $f_j(w) = h_{j0} + h_{j1}w + h_{j2}w^2 + ...$ are analytic in a neighborhood of $w = 0$,
3) *All other dominant singularities are of the form $\rho\omega$ with ω being a root of unity.*
4) *If the system is a-aperiodic then all f_j have ρ as the unique dominant singularity. In that case, the coefficients admit a complete asymptotic expansion of the form (6) below.*

$$[z^n]f_j(z) = \frac{-h_{j1}}{2\sqrt{\pi\tau}} \frac{\tau^{\frac{n}{2}}}{n^{\frac{3}{2}}} \left(1 + \sum_{i=1}^{s} \frac{c_i}{n^i} + O_s\left(\frac{1}{n^{s+1}}\right)\right) \qquad (6)$$

where $\tau = \rho^{-1}$.

The formula (6) is obtained from the Darboux lemma (see [19] and [20]). It is transformed to a formula approximating the value of the coefficients $[z^n]f_j(z)$. This transformation is known as the transfer lemma (see [4]). Let f_A and f_F be the generating functions determined by a set A of formulas and the set of all formulas, correspondingly. Suppose that they have the same dominant singularity ρ and there are suitable constants α_1, α_2, β_1, β_2 such that:

$$f_A(z) = \alpha_1 - \beta_1\sqrt{1-z/\rho} + O(1-z/\rho), \tag{7}$$

$$f_F(z) = \alpha_2 - \beta_2\sqrt{1-z/\rho} + O(1-z/\rho). \tag{8}$$

From (6) we obtain that

$$\frac{[z^n]f_A(z)}{[z^n]f_F(z)} = \frac{\frac{\beta_1}{2\sqrt{\pi\tau}}\frac{\tau^{\frac{n}{2}}}{n^{\frac{3}{2}}}\left(1+O\left(\frac{1}{n}\right)\right)}{\frac{\beta_2}{2\sqrt{\pi\tau}}\frac{\tau^{\frac{n}{2}}}{n^{\frac{3}{2}}}\left(1+O\left(\frac{1}{n}\right)\right)} \sim \frac{\beta_1}{\beta_2} \tag{9}$$

where $\tau = \rho^{-1}$. Then the asymptotic density of the class A is given by:

$$\mu(A) = \lim_{n\to\infty}\frac{[z^n]f_A(z)}{[z^n]f_F(z)} = \frac{\beta_1}{\beta_2}. \tag{10}$$

We will apply this approach in the next part of our paper.

VIII. Partial results for quotient algebras

For technical reasons we shall consider three quotient algebras obtained from $\mathfrak{L}_{12}$ by appropriate identifications given as projections of the full 12 element poset $\mathfrak{L}_{12}$ from Fig. 1. First identification is the following:

$$\mathbf{0}\cup VI = Z, \qquad VII\cup VIII = R$$
$$II\cup IX = S, \qquad I\cup V = U$$
$$IV\cup X = W, \qquad \mathbf{T}\cup III = J$$

This identification is in fact the projection of the poset $\mathfrak{L}_{12}$ in Fig. 1 to the right back plane surface. In this way we receive two dimensional six-element lattice $\mathfrak{L}_6$, see Fig. 2. Then we make the further identification gluing: $WJ = W\cup J$, $SU = S\cup U$, and $ZR = Z\cup R$ and obtain three-element one dimensional lattice $\mathfrak{L}_3$ also shown in Fig. 2. By the global identification of $WSZ = W\cup S\cup Z$ and $JUR = J\cup U\cup R$ we obtain minimal two element lattice. The lattice operations of implication and negation $\{\neg,\to\}$ induced on new classes in the new posets are given by the following truth tables. In Table VI we may recognize the truth table for classical logic with the class JUR standing for *truth* and class WSZ standing for *falsity*. The detailed quantitative analysis of the density of truth may be found in [15], [11] and [21]. Similarly, Table V is the truth table, already seen, for the Łukasiewicz logic.

TABLE IV
TRUTH TABLE FOR INDUCED 6 ELEMENT QUOTIENT ALGEBRA.

→	Z	R	S	U	W	J	¬
Z	J	J	J	J	J	J	J
R	W	J	W	J	W	J	W
S	U	U	J	J	J	J	U
U	S	U	W	J	W	J	S
W	R	R	U	U	J	J	R
J	Z	R	S	U	W	J	Z

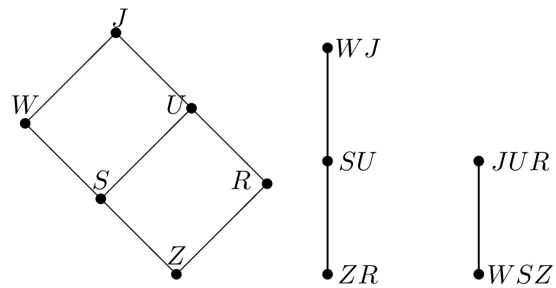

Fig. 2. Three projections of the lattice $\mathfrak{L}_{12}$.

TABLE V
TRUTH TABLE FOR INDUCED 3 ELEMENT QUOTIENT ALGEBRA.

→	ZR	SU	WJ	¬
ZR	WJ	WJ	WJ	WJ
SU	SU	WJ	WJ	SU
WJ	ZR	SU	WJ	ZR

A. Solving classical logic of implication and negation

We will solve successively the systems of functional equations. From now on we will use the following notation. Functions calculated earlier and having an analytic form will be marked with boldface letters with subscripts denoting the class of formulas. Unknown functions in the system of equations are marked with regular characters also with subscripts denoting the class of formulas.

Lemma 6. *Generating functions for two complementary classes WSZ and JUR are the following:*

$$f_{WSZ}(z) = \frac{-1+\boldsymbol{f}-z}{2} + \frac{\sqrt{(1-\boldsymbol{f}+z)^2-4\boldsymbol{f}z}}{2} \tag{11}$$

$$f_{JUR}(z) = \frac{1+\boldsymbol{f}+z}{2} - \frac{\sqrt{(1-\boldsymbol{f}+z)^2-4\boldsymbol{f}z}}{2} \tag{12}$$

Proof. Notice that Table VI describes the classical implicational-negational logic. Therefore the direct translation of recursion

$$|WSZ_0| = 0,\ \ |WSZ_1| = 0,\ \ |WSZ_2| = 0, \tag{13}$$

$$|JUR_0| = 0,\ \ |JUR_1| = 0,\ \ |JUR_2| = 1, \tag{14}$$

$$|WSZ_n| = |JUR_{n-1}| + \sum_{i=1}^{n-2}(|JUR_i|)\cdot|WSZ_{n-i}|, \tag{15}$$

becomes the equation $f_{WSZ} = z\cdot f_{JUR} + f_{JUR}\cdot f_{WSZ}$. Since $f_{JUR} = \boldsymbol{f} - f_{WSZ}$ we get the quadratic equation $f_{WSZ} = (\boldsymbol{f}-f_{WSZ})\cdot f_{WSZ} + z\cdot(\boldsymbol{f}-f_{WSZ})$. Solving it with the boundary condition $f_{WSZ}(0) = 0$ we obtain an analytic

TABLE VI
TRUTH TABLE FOR INDUCED 2 ELEMENT QUOTIENT ALGEBRA.

→	WSZ	JUR	¬
WSZ	JUR	JUR	JUR
JUR	WSZ	JUR	WSZ

formula enumerating the class WSZ. The complement of it is the function $f_{JUR}(z) = \boldsymbol{f}(z) - f_{WSZ}(z)$. For more elaborated treatment of counting formulas in classical logic see [21] and [15]. □

B. Solving 3 element lattice

Let us introduce the symbols: $WJ_n = WJ \cap F_n$, $SU_n = SU \cap F_n$ and $ZR_n = ZR \cap F_n$. From Table V we obtain following recursions:

Lemma 7. *The numbers $|SU_n|$ are given by the following recursion*

$$|SU_0| = 0, \ \ |SU_1| = 0, \ \ \ |SU_2| = 1, \tag{16}$$

$$|SU_n| = |SU_{n-1}| + \sum_{i=1}^{n-1} |SU_i||ZR_{n-i}| + + \sum_{i=1}^{n-1} |WJ_i||SU_{n-i}|. \tag{17}$$

Proof. This recursion is a direct translation of Table V. □

Lemma 8. *The generating functions f_{SU} for the numbers $|SU_n|$ is the following:*

$$f_{SU} = \frac{1}{4}\left(-1 - X + z + \sqrt{2}Y\right) \tag{18}$$

where $X = \sqrt{1 - 2z - z^2}$ $\ \ Y = \sqrt{1 + X - 2z - Xz + 7z^2}$

Proof. From recursion (16), and (17) we get that the generating function f_{SU} fulfils the equation:

$$f_{SU} = zf_{SU} + f_{SU}f_{ZR} + f_{WJ}f_{SU} + z^2.$$

Then: $f_{SU} = zf_{SU} + f_{SU}(f_{ZR} + f_{WJ}) + z^2$. Because $f_{ZR} + f_{WJ} = \boldsymbol{f} - f_{SU}$ we get again a quadratic equation with respect to unknown function f_{SU}.

$$f_{SU} = zf_{SU} + f_{SU}(\boldsymbol{f} - f_{SU}) + z^2.$$

Solving this equation with the boundary condition $f_{SU}(0) = 0$ we obtain generating function (18). □

Lemma 9. *The numbers $|ZR_n|$ are given by the following recursion*

$$|ZR_0| = |ZR_1| = |ZR_2| = |ZR_3| = |ZR_4| = 0, \tag{19}$$

$$|ZR_5| = 1, \tag{20}$$

$$|ZR_n| = |WJ_{n-1}| + \sum_{i=1}^{n-1} |WJ_i||ZR_{n-i}|. \tag{21}$$

Proof. This recursion is a direct translation of the Table V.

Lemma 10. *The generating functions f_{ZR} for the sequence of numbers $|ZR_n|$ is the following:*

$$f_{ZR} = \frac{1}{8}\left(-1 - X - \sqrt{2}Y - 7z + \sqrt{Z}\right) \tag{22}$$

where both X and Y are functions defined in Lemma 8 and $Z = 1 + X^2 + 2Y^2 + 2X(1 + \sqrt{2}Y - z) - 2\sqrt{2}Y(z-1) + 62z + z^2$.

Proof. From recursion (19), (20) and (21) we get that the generating function f_{ZR} fulfils the equation:

$$f_{ZR} = zf_{WJ} + f_{WJ}f_{ZR}.$$

Then: $f_{ZR} = f_{WJ}(f_{ZR} + z)$. Because of $f_{WJ} = f - f_{SU} - f_{ZR}$ we finally get

$$f_{ZR} = (\boldsymbol{f} - \boldsymbol{f}_{SU} - f_{ZR})(f_{ZR} + z).$$

This is also a quadratic equation with respect to the unknown function f_{ZR}. After solving this equation with the boundary condition $f_{ZR}(0) = 0$ and with the already known function $\boldsymbol{f}_{SU}$ we get (22). □

Since we already have analytic formulas for functions $\boldsymbol{f}_{ZR}$ in (22) and $\boldsymbol{f}_{SU}$ in (18) and because of the global equation $f_{WJ} + \boldsymbol{f}_{SU} + \boldsymbol{f}_{ZR} = \boldsymbol{f}$ we conclude:

Corollary 11. *The function f_{WJ} is the following:*

$$f_{WJ} = \frac{1}{8}\left(7 - X - \sqrt{2}Y + z - \sqrt{Z}.\right) \tag{23}$$

where both X and Y are functions defined in Lemma 8 and Z is function defined in Lemma 10.

Lemma 12. *The generating functions $\boldsymbol{f}_{SU}$, $\boldsymbol{f}_{ZR}$, $\boldsymbol{f}_{WJ}$ have the following expansions around $z_0 = \frac{1}{3}$:*

$$f_{SU} \approx 0,2060 - 0,1595\sqrt{1-3z} + O(1-3z) \tag{24}$$

$$f_{ZR} \approx 0,0342 - 0,1204\sqrt{1-3z} + O(1-3z) \tag{25}$$

$$f_{WJ} \approx 0,0931 - 0,2972\sqrt{1-3z} + O(1-3z) \tag{26}$$

Proof. By the *Mathematica* package. □

From Lemma 12 and formulas (5) and (10) we conclude:

Corollary 13. *The asymptotic densities of the classes SU, ZR and WJ are the following:*

$$\mu(SU) \approx 0,2764, \ \ \mu(ZR) \approx 0,2087, \ \ \mu(WJ) \approx 0,5149$$

C. Solving 6 element lattice

As we see the generating function for the classes ZR and WJ are quite complicated.The whole lattice $\mathfrak{L}_6$ is much more complicated. Analogously as above, from Table IV we obtain the suitable recurrences for the numbers $|Z_n|$, $|R_n|$,$|S_n|$, $|U_n|$, $|W_n|$, $|J_n|$. Then they are translated into the system of the following six functional equations:

$$f_Z = f_J f_Z + zf_J, \tag{27}$$

$$f_R = f_W(f_R + f_Z) + f_J f_R + zf_W, \tag{28}$$

$$f_S = f_U f_Z + f_J f_S + zf_U, \tag{29}$$

$$f_U = f_S(f_Z + f_R) + f_U f_R + f_W(f_S + f_U) + + f_J f_U + zf_S + z^2, \tag{30}$$

$$f_W = f_R(f_Z + f_S + f_W) + f_U(f_S + f_W) + + f_J f_W + zf_R, \tag{31}$$

$$f_J = f_Z(f_Z + f_R + f_S + f_U + f_W + f_J) + + f_R(f_R + f_U + f_J) + f_S(f_S + f_U + f_W + f_J) + f_U(f_R + f_J) + f_W(f_W + f_J) + f_J^2 + zf_Z. \tag{32}$$

Furthermore, we have that:

$$\boldsymbol{f} = f_Z + f_R + f_S + f_U + f_W + f_J, \quad (33)$$
$$\boldsymbol{f_{JUR}} = f_J + f_U + f_R, \quad (34)$$
$$\boldsymbol{f_{WSZ}} = f_W + f_S + f_Z, \quad (35)$$
$$\boldsymbol{f_{ZR}} = f_Z + f_R, \quad (36)$$
$$\boldsymbol{f_{SU}} = f_S + f_U, \quad (37)$$
$$\boldsymbol{f_{WJ}} = f_W + f_J. \quad (38)$$

Application of (33)-(38) into (27)-(32) gives us much simpler system which may be turned after simplifications to just one functional polynomial equation of order 4 with the only one unknown functional variable f_J

$$f_J^4 + f_J^3(-3 - \boldsymbol{f_{WJ}} + \boldsymbol{f_{WSZ}} + z) + \\ + f_J^2(3 + 3\boldsymbol{f_{WJ}} - 3\boldsymbol{f_{WSZ}} - z) + \\ + f_J(-1 - 3\boldsymbol{f_{WJ}} + 3\boldsymbol{f_{WSZ}} - z\boldsymbol{f_{SU}} - z\boldsymbol{f_{WJ}} + z\boldsymbol{f_{WSZ}} + \\ + z^2) + \boldsymbol{f_{WJ}} - \boldsymbol{f_{WSZ}} + z\boldsymbol{f_{SU}} + z\boldsymbol{f_{WJ}} - z\boldsymbol{f_{WSZ}} = 0.$$

Solving it with the help of the *Mathematica* package we get analytic formula for f_J and therefore for all functions f_Z, f_R, f_S, f_U and f_W from the 6 element lattice. Anyway, this also leads to extremely complex analytic functions. Even with taking the advantage of the *Mathematica* package, all the needed calculations are not feasible. We may solve also the system numerically. Let us notice that the system of equations (27)-(32) is a-proper, a-positive, a-irreducible and a-aperiodic. So, it is possible to apply the Drmota-Lalley-Woods Theorem 5. Furthermore, all the considered functions have the same dominant singularity $z_0 = \frac{1}{3}$ and there exist their expansions around z_0 in the form:

$$f_i = a_i + b_i\sqrt{1-3z} + O(1-3z). \quad (39)$$

From the system of equations (27)-(32) we are able to count the floating point values of the considered functions at $z_0 = \frac{1}{3}$. We can differentiate all the functions (and the appropriate equations as well) and solve it with respect to the value of the derivatives at $z_0 = \frac{1}{3}$. After numerical computation (20 000 steps) we obtain the following values (rounded to four digits) for the considered functions.

Lemma 14.

$$f_Z(z) \approx 0,0285 - 0,1016\sqrt{1-3z} + O(1-3z),$$
$$f_R(z) \approx 0,0057 - 0,0189\sqrt{1-3z} + O(1-3z),$$
$$f_S(z) \approx 0,0581 - 0,0684\sqrt{1-3z} + O(1-3z)$$
$$f_U(z) \approx 0,1479 - 0,0911\sqrt{1-3z} + O(1-3z),$$
$$f_W(z) \approx 0,0143 - 0,0385\sqrt{1-3z} + O(1-3z),$$
$$f_J(z) \approx 0,0788 - 0,2587\sqrt{1-3z} + O(1-3z).$$

From the above it follows that:

Theorem 15. *The asymptotic densities of the classes of formulas from the lattice $\mathfrak{L}_6$ are the following:*

$$\mu(Z) \approx 0,1760, \quad \mu(R) \approx 0,0327, \quad \mu(S) \approx 0,1185, \quad (40)$$
$$\mu(U) \approx 0,1579, \quad \mu(W) \approx 0,0667, \quad \mu(J) \approx 0,4481. \quad (41)$$

D. Solving the lattice for Łukasiewicz's logic

We repeat the whole procedure for the total lattice for the Łukasiewicz logic drawn in Fig. 1. From Table III we get recurrences for all the numbers $|\mathbf{0}_n|$, $|I_n|$, $|II_n|$, ...,$|X_n|$, $|\mathbf{T}_n|$. Recurrences are transformed into equations on generating functions in the very same way as for 6 element lattice. They form a system of 12 equations, which is a-proper, a-positive, a-irreducible and a-aperiodic. So, we apply the Drmota-Lalley-Woods described in Theorem 5. Also, all the considered functions have the same dominant singularity placed at $z_0 = \frac{1}{3}$ and there exist their expansions around z_0 in the form of (39). From the expansions we obtain:

Theorem 16. *The asymptotic densities of the class of formulas of the lattice $\mathfrak{L}_{12}$ are the following:*

$$\mu(\mathbf{0}) \approx 0,1458, \quad \mu(I) \approx 0,1215, \quad (42)$$
$$\mu(II) \approx 0,1027, \quad \mu(III) \approx 0,0634, \quad (43)$$
$$\mu(IV) \approx 0,0661, \quad \mu(V) \approx 0,0386, \quad (44)$$
$$\mu(VI) \approx 0,0292, \quad \mu(VII) \approx 0,0305, \quad (45)$$
$$\mu(VIII) \approx 0,0023, \quad \mu(IX) \approx 0,0164, \quad (46)$$
$$\mu(X) \approx 0,0011, \quad \mu(\mathbf{T}) \approx 0,3824. \quad (47)$$

Let us emphasize two the most interesting results. The density of tautologies: $\mu(\mathbf{T}) \approx 0,3824$ and the density of satisfiable formulas $\mu(SAT) \approx 0,8378$ (SAT consists of all classes except $\mathbf{0}$ and IX). So randomly chosen huge formula of Łukasiewicz's logic has quite good chances, almost 84% to be satisfiable. This result should be compared with a similar result for fuzzy 3 valued Gödel-Dummett's logic (see [13]), in which we obtained that the density of tautologies is about $0,3953$ while the density of satisfiable formulas is about $0,8364$. We may also compute the relative density of Łukasiewicz logic in the classical logic answering the interesting question.

> *What is the probability that randomly chosen tautology from the classical logic has the Łukasiewicz logic proof or how big is the Łukasiewicz fragment of the classical logic?*

A class consisting of classical tautologies has a density of approximately 0.4232. Our class T of Łukasiewicz tautologies being a proper subset of classical is approximately 0.3824. So, the random long formula chosen from the set of classical tautologies has quite good chances, more than 90% to be provable in Łukasiewicz logic.

IX. Possible use of results

Dealing with extremely long formulas seems to be the everyday challenge for all practitioners of the fuzzy logic system design. They construct systems with the fuzzy description of their behavior. Usually, fuzzy logic formulas describing the meaning and behavior of fuzzy systems are extremely long. So the designer of the fuzzy system may be confronted with the computationally difficult NP-complete (or coNP-complete) problems of finding whether the given fuzzy formula is satisfiable or valid. This paper proposed the kind of probabilistic solution to this problem. Our approach may be helpful by estimating the probability that the formula is satisfiable or also it may determine the probability of the truths of it.

X. Conclusions and problems

A natural continuation of our research is to investigate the implicational-negational fragments of both these logics equipped with k variables. We believe that asymptotically when the number of variables k tends to infinity, the densities of true formulas in the Gödel-Dummett and Łukasiewicz 3 valued logics are equal. More formally, if the density of true formulas with k variables is denoted as $\mu(G_k)$ for Gödel-Dummett's logic and as $\mu(Ł_k)$ for Łukasiewicz's logic and if $\mu(SATG_k)$ is the density of satisfiable formulas for Gödel-Dummett logic and $\mu(SATŁ_k)$ - the one for Łukasiewicz's logic then we believe that

Conjecture 1

$$\lim_{k\to\infty} \frac{\mu(G_k)}{\mu(Ł_k)} = 1, \qquad \lim_{k\to\infty} \frac{\mu(SATG_k)}{\mu(SATŁ_k)} = 1.$$

Thus we believe that for large numbers k of variables there is no substantial difference between use of Gödel-Dummett or Łukasiewicz 3 valued logics for description of fuzzyness. The similar phenomenon appeared when we investigate the relationship between intuitionistic and classical logics of implication with the limited number of variables. It turned out that randomly chosen classically valid formula has asymptotically 100% chances to possess intuitionistic proof. For details consult [5] and [10].

Other natural continuation of our research is changing the level of fuzziness of the logics under consideration. One may consider the implicational-negational fragments of n-valued Łukasiewicz logics, for $n \geq 2$. These logics are characterized by the truth-tables:

$$M(n)_L = \langle \{0, \frac{1}{n-1}, ..., \frac{n-2}{n-1}, 1\}, \to_L, \neg_L, \{1\} \rangle$$

where $p \to_L q = \min\{1, 1-p+q\}$ and $\neg_L p = p \to_L 0$, for any $n \geq 2$. It belongs to logical folklore that n-valued Łukasiewicz's logics form a distributive lattice. We denote these logics as Ł(n). Namely we get:

$$Ł(k) \subset Ł(n) \quad \text{iff} \quad (n-1) \text{ divides } (k-1).$$

Then we get the following sequences of inclusions:

$$... \subset Ł(9) \subset Ł(5) \subset Ł(3), \quad ... \subset Ł(13) \subset Ł(7) \subset Ł(3),$$
$$... \subset Ł(21) \subset Ł(11) \subset Ł(3), \quad ... .$$

We see that all the Łukasiewicz logics with an odd number of logical values are sub-logics of all Ł(3). Then our main result can be expanded also on many other Łukasiewicz logics. We get:

$$\mu(Ł(2n+1)_p^{\to,\neg}) \leq 0,3824 \quad \text{for} \ n \geq 1 .$$

On the other hand, the implicational-negational fragments with one variable of the n-valued Gödel-Dummett logics are identical. Then we also get:

$$\mu(Ł(2n+1)_p^{\to,\neg}) \leq \mu(\mathrm{G}(2n+1)_p^{\to,\neg}) \quad \text{for} \ n \geq 1 .$$

We would like to generalize the above inequalities for all many-valued Łukasiewicz logics.

Conjecture 2

$$\mu(Ł(n)_p^{\to,\neg}) \leq 0,3824 \quad \text{for} \ n \geq 3 .$$

Conjecture 3

$$\mu(Ł(n)_p^{\to,\neg}) \leq \mu(\mathrm{G}(n)_p^{\to,\neg}) \quad \text{for} \ n \geq 3 .$$

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