Multi-dimensional data aggregation utilizing extended partitioned Bonferroni mean Operator

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***Abstract*—In this contribution, we develop the concept of an Extended Partitioned Bonferroni Mean ($\mathcal{EPBM}$) operator, which is efficient enough to aggregate input vectors with a varying number of components integrated with some dependence pattern. The global monotonicity for the $\mathcal{EPBM}$ is analyzed by defining a new partition for each arity. Further to illustrate the applicability and feasibility of the proposed extended aggregation operator, an example based on medical device selection is demonstrated. Finally, we present a way to obtain the weights associated with the corresponding $\mathcal{EPBM}$ operator employing the Max-Entropy technique.**

***Index Terms*—Extended aggregation function, Partitioned Bonferroni mean, Global monotonicity.**

I. Introduction

Aggregation is a process of combining several inputs to obtain a single representative output value. A wide range of aggregation operators have been introduced to obtain the representative values of the accumulated information. Although, the conventional aggregation operators usually consider a fixed number of input arguments. For some applications, it may be too restrictive as it is often the case that aggregation of inputs of various sizes has to be considered under the same framework. Thus we need to adopt the concept of extended aggregation function where one is efficient enough to aggregate input vectors with varying numbers of components. In 1997, Mayor and Calvo [1] first introduced the concept of Extended aggregation function (EAF). The idea of extended aggregation operators allows decision-makers to make a comparison between two lists of input arguments with different dimensions. Further, this notion has been put forwarded by many researchers over different aggregation operators [2]–[6]. Among them, the extended ordered weighted averaging (EOWA) operator and extended quasi-linear arithmetic mean (EQLWM) are widely acclaimed operators which draw the attention of most of the researchers. But these two operators are incompetent in capturing any kind of inter-relationship among the aggregated arguments.

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Bonferroni mean (BM) operator [7] was first introduced to address the homogenous conjunction among each pair of input arguments in the aggregation scenario. Further several variants of the BM operators have been developed where instead of assuming that each input argument is related to rest all input arguments, decision-makers emphasized a specific inter-relationship structure amongst the criteria set, which has a great importance in the process of aggregation [8], [15]. For instance, in [8] Dutta and Guha proposed the concept of partitioned Bonferroni mean (PBM) operator, where the set of criteria are partitioned into several groups and the members of each partition are interrelated while no inter-relationships exist among the intra-partitions. In our proposal, we shed light on some aspects of PBM operator, which has not been taken into account so far. Till now, the connection of extended aggregation function with PBM has not yet been attended. Defining extended-PBM operator was a challenging issue as for each arity, we need to define a new partition. With this view, in this article, we present the mathematical interpretation of the extended PBM operator ($\mathcal{EPBM}$). Further the applicability of the newly proposed operators are explored in fusing the interconnected data in the context of the multi-layer hierarchy process, where the input vectors may have a varying number of components. Finally the applicability of the proposed extended PBM operator is illustrated in medical decision making scenario, to elect a medical device like blood pump associated with multiple evaluation parameters.

Another important aspect associated with the newly develop extended aggregation operator is the choice of associated weights. To estimate the weight vector of an aggregation operator, the Max-Entropy technique proposed by O'Hagan [24]–[26] is the most popular and elegant one. By investigating the solution of a mathematical programming problem based on two fundamental measures: orness measure and dispersion [9], we can estimate the corresponding weights relating to the given set of criteria. Since the measure of orness is associated with the dimension of the input arguments with multiple integral, thus instead of deriving the analytical formula of the orness measure for our proposed $\mathcal{EPBM}$ operator, we attempt to numerically estimate the integral value using the Monte Carlo integration approach [28].

The paper is structured as follows: In section 2, we recall

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the basic concept of extended aggregation function and PBM operators, which will be used in the rest of the papers. In section 3, the modeling capability of extended partitioned Bonferroni mean operator is presented with a systematic investigation of its behavior and properties. Along with that an algorithm to find the desirable alternative based on the proposed operator is developed and justified with an example. In Section 4, Weighted representation of the proposed operator with a brief weight determination procedure is presented. Section 5 concludes with some future work.

II. Preliminaries

Here we begin by recalling the concept of Extended Aggregation Function (EAF) first.

A. Extended aggregation function

Suppose, $\bigcup_{n\geq 1}[0,1]^n$ represents the set of all ordered lists that can be constructed from $[0,1]$. In order to compare two ordered list with a different dimension, we first need to present the binary relations on $\bigcup_{n\geq 1}[0,1]^n$ as follows:

Definition 1: [1] Suppose, $\mathbf{x} = (x_1, x_2, ..., x_{n_1})$ and $\mathbf{y} = (y_1, y_2, ..., y_{n_2})$ be the two elements from $\bigcup_{n\geq 1}[0,1]^n$. Then the orderings on $\bigcup_{n\geq 1}[0,1]^n$ can be considered as,

(i) for $n_1 = n_2$, $\mathbf{x} \leq_\pi \mathbf{y}$ if $x_i \leq y_i$ for all $i = 1, 2, ..., n_1$.

(ii) for $n_1 \leq n_2$, $\mathbf{x} \leq_\alpha \mathbf{y}$ if $x_i \leq y_i$ for all $i = 1, 2, ..., n_1$ and if $n_1 < n_2$, then $max(x_1, x_2, ..., x_{n_1}) \leq min(y_{n_1+1}, y_{n_1+2}, ..., y_{n_2})$.

(iii) for $n_1 \geq n_2$, $\mathbf{x} \leq_\beta \mathbf{y}$ if $x_i \leq y_i$ for all $i = 1, 2, ..., n_2$ and if $n_1 > n_2$, then $max(x_{n_2+1}, x_{n_2+2}, ..., x_{n_1}) \leq min(y_1, y_2, ..., y_{n_2})$.

So, the binary relations $\leq_\mathbf{s}$, $\mathbf{s} \in \{\pi, \alpha, \beta\}$ are orderings on $\bigcup_{n\geq 1}[0,1]^n$. The first order is the standard partial order on Cartesian products of $[0,1]$ related to the considered dimensions. As an extension of this order, two other partial orders (α- and β-order) were introduced in [1], motivated by comparison of weighted arithmetic means with different aggregated entering arities.

In this contribution we will consider extended aggregation function in the sense of [4].

Definition 2: [4] A mapping $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \rightarrow [0,1]$ is an extended aggregation function on $([0,1], \leq)$ if

- it is monotonic with respect to $\leq_\pi$, $\leq_\alpha$ and $\leq_\beta$, i.e., for all $\mathbf{x} = (x_1, ..., x_{n_1})$, $\mathbf{y} = (y_1, ..., y_{n_2}) \in \bigcup_{n\geq 1}[0,1]^n$, $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x} \leq_\mathbf{s} \mathbf{y}$, $\mathbf{s} \in \{\pi, \alpha, \beta\}$.
- $A(x) = x$ for any $x \in [0,1]$.

Observe that, for any $x \in [0,1]$,

$$(\overbrace{x, x, ..., x}^{\text{n times}}) \leq_\beta x \leq_\alpha (\overbrace{x, x, ..., x}^{\text{n times}})$$

and thus $A(\overbrace{x, x, ..., x}^{\text{n times}}) \leq A(x) = x \leq A(\overbrace{x, x, ..., x}^{\text{n times}})$.

Hence a mapping $A : \bigcup_{n\geq 1}[0,1]^n \rightarrow [0,1]$ is an extended aggregation function if and only if the restriction of A to $[0,1]^n$, $n \in \mathbb{N}$ is an n-ary idempotent aggregation function [5], [6]. One can interpret it as, $A|_{[0,1]^n} = A_{(n)}$ for all $n \in \mathbb{N}$. Thus the extended aggregation function can be introduced as a family $A = (A_{(n)})_{n\in\mathbb{N}}$ of n-ary aggregation function. Thus by definition 2, we get an extended aggregation function belongs to the class of idempotent extended aggregation function.

Next, we recall the definition of weighting triangle associated with the weighted extended aggregation function that collects the weights of any weighting list $W_n = (w_{1,n}, w_{2,n}, ..., w_{n,n})$ where, $n \geq 1$.

Definition 3: [4] A weighting triangle is a collection of numbers $w_{j,n} \in [0,1]$, for $j = 1, ..., n$ such that $\sum_{j=1}^{n} w_{j,n} = 1$ for each $n > 1$. It can be represented as,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & w_{1,2} & & w_{2,2} & & \\ & w_{1,3} & & w_{2,3} & & w_{3,3} & \\ w_{1,4} & & w_{2,4} & & w_{3,4} & & w_{4,4} \\ & & & \dots & & & \end{array}$$

It can be denoted as, Δ.

In the next section, we will recall the concept of conventional PBM operator proposed by Dutta and Guha [8], for a fixed number of arity n where $n \in \mathbb{N}$.

B. Partitioned Bonferroni Mean (PBM) operator

Suppose, $\mathbf{a} = (a_1, a_2, ..., a_n)$ denotes the degree of satisfaction of the alternative X associated with the criteria set $C = \{C_1, C_2, \dots, C_n\}$ where, $a_i \in [0,1]$ $\forall$ $i = 1, 2, ..., n$. Suppose that on the basis of the inter-relationship pattern, the criteria set $\{C_1, C_2, \dots, C_n\}$ is partitioned into d mutually disjoint partition sets $\{P_1, P_2, \dots P_d\}$ such that $\bigcup_{r=1}^{d} P_r = C$. We further assume that criteria of each partition set P_r is interrelated to each other's and there is no interrelationship among attributes of any two partition sets P_r and P_k whenever $r, k \in \{1, 2, ..., d\}$ and $r \neq k$. With this information in background, the partitioned Bonferroni mean (PBM) operator of the collection of inputs $(a_1, a_2, ..., a_n)$ can be defined as follows:

Definition 4: [8] For $p, q \geq 0$ with p+q>0, the partitioned Bonferroni mean operator is a mapping $PBM^{p,q} : [0,1]^n \rightarrow [0,1]$ such that

$$PBM^{p,q}(a_1, a_2, ..., a_n) = \frac{1}{d}\left(\sum_{r=1}^{d}\left(\frac{1}{|P_r|}\sum_{i\in P_r} a_i^p \left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)\right)^{\frac{1}{p+q}}\right) \quad (1)$$

with the convention $\frac{0}{0} = 1$ and $|P_r|$= cardinality of P_r.
From Eq.(1), the expression $\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q$ indicates the average satisfaction of the inputs belonging to partition set P_r except a_i. Now, $a_i^p\left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)$ captures the conjunction of the satisfaction of input a_i with the average satisfaction of the rest of inputs of the partition set P_r. Then

the expression $\frac{1}{|P_r|}\sum_{i\in P_r} a_i^p\left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)$ gives the satisfaction of the inter-related inputs of the partition set P_r and by $\frac{1}{d}\sum_{r=1}^{d}\left(\frac{1}{|P_r|}\sum_{i\in P_r} a_i^p\left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)\right)^{\frac{p}{p+q}}$ we obtain the average satisfaction of all the inputs of the d distinct partition sets.

From the construction of the PBM operator, it is clear that the aggregated value computed by the PBM depends on the inter-relationships among the inputs. Apart from that, one can easily verify that the PBM operator satisfies the idempotency, monotonicity, and boundedness properties.

In the next section, we extend the concept of the PBM operator to aggregate input of various sizes.

III. Construction of Extended-PBM Operator

In this section, we develop the concept of Extended-PBM Operator where interconnected inputs of various sizes are aggregated under the same framework. To do so, we first need to define extended-PBM with possible different arities.

We are starting the process with the similar background and notations used in the definition of PBM operator (Definition 4). We now include one new criteria C_{n+1} in the old criteria set C. Hence the old criteria set is updated to $C^* = \{C_1, C_2, \ldots, C_n, C_{n+1}\}$. Suppose a_{n+1} is the degree of satisfaction of the alternative X under the criteria C_{n+1}, with the assumption that $a_{n+1} \geq max\{a_1, a_2, ..., a_n\}$. Thus with this assumption we can define the α-order between $\mathbf{a}$ and $\mathbf{a}^*$ i.e., we can say that $\mathbf{a} \leq_\alpha \mathbf{a}^*$ where $\mathbf{a}^* = (a_1, a_2, ..., a_n, a_{n+1})$ is the new input set. We primarily need to show that $PBM^{p,q}(a_1, a_2, ..., a_n) \leq PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1})$.

As we are updating the old criteria set C to C^*, consequently the corresponding new partition structure can be constructed from old partition structure in this way:

- Either the new criterion C_{n+1} is not interrelated with any of the other criteria $\{C_1, .., C_n\}$. In that case, the new partition structure is $\{P_1, P_2, ..., P_d, \{C_{n+1}\}\}$.
- Or, C_{n+1} is interrelated with all the criteria of a particular partition set, say for example, P_k and then the new partition structure is $\{P_1, ..., P_{k-1}, P_k \cup \{C_{n+1}\}, P_{k+1}, ..., P_{d_n}\}$.

Now analyzing both the cases one can get,

Case 1. First considering that, the new criterion C_{n+1} is not interrelated with any of the other criteria $\{C_1, .., C_n\}$. Hence, for the input arguments $(a_1, a_2, ..., a_n, a_{n+1})$ we obtain the aggregated values of the alternative X as follows:

$$PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1}) = \left(\frac{d}{d+1}\left(\frac{1}{d}\sum_{r=1}^{d}\left(\frac{1}{|P_r|}\sum_{i\in P_r} a_i^p\left(\frac{1}{|P_r|-1}\sum_{\substack{j\neq i\\ j\in P_r}} a_j^q\right)\right)^{\frac{p}{p+q}}\right) + \frac{1}{d+1}\left(a_{n+1}\right)^p\right)^{\frac{1}{p}} \quad (2)$$

Now, Eq (2) is equivalent to,

$$PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1}) = \left(\frac{d}{d+1}\left(PBM^{p,q}(a_1, a_2, ..., a_n)\right)^p + \frac{1}{d+1}\left(a_{n+1}\right)^p\right)^{\frac{1}{p}}$$

Since, $PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1})$ is the convex combination of $PBM^{p,q}(a_1, a_2, ..., a_n)$ and a_{n+1} with $a_{n+1} \geq max\{a_1, a_2, ..., a_n\}$ and $\frac{d}{d+1}, \frac{1}{d+1} \geq 0$, $\frac{d}{d+1} + \frac{1}{d+1} = 1$.
Hence,

$$PBM^{p,q}(a_1, a_2, ..., a_n) \leq PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1}).$$

Case 2. If C_{n+1} is interrelated with all the criteria of a particular partition set, say, P_k then, partition set P_k is updated to $P_k^* = P_k \cup \{C_{n+1}\}$ while the construction of the rest of the partition sets will remain same. In this instance, instead of proving the monotonicity condition for whole set of input arguments we just need to prove it for the k-th partition set only. Now as we know each member of partition set is interrelated to the rest of the member of that particular partition set and thus form a homogeneous kind of inter-relationship structure within that particular partition set.
Suppose $\mathbf{x} = (x_1, x_2, ..., x_m)$ be the collection of inputs associated with the k-th partition set P_k where $\mathbf{x} \subset \mathbf{a}$. Denoting $a_{n+1} = x_{m+1}$, $\mathbf{x}^* = (x_1, x_2, ..., x_m, x_{m+1})$ will be the collection of the input arguments of partition set P_k^*. Then to prove $PBM^{p,q}(a_1, a_2, ..., a_n) \leq PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1})$ we just need to establish,

$$\left(\frac{1}{|P_k|}\sum_{i\in P_k} x_i^p\left(\frac{1}{|P_k|-1}\sum_{j\neq i,\ j\in P_k} x_j^q\right)\right)^{\frac{1}{p+q}} \leq \left(\frac{1}{|P_k^*|}\sum_{i\in P_k^*} x_i^p\left(\frac{1}{|P_k^*|-1}\sum_{j\neq i,\ j\in P_k^*} x_j^q\right)\right)^{\frac{1}{p+q}}$$

Thereby, from the definition of α-order we can write $x_{m+1} \geq max\{x_1, x_2, ..., x_m\}$.
Consequently we have,

$$\frac{1}{|P_k|-1}\sum_{j\neq i,\ j\in P_k} x_j^q \leq x_{m+1}^q$$

i.e.,

$$\sum_{j\neq i,\ j\in P_k} x_j^q \leq (|P_k|-1)x_{m+1}^q$$

Hence,

$$(|P_k|-1)\sum_{j\neq i,\ j\in P_k} x_j^q + \sum_{j\neq i,\ j\in P_k} x_j^q \leq (|P_k|-1)\sum_{j\neq i,\ j\in P_k} x_j^q + (|P_k|-1)x_{m+1}^q$$

Thus,

$$|P_k|\sum_{j\neq i,\ j\in P_k} x_j^q \leq (|P_k|-1)\sum_{j\neq i,\ j\in P_k^*} x_j^q$$

As, $|P_k| = |P_k^*| - 1$. So,

$$\frac{1}{|P_k|-1}\sum_{j\neq i,\ j\in P_k} x_j^q \leq \frac{1}{|P_k^*|-1}\sum_{j\neq i,\ j\in P_k^*} x_j^q$$

In the similar manner we can prove,

$$\left(\frac{1}{|P_k|}\sum_{i\in P_k} x_i^p\left(\frac{1}{|P_k|-1}\sum_{j\neq i,\ j\in P_k} x_j^q\right)\right)^{\frac{1}{p+q}} \leq$$

$$\left(\frac{1}{|P_k^*|}\sum_{i\in P_k^*} x_i^p\left(\frac{1}{|P_k^*|-1}\sum_{j\neq i,\ j\in P_k^*} x_j^q\right)\right)^{\frac{1}{p+q}}$$

Thus combining case 1. and case 2., for any two collections of input arguments $\mathbf{a} = (a_1, a_2, ..., a_n)$ and $\mathbf{a}^* = (a_1, a_2, ..., a_n, a_{n+1})$ with varying numbers of components we can conclude that, if $\mathbf{a} \leq_\alpha \mathbf{a}^*$ then,

$$PBM^{p,q}(a_1, a_2, ..., a_n) \leq PBM^{p,q}(a_1, a_2, ..., a_n, a_{n+1}).$$

Similarly, the case when $\leq_\beta$ is considered can be discussed. In general, we can say that a function $\mathcal{EPBM}$ is a mapping $\mathcal{EPBM} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ that satisfies the monotonicity condition with respect to $\leq_\pi$, $\leq_\alpha$ and $\leq_\beta$ i.e, for all $\mathbf{x} = (x_1, ..., x_{n_1})$, $\mathbf{y} = (y_1, ..., y_{n_2}) \in \bigcup_{n\geq 1}[0,1]^n$, $\mathcal{EPBM}(\mathbf{x}) \leq \mathcal{EPBM}(\mathbf{y})$ whenever $\mathbf{x} \leq_\mathbf{s} \mathbf{y}$, $\mathbf{s} \in \{\pi, \alpha, \beta\}$.

Further one can easily prove the idempotency condition of $\mathcal{EPBM}$ operator for any fixed number of input argument i.e., $PBM^{p,q}(\overbrace{x, x, ..., x}^{\text{n times}}) = x \quad$ for all $\quad x \in [0,1], n \geq 1$. Following these two characterizations, our proposed $\mathcal{EPBM}$ operator belongs to the class of idempotent EAF on $\bigcup_{n\geq 1}[0,1]^n$.

Finally, we introduce the formal definition the $\mathcal{EPBM}$ operator as:

Definition 5: An extended aggregation function $\mathcal{EPBM} : \bigcup_{n\geq 1}[0,1]^n \to [0,1]$ is called an extended partitioned Bonferroni mean if $\mathcal{EPBM}(x) = x$ where $x \in [0,1]$ and there are $p, q \geq 0$, $p + q > 0$ such that for each $n \geq 2$ the restriction $\mathcal{EPBM}|_{[0,1]^n}$ is a $PBM^{p,q}_{(n)}$ operator related to partition $\{P_1^n, P_2^n, ..., P_{d_n}^n\}$ of $\{1, 2, ..., n\}$ so that for any $n < m$ and any $i, k \in \{1, 2, ..., d_n\}$ with $i \neq k$, there exists some j, s element in $\{1, 2, ..., d_m\}$ with $j \neq s$ so that $P_i^n \subset P_j^m$ and $P_k^n \subset P_s^m$.

Now depending on the nature of the criteria set, our proposed operator $\mathcal{EPBM}$ relates with some classical EAF. If all the criteria belong to the same class i.e. d=1, then the $\mathcal{EPBM}$ operator transforms to the extended-BM operator defined in [6].

Remark 1: More precisely we can depict the idea of the partition set defined for a fixed n as follows:
for any countable criteria set $\{C_1, C_2, ...\}$ we always have a set of partition $\{P_1, P_2, ...\}$ which is possibly infinite in number but for sure countable, then, for a fixed n, we can consider finite partition, omitting empty sets, of the form $P^{(n)} = \{P_1 \cap N, P_2 \cap N, ...\}$, where $N = \{1, .., n\}$.

Now to aggregate the result in the presence of inter-related data, modeling inter-relationship is very important. Based on the real-life decision situations, experts first have to construct the relationship pattern among the attributes and the proposed operator and its different reduced versions can be used.
In the present study, the relationship pattern between attributes is given beforehand. However, one can identify the underlying inter-relationship pattern directly from the data set using the concept of similarity measure. In this regard few attempts have been done [14], [17].

A. Handling the hierarchy of criteria with extended PBM operator

Hierarchical decomposition basically assists decision maker by providing a ranking of alternatives not only considering the whole set of criteria but also with respect to any intermediate higher-level point of view. Hierarchical system can be observed as a simple linear chain of interactions where the output of each level is dependent on another in a sequential manner. In this segment, we develop a general aggregation approach which can not only evaluate the ultimate goal, but also produces partial results by characterizing the situations with the possible partition.

In a consequence, we propose an efficient algorithm based on the newly developed $\mathcal{EPBM}$ operator (developed in the section III.), to find the most desirable alternative. To carry out the computation two types of information are required; one is the performance of the alternatives against the assessed set of criteria and the another one is the specific structure of the interrelationship among the criteria set.

Step 1. Suppose a set of alternatives $A = \{A_i | i \in I\}$ (where $I = \{1, 2, ..., m\}$) are assessed over a criteria set structured into hierarchical fashion. The set of sub-criteria at elementary level (i.e., the set of criteria lies at the lower tier of the multi-layer hierarchical structure) are denoted by $\{C_j | j \in J\}$ where $J = \{1, 2, ..., n\}$. The performance of the alternative A_i under the elementary criterion C_j is given by a real entity a_{ij}. The evaluations are summarized in the following decision matrix:

$$D = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \end{array}.$$

Step 2. Next we need to provide the specific structure of the interrelationship among the criteria set to implement our proposed operator. In the spirit of [16], we assume that the elementary criteria set follows a partition structure interrelationship pattern where each class of the partition comprises of the elementary sub-criteria belonging to the same criteria of the immediately upper level.

Step 3. Based on the interrelationship pattern, we utilize our newly developed $\mathcal{EPBM}$ operator to find the alterna-

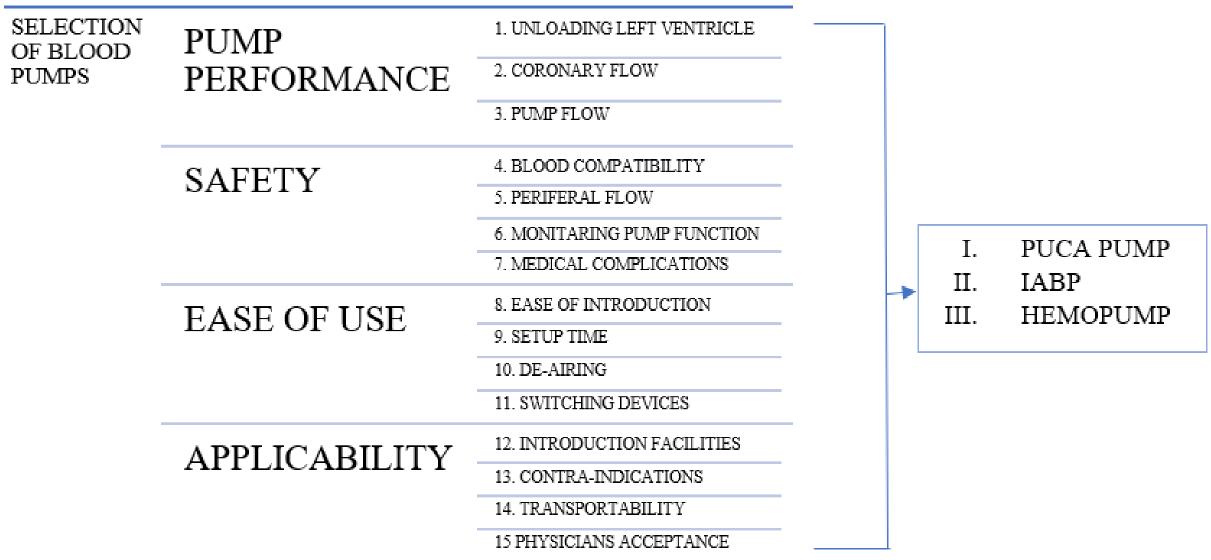

Fig. 1. Hierarchical structure of the criteria set for evaluation of medical device

tives A_i's overall performance η_i for all $i \in \{1, 2, ..., m\}$ as:

$$\eta_i = PBM^{p,q}_{(n)}(a_{i1}, a_{i2}, ..., a_{in})$$

Based on the grounds that decision makers can compute the decision output for different dimensions of criteria set utilizing $\mathcal{EPBM}$ operator, thus one can easily exhibit the final comprehensive result with respect to the whole set of criteria or partial result for any set of elementary sub-criteria.

Step 4. Finally, by comparing the overall performances we find out the elite one in the sense that higher overall performance indicates better rank.

In the next example, the applicability and feasibility of the proposed extended aggregation operator is demonstrated.

Example 3.1: Here we are presenting a sample problem (Adopted from [30]) where the various aspects of an medical device can be examined based on a set of criteria, presented in a hierarchical evaluation structure. To provide a temporary support to the circulatory system of a patient, a group of decision support system team has offer three choices for left ventricular assist devices which might be adapted. These are

1) Pulsatile Catheter Pump/ PUCA Pump (A_1)
2) Intra-Aortic Balloon Pump / IABP (A_2)
3) Hemopump (A_3).

The comparison among the different blood pump machines has been conducted based on the parameters that are organized into 4 broad categories $\mathbf{C}_1$: Pump Performance, $\mathbf{C}_2$: Safety, $\mathbf{C}_3$: Ease of Use, $\mathbf{C}_4$: Applicability; which have been further grouped into a number of subcategories $\{C_j | j \in J\}$ where $J = \{1 : 15\}$ as shown in Fig. 1. For ease, we consider here a 2-layer hierarchical structure of the criteria set. We have assumed that the elementary criteria set follows a partition structure interrelationship pattern in the sense that every subset of elementary sub-criteria set belonging to only one criterion of the level immediately above follow homogeneous dependency relationship.

The effectiveness of different ventricular assist devises $A_i (i = 1, 2, 3)$ estimated by a multi-disciplinary group of developers, manufacturers and end-users (i.e., cardiologists and thoracic surgeons) based on sub-criteria C_j where $j \in J$ are collected and summarized in Fig. 2.

With this available estimations, we employ the proposed decision-making algorithm to sort the medical devices and finally find out the elite one. For the sake of simplicity, we have set the parameters associated with $\mathcal{EPBM}$ as $p = q = 1$. However choice of the parameters p and q associated with $\mathcal{EPBM}$ may have an influence on the final ranking of the different ventricular assist devises. Now utilizing $\mathcal{EPBM}$ operator, aggregate all the estimated values a_{ij}, $(j = 1, 2, ..., 15)$ of the i-th line and get the overall decision outcome η_i, $(i = 1, 2, 3)$ corresponding to the alternative A_i as:

$\eta_1 = 0.2378;$
$\eta_2 = 0.4986;$
$\eta_3 = 0.2379.$

Thus the overall performance for the IABP pump is respectively much higher than the rest two. Thus from the aforementioned numerical analysis, the proposed $\mathcal{EPBM}$ operator produces the ranking of all the alternatives at the comprehensive level (root node of the hierarchy) as $A_2 > A_3 > A_1$ in which the alternative A_2 ranks the first.

With conventional methodologies it is not possible to com-

ALTERNATIVE	Pump Performance			Safety				Ease Of Use				Applicability			
	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
PUCA Pump	0.50	0.58	0.21	0.19	0.13	0.30	0.19	0.14	0.15	0.10	0.17	0.14	0.20	0.33	0.14
IABP	0.06	0.21	0.03	0.65	0.43	0.58	0.62	0.76	0.81	0.75	0.65	0.73	0.60	0.33	0.78
Hemopump	0.44	0.21	0.76	0.16	0.44	0.12	0.19	0.11	0.04	0.15	0.17	0.14	0.20	0.33	0.08

Fig. 2. Degree of satisfaction of the alternatives under hierarchical criteria set

TABLE I
RANKING OF BLOOD PUMPING DEVICES AT ALL THE LEVEL OF THE HIERARCHICAL STRUCTURE OF CRITERIA SET

Ranking	1	2	3
Comprehensive	A_2	A_3	A_1
Pump Performance	A_3	A_1	A_2
Safety	A_2	A_3	A_1
Ease Of Use	A_2	A_1	A_3
Applicability	A_2	A_1	A_3

pute the aggregated evaluation of the inputs of various sizes under the same framework. The idea of extended aggregation operators allow us to make a comparison between two lists of input arguments with different dimensions. The optimal ranking obtained by utilizing proposed $\mathcal{EPBM}$ operator for each criteria/ sub-criterion of the hierarchy is summarized in Table I. As from the ranking results of the devices one can see that A_1 amuse the last position, thus it may be possible that the group of decision support system team wants to analyze the performances of A_1 based on each broad category distinctly. Employing proposed $\mathcal{EPBM}$ operator we acquire the effectiveness of device A_1 based on different criteria as; the performance of A_1 based on the criteria pump performance is computed as, 0.4151 and for criteria safety is, 0.1994. So, the committee finds that the efficiency of A_1 with respect to the criteria safety, which is one of the vital criteria to elect a medical devise, is not up to the mark. Thus they can take initiative to improve the preference of the alternative A_1 with respect to that particular criteria.

IV. WEIGHTED FORM OF EXTENDED PBM OPERATOR

As all the evaluation criteria are not equally important, thus to take into account the variability among them we need to consider the weight vectors associated with the criteria set. In this section, we introduce the definition of the weighted extended PBM operator. By assigning weight vectors, one can rewrite the definition of extended-PBM operator as follows,

Definition 6: An extended aggregation function $\mathcal{WEPBM}:\bigcup_{n\geq 1}[0,1]^n \rightarrow [0,1]$ is called an weighted extended partitioned Bonferroni mean if $\mathcal{WEPBM}(x)=x$ where $x\in[0,1]$ and there are $p,q\geq 0$, $p+q>0$ such that for each $n\geq 2$ the restriction $\mathcal{WEPBM}|_{[0,1]^n}$ is a $WPBM^{p,q}_{(n)}$ operator related to partition $\{P^n_1,P^n_2,...,P^n_{d_n}\}$ of $\{1,2,...,n\}$ and defined as

$$WPBM^{p,q}_{(n)}(a_1,a_2,...,a_n)=\frac{1}{d_n}\Bigg(\sum_{r=1}^{d_n}\Bigg(\frac{1}{\sum_{i\in P^n_r}w_{i,n}}\sum_{i\in P^n_r}w_{i,n}a_i^p\Bigg(\frac{1}{\sum_{\substack{j\neq i\\ j\in P^n_r}}w_{j,n}}\sum_{\substack{j\neq i\\ j\in P^n_r}}w_{j,n}a_j^q\Bigg)\Bigg)^{\frac{1}{p+q}}\Bigg) \quad (3)$$

where for each $n\in\mathbb{N}$, there exists a $W_n=(w_{1,n},w_{2,n},...,w_{n,n})$ with $\sum_{j=1}^n w_{j,n}=1$, so that for any $n<m$ and any i,k element in $\{1,2,...,d_n\}$, $i\neq k$, there are j,s element in $\{1,2,...,d_m\}$ with $j\neq s$ so that $P^n_i\subset P^m_j$ and $P^n_k\subset P^m_s$.

Observe that, the above defined weighted extended-PBM operator $\mathcal{WEPBM}$ as a function on $\bigcup_{n\geq 1}[0,1]^n$ is idempotent, bounded and monotonic with respect to $\leq_\pi$.

In the next segment, we try to estimate the weight associated with the proposed EAF by implementing Max-Entropy technique.

A. Learning of weights in the aggregation operator

In this section our main focus is to identifying the weights of criteria set based on the associated subjective knowledge about the problem and regardless of the empirical data. The decision maker's subjective view regarding any aggregation operator is a necessary and crucial parameter in the decision making system which can be expressed by the behavioral property of the aggregation operator. In this regard, to measure the continuous transition of an aggregation operator from conjunction to disjunction Yager first introduced the orness measure [9] in 1988 associated with ordered weighted averaging (OWA) operator. It reflects how much an aggregation operator is turned to become more or-like, i.e. the degree to which an aggregation function is close to maximum function. Initially, it was introduced for power mean by Dujmovic [10] and was named as disjunction degree. For any set of aggregation arguments, the aggregated value always monotonically increases with the level of orness, i.e., the aggregated values are consistent with the orness level. Thus for any parameterized aggregation operator orness measure is used to control the parameters and to represent the opinion of a decision-maker. A decision maker with a pessimistic view may prefer high andness (more and-like, i.e. close to minimum), whereas a decision-maker with an optimistic perspective may prefer high orness [14]. In [13], Salido extends Yager's orness concept for the OWA operator to other mean operators. Some intrinsic properties of orness measure are studied in [11], [12]. However, in literature there exists a massive range of aggregation operators, which are essential in both theoretical and application purposes. Of which researchers developed the analytical formula of orness measure for some primary aggregation operators [18]–[22]. The orness measure associated with any aggregation operator can be presented in the following way,

Definition 7: [10], [23] Suppose, $\mathbf{a} \in [0,1]^n$ denotes the collection of input arguments. Then, the orness of an aggregation function $A : [0,1]^n \rightarrow [0,1]$ is defined as:

$$Orness(A) = \frac{\int_{[0,1]^n} A(\mathbf{a})\ d\mathbf{a} - \int_{[0,1]^n} Min(\mathbf{a})\ d\mathbf{a}}{\int_{[0,1]^n} Max(\mathbf{a})\ d\mathbf{a} - \int_{[0,1]^n} Min(\mathbf{a})\ d\mathbf{a}} \quad (4)$$

It is easy to compute the orness measure of fundamental aggregation functions like arithmetic mean, geometric mean, or OWA operator etc. by calculating the simple n-fold integral $\int_{[0,1]^n} A(\mathbf{a})\ \ d\mathbf{a}$. Whereas in case of complex aggregation operators like extended BM operator or our proposed $\mathcal{EPBM}$ operator, determination of the associated n-fold integration is quite impossible. In this instance, instead of deriving the analytical formula of the orness measure one can numerically estimate the integral value using Monte Carlo integration approach with an estimate of error [29].

In Monte Carlo simulation method, for any complicated operator the value of the n-fold integral $\int_{[0,1]^n} A(\mathbf{a})d\mathbf{a}$ is approximated by using the random sampling of the domain of integrands for evaluating the integrand values. For any n-dimensional unit hyper cube $[0,1]^n$, the Monte Carlo estimated value for the integral $\int_{[0,1]^n} A(\mathbf{a})d\mathbf{a}$ is $\frac{1}{N}\sum_{k=1}^{N} A(\mathbf{a})$ where N is the number of sampled points in $[0,1]^n$. The detailed can be found out in [27]–[29].

Thus, with $\int_{[0,1]^n} Max(\mathbf{a})\ \ d\mathbf{a} = \frac{n}{n+1}$ and $\int_{[0,1]^n} Min(\mathbf{a})\ \ d\mathbf{a} = \frac{1}{n+1}$, for $WPBM_{(n)}^{p,q}$ operator eq.(4) can be expressed more specifically as,

$$Orness(PBM_{(n)}^{p,q}) = \frac{\frac{1}{N}\sum_{k=1}^{N} WPBM_{(n)}^{p,q}(\mathbf{a}) - \frac{1}{n+1}}{\frac{n}{n+1} - \frac{1}{n+1}}$$
$$= \frac{n+1}{n-1}\frac{1}{N}\sum_{k=1}^{N} WPBM_{(n)}^{p,q}(\mathbf{a}) - \frac{1}{n-1} \quad (5)$$

Now from Equality (5), it is clear that the orness measure of $WPBM_{(n)}^{p,q}$ operator depends on four parameters: length of the input set n, associated parameters p and q, partition structure of the criteria set and finally, on its associated weight vector.

Next, to measure the amount of information used by the aggregation operator, Yager introduced a new measure named as dispersion in [9] as,

$$disp(W) = (-\sum_{j} w_j.ln\ w_j) \quad (6)$$

This measures a kind of entropy associated with the aggregation operator. This measure of dispersion uses the Shannon entropy concept in a certain sense that the more disperse the W the more of the information about the individual criteria is being used in the aggregation of the aggregate value.

Based on these two measures introduced by Yager, O'Hagan proposed a maximal entropy technique [24]–[26] in order to evaluate the unique weight vector for some aggregation operator by solving the mathematical programming problem just by specifying a desired value for the single parameter α which is the decision maker's view towards the specific aggregation operator and maximizing the entropy subject to the constraint.

Given $\alpha \in (0,1)$, for each $n \geq 2$ solving the following mathematical programming problem, we can find the weight vector associated with our proposed operator $WPBM_{(n)}^{p,q}$.

$$\begin{cases} Max\ (-\sum_{j=1}^{n} w_{j,n}.\ ln\ w_{j,n}) \\ Subject\ to, \\ \frac{n+1}{n-1}\frac{1}{N}\sum_{k=1}^{N} WPBM_{(n)}^{p,q}(\mathbf{a}) - \frac{1}{n-1} = \alpha \\ \sum_{j=1}^{n} w_{j,n} = 1 \\ w_{j,n} \in [0,1]\ \forall\ j \in \{1,2,...,n\}. \\ \alpha \in [0,1]. \end{cases} \quad (7)$$

If the values of the set of parameters (p,q) and structure of the partition set is known to the decision maker, then for each $n \geq 2$, the weight vector associated with our proposed operator $\mathcal{EPBM}$ could be determined solving the above mathematical programming problem (7). However in literature there exists several non-linear optimization techniques to solve non-linear optimization problem. But the aforementioned optimization problem consist of a complex non-linear constraints due to the semantic representation of the orness measure of the proposed operator. With the increasing number of parameters, the complexity of the problem get increased in proportionate linear O(n) manner. Future work concerns deeper analysis of solving this particular mathematical programming problem.

V. Conclusion

In this study, we have proposed a new concept of Extended Partitioned Bonferroni Mean ($\mathcal{EPBM}$) operator, to model the interconnections among the varying dimensional real data involved in the MCGDM problem. By developing this new aggregation operator, we have tried to overcome the drawbacks of the traditional PBM operator, which usually considers a fixed number of input arguments. We have investigated the global monotonicity and idempotence properties of the newly developed operator. Further, we have implemented this new concept of Extended Partitioned Bonferroni Mean ($\mathcal{EPBM}$) operator to handle the hierarchical structure of criteria set for evaluation of a medical device, where decision-maker is able to ranking the alternatives not only considering the whole set of criteria but also with respect to any intermediate higher-level point of view. In any decision-making process, the results at each level of the hierarchy is considered as a handy tool. Hence to analyze the hierarchical model, the aggregation operators defined for varying number of components can be taken as a significant constituent. Finally, utilizing the Max-Entropy technique we have learned the weight vector associated with our proposed operator.

In future, we hope to apply the $\mathcal{EPBM}$ operator to other fields and study more interesting properties of this operator. Moreover, we can extend the proposed aggregation operator to handle real-life problems where uncertainty is involved in the decision process. Along with that, our another motive is to establish the condition of weight vectors satisfies by the weighting triangle Δ associated with the $\mathcal{WEPBM}$ operator. That is, under which condition the above defined weighted $\mathcal{WEPBM}$ operator is an EAF or, under which condition of weight vectors it is monotonic with respect to $\leq_\alpha$ and $\leq_\beta$.

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