A new method to measure the knowledge amount of Atanassov's intuitionistic fuzzy sets

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Abstract—**It is of great significance to measure the knowledge amount conveyed by Atanassov's intuitionistic fuzzy sets (AIFSs). Many efforts have been done to define a suitable knowledge measure for AIFSs, or uncertainty measure, named as a dual measure of knowledge measure. However, many of these measures are developed from the view of point of intuitionistic fuzzy entropy, which cannot well reflect the knowledge amount associated with an AIFS. Other knowledge measures developed based on the difference between an AIFS and its complement may lead to information loss in the scenario of decision making. This paper proposed a new knowledge measure for AIFSs. The axiomatic definition of knowledge measure is extended to a more general level. The properties of the new developed knowledge measure are investigated through mathematical analysis and numerical examples. Further discussion on the relation between knowledge measure and entropy measure is proposed to clear up the relation and distinction between them.**

Keywords—Atanassov's intuitionistic fuzzy sets, knowledge measure, uncertainty measure

I. INTRODUCTION

By relaxing the condition that the non-membership degree is necessarily equal to one minus the membership degree, Atanassov [1]–[4] extended Zadeh's fuzzy set [44] to intuitionistic fuzzy set, which is known as Atanassov's intuitionistic fuzzy set (AIFS). For an AIFS, the gap between one and the sum of membership degree and non-membership degree is named as hesitation degree. The introduction of hesitation degree brings much convenience in depicting uncertain information. Since its inception, AIFSs have received much attention from researchers because of its advantage in modelling uncertain information systems. The theory of AIFS is playing an important role in many fields such as intelligent reasoning [10],[25], decision making [9],[11],[38],[42], and so on. The relationship between intuitionistic fuzzy set theory and other theories for uncertainty reasoning such as belief function theory [18],[19],[27],[28],[29],[30],[31],[41],[48] is also attracting more and more interest.

The concept of entropy is firstly introduced to fuzzy set by Zadeh [44] to measure the uncertainty or fuzziness in a fuzzy set. The name of fuzzy entropy for a fuzzy set is partially similar to the concept of Shannon entropy [26] which was initially defined in probability theory. Non-probabilistic entropy was proposed by De Luca and Termini [20], who also developed the axiomatic definition of entropy measure. After Manko's [21] introduction of fuzziness and nonfuzziness for AIFSs, Szmidt and Kacprzyk [32] presented the measure of intuitionistic fuzzy entropy, together with its axiomatic definition. Following the work of Szmidt and Kacprzyk [32], many authors [5],[32],[37],[38],[42],[43] have done a lot work concentrating on the definition of entropy measures. There is also some research focusing on the entropy of AIFSs and its application in the evaluation of attribution weighting vector [38],[42],[43]. It has been pointed out by Szmidt et al. [35] that entropy measure cannot capture all uncertainty hidden in an AIFS. Thus it may be difficult for us to develop a satisfactory uncertainty measure for AIFSs merely by entropy measure. The difference between entropy and hesitation in measuring uncertainty of AIFSs has been realized by Pal et al. [24]. In [24], it was claimed that the combination of entropy and hesitation may furnish an effective way to measure the total uncertainty hidden in an AIFS.

Generally, knowledge measure is related to the useful information provided by an AIFS. From the viewpoint of information theory, much information indicates much amount of knowledge, which is helpful for making a decision. So the measure of knowledge can be regarded as a dual measure of total uncertainty measure, rather than entropy measure. This means that the less total uncertainty always accompanies with greater knowledge amount. With the purpose of making an evident distinction between intuitionistic fuzzy information, Szmidt et al. [35] took both intuitionistic fuzzy entropy and hesitation into consideration to develop a knowledge measure for AIFS, where the intuitionistic fuzzy entropy was defined based on the nearer distance and farer distance. This knowledge measure has been used to estimate attribute weights in multi-attribute decision making (MADM) problem [9]. Nguyen [23] developed a novel knowledge measure from the view of measuring the distance between an AIFS and the most uncertain AIFS. It seems that this knowledge measure can well describe fuzziness and intuitionism in AIFSs. However, the use of normalized Euclidean distance may bring

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another problem that the relation between fussiness and knowledge cannot be completely reflected. Recently, Guo [14] put forward an axiomatic definition for the knowledge measure of AIFS. In [14], a new model with great robustness was introduced to measure the knowledge amount associated with an AIFS. The new model proposed by Guo [14] is based on the difference between an AIFS and its complement, which has been widely used to defined entropy measure for AIFSs [22],[38],[40]. Moreover, the combination of the two parts in Guo's model [14] is lack of clear physical interpretation. Recently, Das el al. [12] made a comprehensive review on axiomatic definitions of information measures of AIFSs and investigated their relationships, where entropy measure, knowledge measure, distance measure and similarity measure are all concerned.

Above analysis demonstrates that the knowledge measure of AIFSs is still an open topic attracting much attention. Most research on knowledge and uncertainty measures of AIFSs mainly focus on the difference between AIFS and its complement. Only few knowledge measures are defined based on the distance between an AIFS and the most uncertain one. Although, Nguyen [23] opened up this new way to study knowledge measure of AIFSs, further exploration is needed to improve this kind of knowledge measure and get a desirable knowledge measure for AIFSs. This motivates us to present a new knowledge measure for AIFSs. Axiomatic definition of knowledge measure of AIFSs will also be formulated in a more general view of point. Moreover, the properties of the proposed knowledge measure will be further investigated. To illustrate the performance our proposed knowledge measure, we compare it with other measures based on numerical examples.

The rest parts of this paper are organized as following. Some concepts related to AIFSs are introduced in Section II. A new distance measure for AIFSs is developed in Section III, which is followed by the proposal and discussion of distance-based knowledge measure in Section IV. In section V, this paper is summarized.

II. Preliminaries

In this section, we briefly recall some basic knowledge related to AIFSs to facilitate subsequent exposition.

Definition 2.1. [44] Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, then a fuzzy set A in X is defined as follows:

$$A=\{\langle x,\mu_A(x)\rangle | x\in X\} \quad (1)$$

where $\mu_A(x): X\to[0,1]$ is the membership degree.

Definition 2.2. [1] An intuitionistic fuzzy set A in X defined by Atanassov can be written as:

$$A=\{\langle x,\mu_A(x),v_A(x)\rangle | x\in X\} \quad (2)$$

where $\mu_A(x): X\to[0,1]$ and $v_A(x): X\to[0,1]$ are membership degree and non-membership degree, respectively, with the condition:

$$0\le \mu_A(x)+v_A(x)\le 1 \quad (3)$$

The hesitation degree of AIFs *A* defined in X is denoted as $\pi_A(x)$. It is determined by the following expression:

$$\pi_A(x)=1-\mu_A(x)-v_A(x) \quad (4)$$

Apparently, we can get $\pi_A(x)\in[0,1]$, $\forall x\in X$. $\pi_A(x)$ is also called the intuitionistic index of x to A. Greater $\pi_A(x)$ indicates more vagueness. Obviously, when $\pi_A(x)=0$, $\forall x\in X$, the AIFS degenerates into an ordinary fuzzy set.

Generally, the couple $\langle\mu_A(x),v_A(x)\rangle$ is also called an intuitionistic fuzzy value (IFV) for clarity. In the following, we use $AIFSs(X)$ to denote the set of all AIFSs in X.

It is worth noting that besides Definition 2.2, there are other possible representations of AIFSs proposed in the literature. Hong and Kim [29] proposed to use an interval representation $[\mu_A(x),1-v_A(x)]$ instead of the couple $\langle\mu_A(x),v_A(x)\rangle$ to express Atanassov's intuitionistic fuzzy set A in X. This approach is equivalent to the interval valued fuzzy sets interpretation of AIFS, where $\mu_A(x)$ and $1-v_A(x)$ represent the lower bound and upper bounds of membership degree, respectively. Obviously, $[\mu_A(x),1-v_A(x)]$ is a valid interval, since $\mu_A(x)\le 1-v_A(x)$ always holds for $\mu_A(x)+v_A(x)\le 1$.

Definition 2.3. **[1]** For $A\in AIFSs(X)$ and $B\in AIFSs(X)$, some relations between them are defined as:

(R1) $A\subseteq B$ iff $\forall x\in X$ $\mu_A(x)\le\mu_B(x), v_A(x)\ge v_B(x)$;

(R2) $A=B$ iff $\forall x\in X$ $\mu_A(x)=\mu_B(x), v_A(x)=v_B(x)$;

(R3) $A^C=\{\langle x,v_A(x),\mu_A(x)\rangle | x\in X\}$,where A^C is the complement of A.

Definition 2.4. For two IFVs $a=\langle\mu_a,v_a\rangle$, $b=\langle\mu_b,v_b\rangle$, the partial order between them is defined as: $a\le b \Leftrightarrow \mu_a\le\mu_b, v_a\ge v_b$.

Based on the partial order, in the space of IFV, we can get the smallest IFV as $\langle 0,1\rangle$, denoted by **0,** and the largest IFV is $\langle 1,0\rangle$, denoted by **1**.

For a linear order of IFVs, Chen and Tan [8] defined the score function of IFV as $S(a)=\mu_a-v_a$ to rank multiple IFVs. Following the introduction of score function, Hong and Choi [15],[25] developed an accuracy function $H(a)=\mu_a+v_a$ to evaluate the accuracy of an IFV. Then, Xu and Yager [39] proposed a linear order relation between IFVs. For two IFVs $a=\langle\mu_a,v_a\rangle$ and $b=\langle\mu_b,v_b\rangle$, we have:

If $S(a)>S(b)$, then $a>b$;

If $S(a)=S(b)$, $H(a)=H(b)$, then $a=b$;

If $S(a)=S(b)$, $H(a)>H(b)$, then $a>b$.

Another two important concepts for AIFSs are the distance measure and similarity measure, which are usually used to compare the intuitionistic fuzzy information.

Definition 2.5. If a mapping $K: AIFS\to[0,1]$ satisfies the following properties, it is called a knowledge measure of an AIFS *A* defined in $X=\{x_1,x_2,\cdots,x_n\}$.

(KP1) $K(A)=1$ if and only if *A* is a crisp set.

(KP2) $K(A)=0$ if and only if $\pi_A(x_i)=1$, $\forall i\in\{1,2,\cdots,n\}$.

(KP3) $K(A)$ is increasing with $|\mu_A(x_i)-v_A(x_i)|$ and decreasing with $\pi_A(x_i)$, $i=1,2,\cdots,n$.

(KP4) $K(A^C)=K(A)$.

III. A New Knowledge Measure for AIFSs

For an AIFS $A=\{\langle x,\mu_A(x),v_A(x)\rangle | x\in X\}$ defined in $X=\{x_1,x_2,\cdots,x_n\}$, its knowledge amount can be measured by:

$$K^I(A)=\frac{\sqrt{3}}{2n}\sum_{i=1}^{n}\sqrt{(\mu_A(x_i)-v_A(x_i))^2+\frac{1}{3}(\mu_A(x_i)+v_A(x_i))^2} \quad (5)$$

Theorem 3.1. For an AIFS A defined in $X=\{x_1,x_2,\cdots,x_n\}$, the function $K^I(A)$ defined by Eq. (5) is a knowledge measure of AIFS A.

Proof.

To be a strict knowledge measure of AIFSs, $K^I(A)$ defined in Eq. (5) must satisfy all axiomatic properties defined in Definition 2.5.

(KP1) Let *A* be a crisp set. Then we have $\mu_A(x_i)=1$, $v_A(x_i)=0$ or $\mu_A(x_i)=0$, $v_A(x_i)=1$, $i=1,2,\cdots,n$, which implies that $\mu_A(x_i)+v_A(x_i)=1$ and $|\mu_A(x_i)-v_A(x_i)|=1$. Thus, $K^I(A)=1$.

In the condition of $0\le\mu_A(x_i)\le 1$, $0\le v_A(x_i)\le 1$, and $0\le\mu_A(x_i)+v_A(x_i)\le 1$, $K^I(A)=1$ can only be obtained in the case of $\mu_A(x_i)+v_A(x_i)=1$ and $|\mu_A(x_i)-v_A(x_i)|=1$, $\forall i\in\{1,2,\cdots,n\}$. This indicates that $\forall i\in\{1,2,\cdots,n\}$, $\mu_A(x_i)=1$, $v_A(x_i)=0$ or $\mu_A(x_i)=0$, $v_A(x_i)=1$, which means that *A* is a crisp set.

Hence, $K^I(A)=1$ if and only if *A* is a crisp set.

(KP2) In the case where $\forall i\in\{1,2,\cdots,n\}$, $\pi_A(x_i)=1$, we have $\mu_A(x_i)=v_A(x_i)=0$, $\forall i\in\{1,2,\cdots,n\}$. Then $K^I(A)=0$ can be gotten by Eq. (5).

The form of Eq. (5) indicates that only in the case of $(\mu_A(x_i)-v_A(x_i))^2=0$ and $(\mu_A(x_i)+v_A(x_i))^2$ $\forall i\in\{1,2,\cdots,n\}$, can we get $K^I(A)=0$. This implies that: $\mu_A(x_i)-v_A(x_i)=0$ and $\mu_A(x_i)+v_A(x_i)=0$, $\forall i\in\{1,2,\cdots,n\}$. So we have $\mu_A(x_i)=v_A(x_i)=0$, $\pi_A(x_i)=1$, $\forall i\in\{1,2,\cdots,n\}$.

So $K^I(A)$ complies with the property of KP2.

(KP3) The expression of $K^I(A)$ can be rewritten as:

$$K^I(A)=\frac{\sqrt{3}}{2n}\sum_{i=1}^{n}\sqrt{|\mu_A(x_i)-v_A(x_i)|^2+\frac{1}{3}(1-\pi_A(x_i))^2}\ .$$

It is explicit that $K^I(A)$ is monotonously increasing with $|\mu_A(x_i)-v_A(x_i)|$.

By $0\le\pi_A(x_i)\le 1$ and $0\le 1-\pi_A(x_i)\le 1$, we can easily prove that $K^I(A)$ is monotonously decreasing with $\pi_A(x_i)$.

Then, $K^I(A)$ complies with the property of KP3.

(KP4) By the definition of A^C, it is evident that $K^I(A^C)=K^I(A)$.

We note that $K^I(A)$ defined in Eq. (5) complies with all properties in the axiomatic definition of knowledge measure, so it is a knowledge measure for AIFSs. □

A. Discuss its connection with fuzzy entropy

We note that in former definitions on knowledge measure, there is a property listed as [14]: $K(A)\ge K(B)$ if A is less fuzzy than B, i.e., $A\subseteq B$ for $\mu_B(x_i)\le v_B(x_i)$ $\forall i\in\{1,2,\cdots,n\}$, or $A\supseteq B$ for $\mu_B(x_i)\ge v_B(x_i)$ $\forall i\in\{1,2,\cdots,n\}$. While in the properties of fuzzy entropy [34], it is stated as: $E(A)\le E(B)$ if A is less fuzzy than B, i.e., $A\subseteq B$ for $\mu_B(x_i)\le v_B(x_i)$ $\forall i\in\{1,2,\cdots,n\}$, or $A\supseteq B$ for $\mu_B(x_i)\ge v_B(x_i)$ $\forall i\in\{1,2,\cdots,n\}$. This seems that knowledge measure behaves dually to the fuzzy entropy. Next, we will take a further investigation on the connection between knowledge measure and fuzzy entropy measure.

For simplicity, let us consider two AIFSs *A*={< *x*,0,0.5>} and *B*={< *x*,0.3,0.5>} defined in *X*={*x*}. We can note that $A\subseteq B$, and $\mu_B(x)<v_B(x)$. According to the axiomatic definition about knowledge measure proposed in [35], the knowledge amount conveyed by *A* is greater than that conveyed by *B*.

In [35], they used the entropy proposed in [34] to define the knowledge measure. For an AIFS A defined in $X=\{x_1,x_2,\cdots,x_n\}$, its fuzzy entropy is defined as [34]:

$$E_{SK}(A)=\frac{Dis(A,A_{near})}{Dis(A,A_{far})}=\frac{\min\{Dis(A,F^*),Dis(A,F_*)\}}{\max\{Dis(A,F^*),Dis(A,F_*)\}} \quad (6)$$

where $F_*=\{\langle x,0,1\rangle|x\in X\}$ and $F^*=\{\langle x,1,0\rangle|x\in X\}$.

Based on the Hamming distance [33], the entropy is calculated as:

$$E_{SK}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min(\mu_A(x_i),v_A(x_i))+\pi_A(x_i)}{\max(\mu_A(x_i),v_A(x_i))+\pi_A(x_i)} \quad (7)$$

Then, the knowledge measure proposed in [35] is defined as:

$$K_{SKB}(A)=1-\frac{1}{2}(E_K(A)+\frac{1}{n}\sum_{i=1}^{n}\pi_A(x_i)) \quad (8)$$

The knowledge measure proposed by Nguyen [23] is:

$$K_N(A)=\frac{1}{n\sqrt{2}}\sum_{i=1}^{n}\sqrt{(\mu_A(x_i))^2+(v_A(x_i))^2+(\mu_A(x_i)+v_A(x_i))^2} \quad (9)$$

The knowledge measure defined by Guo [14] is:

$$K_G(A)=1-\frac{1}{2n}\sum_{i=1}^{n}(1-|\mu_A(x_i)-v_A(x_i)|)(1+\pi_A(x_i)) \quad (10)$$

For *A*={< *x*,0,0.5>} and *B*={< *x*,0.3,0.5>}, we have:

$$E_{SK}(A)=0.5,\ E_{SK}(B)=0.714,\ E_{SK}(A)<E_{SK}(B);$$

That is to say, AIFS *B* is fuzzier than *A*. So according to the axiomatic property presented in [35] and [14], the knowledge amount of *A* is greater than that of *B*. But what is the fact?

According to Eqs. (7) ~ (10) and Eq. (5), we can get:

$$K_{SKB}(A)=0.5,\ K_{SKB}(B)=0.543.$$

$$K_N(A)=0.5,\ K_N(B)=0.7$$

$$K_G(A)=0.625,\ K_G(B)=0.520.$$

$$K^I(A)=0.5,\ K^I(B)=0.436.$$

We notice that $K_{SKB}(A)<K_{SKB}(B)$, $K_N(A)<K_N(B)$, which violates the property that more entropy indicates less knowledge amount. Based on Guo's knowledge measure [14] and our proposed measure, we have $K_G(A)>K_G(B)$ and $K^I(A)>K^I(B)$, which complies with the property. But this cannot demonstrate the necessary connection between knowledge measure and fuzzy entropy measure.

Let us see another example where two AIFSs are *C*={< *x*,0,0.2>} and *D*={< *x*,0.3,0.2>}. It is evident that $C\subseteq D$, $\mu_D(x)>v_D(x)$. This does not satisfy the condition of K_{AIFS} and E_{AIFS}. So the fuzzy entropy and knowledge measure associated to each AIFS are not explicit. We need to calculate them to address the relation between them. So we can get:

$$E_{SK}(C)=0.800,\ E_{SK}(D)=0.875,\ E_{SK}(C)<E_{SK}(D).$$

$$K_{SKB}(C)=0.200,\ K_{SKB}(D)=0.312,\ K_{SKB}(C)<K_{SKB}(D).$$

$$K_N(C)=0.200,\ K_N(D)=0.436,\ K_N(C)<K_N(D).$$

$$K_G(C)=0.280,\ K_G(D)=0.325,\ K_G(C)<K_G(D).$$

$$K^I(C)=0.200,\ K^I(D)=0.265,\ K^I(C)<K^I(D).$$

So we can see that both the fuzzy entropy and knowledge amount of AIFS *C* are less than those of AIFS *D* with respect to all knowledge measures. This cannot be explained by the axiomatic definition proposed by Guo [14]. So it is irrational to regard knowledge measure and fuzzy entropy as dual concepts. However, the property KP3 in Definition 2.5 can be used to explain the case that less entropy implies less knowledge amount. We see that the amount of knowledge is increasing with the difference between membership and non-membership, but it also decreasing with the hesitance degree. Look at AIFSs *C* and *D*, we can see $|\mu_C(x)-v_C(x)|>|\mu_D(x)-v_D(x)|$, $\pi_C(x)>\pi_D(x)$. The different monotonicity on two variables makes it difficult to compare the knowledge amount conveyed by *C* and *D*. And the comparison between them is helpless for investigating the feature of knowledge measure. Moreover, we can regard the property proposed in [14] is a special case of the property KP3 in Definition 2.5. Thus, our proposed axiomatic definition for knowledge measure is more general than that proposed by Guo [14].

The ambiguous relation between fuzzy entropy and knowledge measure is caused by the fact there is another kind of uncertainty related to the lack of knowledge in an AIFS. If we consider the hesitancy and fuzzy entropy simultaneously to construct au uncertainty measure for AIFSs, the knowledge amount conveyed by an AIFS and the uncertainty hidden in it are complementary. Knowledge measure and uncertainty measure are two concepts sit on opposite ends of a spectrum. As is pointed in [35], when the value of hesitancy degree is fixed, knowledge measure behaves oppositely to the fuzzy entropy. So it may be a bit dogmatic to claim that the less fuzzy entropy, the more knowledge amount.

B. Numerical examples

In this sub-section, the performance of the proposed knowledge measure KI will be examined based on some numerical examples.

Example 1. Four AIFSs defined in *X*={*x*} are given as:

$A_1=\{\langle x,0.5,0.5\rangle\}$, $A_2=\{\langle x,0.3,0.3\rangle\}$, $A_3=\{\langle x,0.2,0.2\rangle\}$, $A_4=\{\langle x,0,0\rangle\}$.

The entropy measure presented in [17],[32],[45],[46],[47] cannot discriminate these AIFSs, since these measures are defined based on the difference between membership and non-membership degrees. The membership degree and non-membership degree are identical in these four AIFSs. So they may be considered identically with the maximal entropy, which induces minimal knowledge amount conveyed by them. However, according to our proposed knowledge measure K_I, we can have:

$$K_I(A_1)=0.5,\ K_I(A_2)=0.3,\ K_I(A_3)=0.2,\ K_I(A_4)=0.$$

We see that these four different AIFSs differ greatly from each other from the viewpoint of knowledge amount. This is helpful for handling such extreme cases with identical supporting and opposing degree. From the definition of K_I, we can find that when $\mu_A(x_i)=v_A(x_i)$, $\forall i\in\{1,2,\cdots,n\}$, the calculation of K_I becomes to the following form:

$$K^I(A)=\frac{1}{n}\sum_{i=1}^{n}\mu_A(x_i) \qquad (11)$$

Eq. (11) indicates that the knowledge amount is increasing with the variable $\mu_A(x_i)$ in the condition of $\mu_A(x_i)=v_A(x_i)$, $\forall i\in\{1,2,\cdots,n\}$. This useful feature coincides with intuitive analysis.

To further demonstrate the discriminability of the knowledge measure K^I, we present the following figure to show the value of knowledge amount associated to AIFS *A* defined in *X*={*x*}. The value of $K^I(A)$ is denoted by the color assigned on each point $(\mu_A(x),v_A(x))$ in the simplex. It is shown that the figure is symmetric with respect to the line $\mu_A(x)=v_A(x)$, which illustrates the property of $K^I(A^C)=K^I(A)$. On the symmetric line $\mu_A(x)=v_A(x)$, the rising trend of knowledge amount is clear. As shown in Fig.1, the maximum amount of knowledge is obtained in two points (0,1) and (1,0), and in the point (0,0) the knowledge amount is minimum.

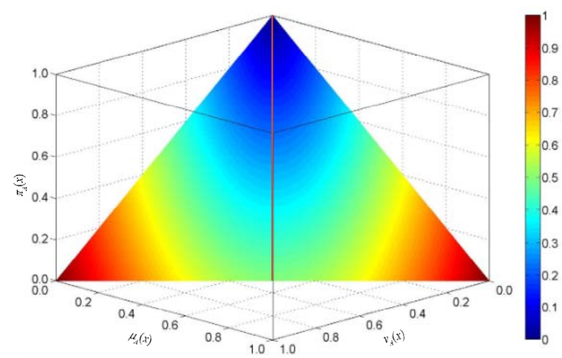

Fig.1. Knowledge amount K_I of AIFSs defined in *X*={*x*}

Example 2. We consider an AIFS *A* defined in *X*={6,7,8,9,10}. The AIFS A is defined as:

$A=\{\langle 6,0.1,0.8\rangle,\langle 7,0.3,0.5\rangle,\langle 8,0.5,0.4\rangle,\langle 9,0.9,0\rangle,\langle 10,1,0\rangle\}$.

De et al. [13] have defined an exponent operations for AIFS A defined in X. Given a non-negative real number *m*, A^m is defined as:

$$A^m=\left\{\left\langle x,(\mu_A(x))^m,1-(1-v_A(x))^m\right\rangle x\in X\right\} \qquad (12)$$

Based on the operations in Eq. (12), we have:

$A^{0.5}=\{\langle 6,0.316,0.553\rangle,\langle 7,0.548,0.293\rangle,\langle 8,0.707,0.225\rangle,\langle 9,0.949,0\rangle,\langle 10,1,0\rangle\}$.

$A^2=\{\langle 6,0.010,0.960\rangle,\langle 7,0.090,0.750\rangle,\langle 8,0.250,0.640\rangle,\langle 9,0.810,0\rangle,\langle 10,1,0\rangle\}$.

$A^3=\{\langle 6,0.001,0.992\rangle,\langle 7,0.027,0.875\rangle,\langle 8,0.125,0.784\rangle,\langle 9,0.729,0\rangle,\langle 10,1,0\rangle\}$.

$A^4=\{\langle 6,0.0001,0.998\rangle,\langle 7,0.008,0.938\rangle,\langle 8,0.062,0.870\rangle,\langle 9,0.656,0\rangle,\langle 10,1,0\rangle\}$.

Considering the characterization analysis on linguistic variables, we can regard the AIFS *A* as ''LARGE'' in *X*. Correspondingly, AIFSs $A^{0.5}$, A^2, A^3 and A^4 can be regarded as "More or less LARGE", "Very LARGE", "Quite very LARGE", and "Very very LARGE", respectively.

Intuitively, from $A^{0.5}$ to A^4, the uncertainty hidden in them becomes less, and the knowledge amount conveyed by them increases. So the following relations hold:

$$E(A^{0.5})>E(A)>E(A^2)>E(A^3)>E(A^4) \qquad (13)$$

$$K(A^{0.5})<K(A)<K(A^2)<K(A^3)<K(A^4) \qquad (14)$$

TO MAKE A COMPARISON, THESE ENTROPY AND KNOWLEDGE MEASURES LISTED IN TABLE I ARE USED. IT IS WORTHY TO BE NOTED THAT SOME OF THE ENTROPY MEASURES IN

Table are initially designed for interval valued fuzzy sets [45],[46]. These entropy measures are modified for AIFSs based on their connection with interval values fuzzy sets. We present the results obtained based on different measures in Table to facilitate comparative analysis.

TABLE I
ENTROPY / KNOWLEDGE MEASURES USED FOR COMPARATIVE ANALYSIS

Authors	Entropy/Knowledge measure
E_{ZL} [45]	$E_{ZL}(A)=1-\frac{1}{n}\sum_{i=1}^{n}\lvert\mu_A(x_i)-v_A(x_i)\rvert$
E_{ZA}[46]	$E_{ZA}(A)=1-\sqrt{\frac{2}{n}\sum_{i=1}^{n}\lvert\mu_A(x_i)-0.5\rvert^2+\lvert 1-v_A(x_i)-0.5\rvert^2}$
E_{ZB}[46]	$E_{ZB}(A)=1-\frac{1}{n}\sum_{i=1}^{n}\left(\lvert\mu_A(x_i)-0.5\rvert+\lvert 1-v_A(x_i)-0.5\rvert\right)$
E_{ZC}[46]	$E_{ZC}(A)=1-\frac{2}{n}\sum_{i=1}^{n}\max\left(\lvert\mu_A(x_i)-0.5\rvert,\lvert 1-v_A(x_i)-0.5\rvert\right)$
E_{ZD}[46]	$E_{ZD}(A)=1-\sqrt{\frac{4}{n}\sum_{i=1}^{n}\max\left(\lvert\mu_A(x_i)-0.5\rvert^2,\lvert 1-v_A(x_i)-0.5\rvert^2\right)}$
E_{ZE}[46]	$E_{ZE}(A)=1-\frac{2}{n}\sum_{i=1}^{n}\left(\frac{\lvert\mu_A(x_i)-0.5\rvert+\lvert 1-v_A(x_i)-0.5\rvert}{4}+\frac{\max\left(\lvert\mu_A(x_i)-0.5\rvert,\lvert 1-v_A(x_i)-0.5\rvert\right)}{2}\right)$
E_{BB} [6]	$E_{BB}(A)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\mu_A(x_i)-v_A(x_i)\right)$
E_{SK} [32]	$E_{SK}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min\left(\mu_A(x_i),v_A(x_i)\right)+\pi_A(x_i)}{\max\left(\mu_A(x_i),v_A(x_i)\right)+\pi_A(x_i)}$
E_{HC}^2 [16]	$E_{HC}^2(A)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\left(\mu_A(x_i)\right)^2-\left(v_A(x_i)\right)^2-\left(\pi_A(x_i)\right)^2\right)$
E_S [16]	$E_S(A)=-\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A(x_i)\ln\mu_A(x_i)+v_A(x_i)\ln v_A(x_i)+\pi_A(x_i)\ln\pi_A(x_i)\right)$
E_{VS} [36]	$E_{VS}(A)=-\frac{1}{n\ln 2}\sum_{i=1}^{n}\left(\mu_A(x_i)\ln\mu_A(x_i)+v_A(x_i)\ln v_A(x_i)+\left(1-\pi_A(x_i)\right)\ln\left(1-\pi_A(x_i)\right)\right)+\frac{1}{n}\sum_{i=1}^{n}\pi_A(x_i)$
E_{ZJ} [47]	$E_{ZJ}(A)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min\left(\mu_A(x_i),v_A(x_i)\right)}{\max\left(\mu_A(x_i),v_A(x_i)\right)}$
E_{LDL} [17]	$E_{LDL}(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left(\lvert\mu_A(x_i)-v_A(x_i)\rvert^3+\lvert\mu_A(x_i)-v_A(x_i)\rvert\right)$
K_{SKB} [35]	$K_{SKB}(A)=1-\frac{1}{2n}\left(\sum_{i=1}^{n}\frac{\min\left(\mu_A(x_i),v_A(x_i)\right)+\pi_A(x_i)}{\max\left(\mu_A(x_i),v_A(x_i)\right)+\pi_A(x_i)}+\pi_A(x_i)\right)$
K_N [23]	$K_N(A)=\frac{1}{n\sqrt{2}}\sum_{i=1}^{n}\sqrt{\left(\mu_A(x_i)\right)^2+\left(v_A(x_i)\right)^2+\left(\mu_A(x_i)+v_A(x_i)\right)^2}$
K_G [14]	$K_G(A)=1-\frac{1}{2n}\sum_{i=1}^{n}\left(1-\lvert\mu_A(x_i)-v_A(x_i)\rvert\right)\left(1+\pi_A(x_i)\right)$

TABLE II
COMPARATIVE RESULTS OF ALL AIFSS WITH RESPECT TO A (COUNTER-INTUITIVE RESULTS ARE IN BOLD TYPE)

	$A^{0.5}$	A	A^2	A^3	A^4
E_{ZL}	**0.4156**	**0.4200**	0.2380	0.1546	0.1217
E_{ZA}	0.3214	0.3043	0.1974	0.1330	0.0979
E_{ZB}	**0.4156**	**0.4200**	0.2380	0.1546	0.1217
E_{ZC}	0.3338	0.3200	0.1400	0.0612	0.0283
E_{ZD}	0.2777	0.2463	0.1188	0.0562	0.0271
E_{ZE}	0.3747	0.3700	0.1890	0.1079	0.0750
E_{BB}	**0.0818**	**0.1000**	0.0980	0.0934	0.0934
E_{SK}	**0.3446**	**0.3740**	0.1970	0.1309	0.1094
E_{HC}	**0.3416**	**0.3440**	0.2610	0.1993	0.1613
E_S	**0.5811**	**0.5874**	0.4555	0.3489	0.2778
E_{VS}	0.5518	0.5217	0.3491	0.2357	0.1733
E_{ZJ}	**0.2851**	**0.3050**	0.1042	0.0383	0.0161
E_{LDL}	0.5083	0.5019	0.3454	0.2516	0.2001
K_{SKB}	**0.7868**	**0.7630**	0.8525	0.8879	0.8986
K_N	**0.8585**	**0.8471**	0.8738	0.8927	0.8999
K_G	**0.7665**	**0.7610**	0.8651	0.9108	0.9257
K_I	0.7059	0.7098	0.8066	0.8624	0.8858

From Table II, we can see that entropy measures E_{ZL}, E_{ZB}, E_{BB}, E_{SK}, E_{HC}, E_S and E_{ZJ} induce the following relations:

$$E_{ZL}(A)>E_{ZL}(A^{0.5})>E_{ZL}(A^2)>E_{ZL}(A^3)>E_{ZL}(A^4).$$

$$E_{ZB}(A)>E_{ZB}(A^{0.5})>E_{ZB}(A^2)>E_{ZB}(A^3)>E_{ZB}(A^4).$$

$$E_{BB}(A)>E_{BB}(A^{0.5})>E_{BB}(A^2)>E_{BB}(A^3)=E_{BB}(A^4).$$

$$E_{SK}(A)>E_{SK}(A^{0.5})>E_{SK}(A^2)>E_{SK}(A^3)>E_{SK}(A^4).$$

$$E_{HC}(A)>E_{HC}(A^{0.5})>E_{HC}(A^2)>E_{HC}(A^3)>E_{HC}(A^4).$$

$$E_S(A)>E_S(A^{0.5})>E_S(A^2)>E_S(A^3)>E_S(A^4).$$

$$E_{ZJ}(A)>E_{ZJ}(A^{0.5})>E_{ZJ}(A^2)>E_{ZJ}(A^3)>E_{ZJ}(A^4).$$

BECAUSE THE ENTROPY OF AIFS $A^{0.5}$ IS LESS THAN THAT OF AIFS A, ENTROPY MEASURES E_{ZL}, E_{ZB}, E_{ZE}, E_{BB}, E_{SK}, E_{ZJ} DO NOT PERFORM AS WELL AS OTHER ENTROPY MEASURES. FROM THE VIEWPOINT OF KNOWLEDGE AMOUNT, WE NOTE THAT THE RESULTS OBTAINED BY K_{SKB}, K_N AND K_G ARE NOT SO REASONABLE, SINCE COUNTER-INTUITIVE RELATIONS $K_{SKB}(A^{0.5})>K_{SKB}(A)$, $K_N(A^{0.5})>K_N(A)$ AND $K_G(A^{0.5})>K_G(A)$ EXIST. HOWEVER, OUR DEVELOPED KNOWLEDGE MEASURE K^I CAN PRODUCE RATIONAL RESULT AS $K^I\left(A^{0.5}\right)<K^I\left(A\right)<K^I\left(A^2\right)<K^I\left(A^3\right)<^I K\left(A^4\right)$. SO IT IS DEMONSTRATED THAT HALF OF ENTROPY MEASURES IN

Table cannot reflect the uncertainty hidden in these AIFSs. Although several knowledge measures have been presented, they are not competent to distinguish the nuance of knowledge amount in different AIFSs. Thus, our developed knowledge measure outperforms other knowledge measures by providing persuading results complying with intuitive analysis.

For a further investigation on the performance of our proposed knowledge measure, we modify the AIFS "LARGE" defined in X={6,7,8,9,10} by increasing the non-membership degree of element "8", and reducing its hesitant degree. The modified AIFS "LARGE" is given as:

$$B=\{\langle 6,0.1,0.8\rangle,\langle 7,0.3,0.5\rangle,\langle 8,0.5,0.5\rangle,\langle 9,0.9,0\rangle,\langle 10,1,0\rangle\}.$$

By the operation shown in (12), the following AIFSs related to B can be generated:

$$B^{0.5}=\{\langle 6,0.316,0.553\rangle,\langle 7,0.548,0.293\rangle,\langle 8,0.707,0.293\rangle,\langle 9,0.949,0\rangle,\langle 10,1,0\rangle\}.$$

$$B^{2}=\{\langle 6,0.010,0.960\rangle,\langle 7,0.090,0.750\rangle,\langle 8,0.250,0.750\rangle,\langle 9,0.810,0\rangle,\langle 10,1,0\rangle\}.$$

$$B^{3}=\{\langle 6,0.001,0.992\rangle,\langle 7,0.027,0.875\rangle,\langle 8,0.125,0.875\rangle,\langle 9,0.729,0\rangle,\langle 10,1,0\rangle\}.$$

$$B^{4}=\{\langle 6,0.0001,0.998\rangle,\langle 7,0.008,0.938\rangle,\langle 8,0.062,0.938\rangle,\langle 9,0.656,0\rangle,\langle 10,1,0\rangle\}.$$

According to the entropy and knowledge measures listed in Table I, we can get comparative results as shown in Table III. We can see that the AIFS B still has more entropy than the AIFS $B^{0.5}$, when entropy measures E_{ZL}, E_{ZB}, E_{ZE}, E_{BB}, E_{SK} and E_{ZJ} are concerned. The ordered results obtained based on these entropy measures are given as follows:

$$E_{ZL}(B)>E_{ZL}(B^{0.5})>E_{ZL}(B^2)>E_{ZL}(B^3)>E_{ZL}(B^4).$$

$$E_{ZB}(B)>E_{ZB}(B^{0.5})>E_{ZB}(B^2)>E_{ZB}(B^3)>E_{ZB}(B^4).$$

$$E_{ZE}(B)>E_{ZE}(B^{0.5})>E_{ZE}(B^2)>E_{ZE}(B^3)>E_{ZE}(B^4).$$

$$E_{BB}(B)>E_{BB}(B^{0.5})>E_{BB}(B^2)>E_{BB}(B^3)>E_{BB}(B^4).$$

$$E_{SK}(B)>E_{SK}(B^{0.5})>E_{SK}(B^2)>E_{SK}(B^3)>E_{SK}(B^4).$$

$$E_{ZJ}(B)>E_{ZJ}(B^{0.5})>E_{ZJ}(B^2)>E_{ZJ}(B^3)>E_{ZJ}(B^4).$$

We see that these ranked orders do not satisfy intuitive analysis in Eq. (13), while other entropy measures can induce desirable results. In this example, E_{HC} and E_S perform well, but the measure E_{ZE} performs poor. This illustrates that these entropy measures are not robust enough.

Moreover, the results produced by knowledge measures K_{SVB}, K_N and K_G are also not reasonable, shown as:

$$K_{SVB}(B)<K_{SVB}(B^{0.5})<K_{SVB}(B^2)<K_{SVB}(B^3)<K_{SVB}(B^4),$$

$$K_N(B)<K_N(B^{0.5})<K_N(B^2)<K_N(B^3)<K_N(B^4),$$

$$K_G(B)<K_G(B^{0.5})<K_G(B^2)<K_G(B^3)<K_G(B^4).$$

However, our proposed knowledge measure K^I indicates that:

$$K^I(B^{0.5})<K^I(B)<K^I(B^2)<K^I(B^3)<K^I(B^4).$$

So the knowledge measures K_{SVB}, K_N and K_G are still not suitable for differentiating the knowledge amount conveyed by AIFSs. The effectiveness of our proposed knowledge measure K^I is indicated by this example once again.

From above examples we can conclude that entropy measures E_{ZL}, E_{ZB}, E_{ZE}, E_{BB}, E_{HC}, E_S, E_{SK} and E_{ZJ} perform poor because of their lack of robustness and discriminability. Our proposed knowledge measure performs much better than knowledge measures K_{SVB} , K_N and K_G. The performances of entropy measures E_A, E_{ZC}, E_{ZD}, E_{VS}, E_{LDL} and our proposed knowledge measure K^I in Table and Table seem to show that less entropy indicates more knowledge amount. Nevertheless, the relationship between entropy and knowledge measures is limited and conditional, as is discussed previously.

TABLE III
COMPARATIVE RESULTS OF ALL AIFSS WITH RESPECT TO B (COUNTER-INTUITIVE RESULTS ARE IN BOLD TYPE)

	$B^{0.5}$	B	B^2	B^3	B^4
E_{ZL}	**0.4291**	**0.4400**	0.2160	0.1364	0.1082
E_{ZA}	0.3310	0.3072	0.1868	0.1193	0.0859
E_{ZB}	**0.4291**	**0.4400**	0.2160	0.1364	0.1082
E_{ZC}	0.3608	0.3600	0.1400	0.0612	0.0283
E_{ZD}	0.2960	0.2517	0.1188	0.0562	0.0271
E_{ZE}	**0.3950**	**0.4000**	0.1780	0.0988	0.0683
E_{BB}	**0.0683**	**0.0800**	0.0760	0.0752	0.0800
E_{SK}	**0.3518**	**0.4073**	0.1677	0.1101	0.0950
E_{HC}	0.3355	0.3280	0.2328	0.1708	0.1379
E_S	0.5494	0.5374	0.3929	0.2905	0.2295
E_{VS}	0.5640	0.5233	0.3369	0.2212	0.1612
E_{ZJ}	**0.3042**	**0.3450**	0.0927	0.0349	0.0151
E_{LDL}	0.5191	0.5120	0.3279	0.2290	0.1791
K_{SKB}	**0.7899**	**0.7563**	0.8782	0.9074	0.9125
K_N	**0.8680**	**0.8641**	0.8950	0.9108	0.9133
K_G	**0.7633**	**0.7600**	0.8828	0.9230	0.9337
K_I	0.7038	0.7182	0.8272	0.8804	0.8992

IV. CONCLUSION

With the purpose of measuring the knowledge amount of AIFSs much better, we propose a knowledge measure. The axiomatic definition of knowledge measure is refined from a more general view, following which we investigate the properties of the new knowledge measure. Mathematical analysis and numerical examples are presented to illustrate the properties of the proposed knowledge measure. Numerical examples together with comparative analysis demonstrate the effectiveness and rationality of the developed method.

In this paper, we only present a knowledge measure based on our proposed distance measure. The main feature of our proposed knowledge lies in its succinct expression and good properties. There must be other kinds of knowledge measure. Further study on knowledge measure is necessary to develop more reasonable models to solve practical decision making problem.

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