Robust Reference Tracking Control Design for Stochastic Polynomial Fuzzy Control System: A Sum-of-Squares Approach

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***Abstract*—In this study, the robust stochastic H infinity reference tracking control design is proposed for stochastic polynomial fuzzy system (SPFS) under external disturbance and continuous and discontinuous random fluctuations. To simplify the tracking control design, the desired reference trajectory is generated by a reference polynomial system. Under the concept of H infinity control, the designed control strategy aims to attenuate the effect of all possible finite energy disturbance on the tracking error to a prescribed level. Based on the polynomial Lyapunov function, with the help of Itô-Lévy formula, the sufficient conditions for robust stochastic H infinity reference tracking control design of SPFS is transformed to Hamilton-Jacobi inequalities (HJIs) problem. Due to the difficulties in solving HJIs problem, by using quadratic Lyapunov function, the solvable sum-of-squares (SOS) conditions are established for the robust stochastic H infinity reference tracking control design and it can be efficiently solved via MATLAB SOSTOOLS toolbox. An investment tracking strategy design for the stochastic financial system is provided to validate the effectiveness of proposed method.**

***Index Terms*—Polynomial fuzzy system, reference tracking control, stochastic system, sum-of-squares, robust control.**

I. Introduction

For the past two decades, Takagi-Sugeno (T-S) fuzzy model has been widely employed to describe a broad class of nonlinear system [1]. By linearizing the nonliear system at several operation points, the nonlinear system can be represented as a combination of local linearized systems with the corresponding IF-THEN rules. Under the concept of parallel distributed compensation (PDC) design in fuzzy-model-based (FMB) control [2], for each local linearized system, a specific controller is designed according to control purpose. Then, the fuzzy controller can be constructed by combining these local linear controllers with the corresponding IF-THEN rules. Further, these local linear controllers can be easily obtained by solving a set of linear matrix inequalities (LMIs) constrained problem [3]. There have a lot of fruitful results of FMB control on many control issues such as stabilization problem [4], [5], tracking control [6], [7], robust control [8], [9], etc.

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Recently, T-S fuzzy model has been extended to polynomial fuzzy model [10], [11]. Different than the conventional T-S fuzzy models, the polynomial fuzzy model enables the polynomial variables to be included in the local systems and this makes the polynomial fuzzy model be able to describe a more general class of nonlinear system. Further, by using the polynomial Lyapunov function for polynomial fuzzy model, the more relaxed sum-of-squares (SOS) conditions can be obtained for the control design. By applying third-party MATLAB SOSTOOLS toolbox [12], the derived SOS conditions can be numerically solved. Various control design problems of polynomial fuzzy model have been addressed. For example, the robust stabilization problem [13], [14], [15] and the tracking control problem [16], [17].

However, to the best of authors' knowledge, very few researches address the control design of stochastic polynomial fuzzy model. In fact, for the most of physical systems in real world, these systems suffer from not only environmental noise but also stochastic intrinsic fluctuations. For example, the stochastic fluctuations in financial market system are inevitable due to the rumors in the market, change of policy, uncertain tax rate, etc. In general, these effects are always formulated as state-dependent Brownian motion and Poisson process. Hence, from a practical point of view, the stochastic internal fluctuations in system should be considered for the control design. By using the Itô-Lévy type stochastic differential equation [18], the continuous and discontinuous random fluctuations in the system plant can be described as the Wiener process and Poisson process, respectively. The stochastic control becomes a popular field and there have many interesting issues [19], [20].

In this study, the robust stochastic H_∞ reference tracking control design is proposed for the stochastic polynomial fuzzy system (SPFS). At first, the polynomial reference model is employed to generate the desired tracking trajectory. To effectively attenuate the effect of external disturbance during the reference tracking process, the designed robust H_∞ tracking control strategy aims to attenuate the effect of all possible external disturbances on the tracking error to a prescribed

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level. Due to the differential compensation terms of stochastic processes in Itô-Lévy formula, in the case of polynomial Lyapunov function, the robust stochastic H_∞ reference tracking control design problem of SPFS is transformed to solving a set of Hamilton-Jacobi inequalities (HJIs). In general, HJI problem can not be solved numerically or analytically. Thus, by using the quadratic Lyapunov function, the solvable SOS conditions are established for the robust stochastic H_∞ reference tracking control design and it can be effectively solved by MATLAB SOSTOOLS toolbox. In the simulation, an investment tracking strategy design for stochastic financial system is provided to validate the proposed robust stochastic H_∞ reference tracking control design.

Notation: A^T denotes the transpose of matrix A; $A \geq 0$ denotes symmetric semi-positive definite matrix A; $E\{\cdot\}$ is the expectation operator, $\mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_x}) = \{v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_x} \Big| E[\int_0^{t_f} v^T v dt]^{\frac{1}{2}} < \infty\}$; $\frac{\partial f(x)}{\partial x}$ denotes the gradient column vector of differentiable function $f(x) \in \mathbb{R}^1$; $\frac{\partial^2 f(x)}{\partial x^2}$ is the Hessian matrix of twice-differentiable function $f(x(t)) \in \mathbb{R}^1$; A monomial in $x = [x_1, \cdots, x_n]$ is a function of the form $\prod_{i=1}^n x_i^{d_i}$ with non-negative integers $\{d_i\}_{i=1}^n$ and it's degree is defined as $d = \sum_{i=1}^n d_i$; A polynomial $P(x)$ is defined as a finite linear combination of monomials in x with real coefficients. $P(x)$ is sum of squares (SOS) if and only if $P(x) = \sum_{i=1}^m f_i^2(x)$ where $\{f_i(x)\}_{i=1}^m$ is a set of polynomial functions of x and $m \in \mathbb{N}$.

II. Preliminary

A. Stochastic Polynomial Fuzzy Model

Consider the following nonlinear stochastic system:

$$\begin{aligned} dx(t) &= [f(x(t)) + g(x(t))u(t) + h(x(t))v(t)]dt \\ &\quad + i(x(t))dw(t) + j(x(t))dn(t) \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state vector, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $v(t) \in \mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{n_x})$ represents the finite energy external disturbance, $w(t) \in \mathbb{R}^1$ is standard 1-D Wiener process and $n(t) \in \mathbb{R}^1$ is Poisson counting process with jump intensity $\lambda \in \mathbb{R}^+$. The 1-D Wiener process $w(t)$ and Poisson counting process $n(t)$ are defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ where Ω denotes the sample space, $\mathcal{P}$ is probability measure on Ω, the $\sigma - field$ $\mathcal{F}_t$ are generated by $w(s)$ and $n(s)$, for $s \leq t$, and $\mathcal{F} = \{\mathcal{F}_{\bar{t}}\}_{\bar{t} \geq 0}$. It is assumed that two stochastic processes $w(t)$ and $n(t)$ are mutually independent. The functions $f(x(t))$, $g(x(t))$, $h(x(t))$, $i(x(t))$ and $j(x(t))$ in (1) are Lipschitz continuous functions with appropriate dimensions.

Under the concept of sector nonlinearity [1], the following local stochastic polynomial fuzzy system (SPFS) are provided to exactly describe the nonlinear stochastic system in (1):

$$\begin{aligned} &Plant\ Rule\ i \\ &If\ z_1(t)\ is\ \varpi_{1,i},\ and\ \cdots\ and, z_g(t)\ is\ \varpi_{g,i} \\ &Then \\ &dx(t) = [A_i(x(t))x(t) + B_i(x(t))u(t) + B_{v,i}(x(t))v(t)]dt \\ &\qquad + C_i(x(t))x(t)dw(t) + D_i(x(t))x(t)dn(t) \end{aligned} \quad (2)$$

where $i = 1, \cdots, r$, r is the number of plant rules, $z_j(t)$ is the premise variable , for $j = 1, \cdots, g$, g is the number of premise variables, the membership function $\varpi_{j,i}$ is associated with the jth premise variable $z_j(t)$ in the ith plant rule and $A_i(x(t)$, $B_i(x(t))$, $B_{v,i}(x(t))$, $C_i(x(t))$, $D_i(x(t))$ are polynomials matrices in $x(t)$, for $i = 1, \cdots, r$. Then, after the defuzzification process, the nonlinear stochastic system in (1) can be inferred as follows:

$$\begin{aligned} dx(t) = \sum_{i=1}^r \mu_i(z(t))\{&[A_i(x(t))x(t) + B_i(x(t))u(t) \\ &+ B_{v,i}(x(t))v(t)]dt + C_i(x(t))x(t)dw(t) \\ &+ D_i(x(t))x(t)dn(t)\} \end{aligned} \quad (3)$$

with

$$\mu_i(z(t)) = \frac{\prod_{j=1}^g \psi_{j,i}(z_j(t))}{\sum_{k=1}^r \prod_{j=1}^g \psi_{j,k}(z_j(t))}, \mu_i(z(t)) \geq 0$$

$$\sum_{i=1}^r \mu_i(z(t)) = 1,\ for\ i = 1, \cdots, r$$

where $z(t) = [z_1^T(t), \cdots, z_g^T(t)]^T$, $\{\mu_i(z(t))\}_{i=1}^r$ denote the normalized grade membership functions, $\psi_{j,i}(z_j(t))$ is the grade membership function corresponding to the membership function $\varpi_{j,i}$.

To generate the desired tracking trajectory, the following polynomial reference model is employed in this study [16]:

$$dx_r(t) = [A_r(x_r(t))x_r(t) + B_r(x_r(t))r(t)]dt \quad (4)$$

where $x_r(t) \in \mathbb{R}^{n_x}$ denotes the desired trajectory to be tracked, $A_r(x_r(t))$ and $B_r(x_r(t))$ are the polynomial system matrices and $r(t) \in \mathbb{R}^{n_x}$ is the reference signal. It is assumed the reference model in (4) is stable. In (4), according to the designer's requirement, the desired trajectory $x_r(t)$ is generated by the reference signal $r(t)$ with specific system matrices $A_r(x_r(t))$ and $B_r(x_r(t))$. Especially, if $r(t)$ is the desired trajectory to be tracked, then the matrices can be chosen as $A_r(x_r(t)) = -I$ and $B_r(x_r(t)) = I$. In this case, $x_r(t)$ is identical to the $r(t)$ at the steady state, i.e., $x_r(t) = r(t)$ at steady state.

By utilizing PDC technique, the following polynomial state-feedback controller is constructed by utilizing the states information of the SPFS in (3) and reference model in (4):

$$\begin{aligned} &Control\ Rule\ j \\ &If\ z_1(t)\ is\ \varpi_{1,j},\ and\ \cdots\ and, z_g(t)\ is\ \varpi_{g,j} \\ &Then \\ &u(t) = K_j^e(x(t))e(t) + K_j^r(x(t))x_r(t) \end{aligned} \quad (5)$$

where $e(t) = x(t) - x_r(t)$ denotes the tracking error and $K_j^e(x(t))$ and $K_j^r(x(t))$ are polynomial controller gains to be designed, for $i = 1, \cdots, r$. After the defuzzification process, the polynomial state-feedback controller can be written as:

$$u(t) = \sum_{j=1}^r \mu_j(z(t))\{K_j^e(x(t))e(t) + K_j^r(x(t))x_r(t)\} \quad (6)$$

Then, by (3), (4), (6), the SPFS can be written as:

$$\begin{aligned} dx(t) = \sum_{j=1}^{r}\sum_{i=1}^{r} & \mu_j(z(t))\mu_i(z(t))\{[A_i(x(t))x(t) \\ & + B_i(x(t))(K_j^e(x(t))e(t) + K_j^r(x(t))x_r(t)) \\ & + B_{v,i}(x(t))v(t)]dt + C_i(x(t))x(t)dw(t) \\ & + D_i(x(t))x(t)dn(t)\} \end{aligned} \tag{7}$$

B. Problem formulation

In general, the external disturbance $v(t)$ in (3) is inevitable during the reference tracking process and it may deteriorate the tracking performance. Besides, the reference signal $r(t)$ is unpredictable for the stochastic fuzzy polynomial system in (3). Thus, the following robust stochastic H_∞ reference tracking control is proposed to effectively attenuate the aforementioned effect on the tracking process:

$$J_\infty(u(t)) = \sup_{\bar{v}(t) \in \mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n_x})} \frac{\begin{array}{l} E\{\int_0^{t_f}[x(t)-x_r(t)]^T \\ \times Q\ [x(t)-x_r(t)] \\ +u^T(t)Ru(t)dt \\ -V(x(0),x_r(0))\} \end{array}}{E\{\int_0^{t_f}\bar{v}^T(t)\bar{v}(t)dt\}} \tag{8}$$

where $Q \geq 0$ and $R > 0$ denotes the weighting matrices of state tracking performance and control effort, respectively, $\bar{v}(t) = [r^T(t), v^T(t)]^T$ denotes the augmented external disturbance and the positive function $V(x(0), x_r(0))$ denotes the effect of initial condition to be excluded.

For the robust stochastic H_∞ reference tracking control performance in (8), we aims to find a robust stochastic H_∞ tracking control strategy $u(t)$ to attenuate the effect of all finite energy disturbance $\bar{v}(t) \in \mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n_x})$ on the tracking process. If one can specify a $u^*(t)$ such that $J(u^*(t)) \leq \rho^*$, for a prescribed $\rho^* > 0$, then the effect of all finite energy disturbance $\bar{v}(t)$ on the tracking error $x(t) - x_r(t)$ is reduced under a disturbance attenuation level ρ^* from an energy point of view.

III. Robust Stochastic Reference Tracking control design for SPFS

In this section, the robust stochastic H_∞ reference tracking control design is proposed for SPFS in (3). In general, due to the complex stochastic process in (3), the analysis of SPFS is much difficult than the deterministic polynomial fuzzy system (DPFS). By using polynomial Lyapunov function and quadratic Lyapunov function, two sufficient conditions are established for robust stochastic H_∞ reference tracking control design. To lighten the notation, the notation t associated with the variable is dropped. For example, the notation x is employed instead of $x(t)$.

To begin with, by letting $\bar{x} = [x_r^T, e^T]^T$, the following augmented SPFS is proposed:

$$\begin{aligned} d\bar{x} = \sum_{i=1}^{r}\sum_{i=1}^{r} & \mu_j(z)\mu_i(z)\{[\bar{A}_i(\bar{x})\bar{x} + \bar{B}_i(\bar{x})\bar{K}_j(\bar{x})\bar{x} \\ & + \bar{B}_{v,i}(\bar{x})\bar{v}]dt + \bar{C}_i(\bar{x})\bar{x}dw + \bar{D}_i(\bar{x})\bar{x}dn\} \end{aligned} \tag{9}$$

where $\bar{A}_i(\bar{x}) = \begin{bmatrix} A_r(x_r) & 0 \\ A_i(x) - A_r(x_r) & A_i(x) \end{bmatrix}$, $\bar{B}_{v,i}(\bar{x}) = \begin{bmatrix} B_r(x_r) & 0 \\ -B_r(x_r) & B_{v,i}(x) \end{bmatrix}$, $\bar{C}_i(\bar{x}) = \begin{bmatrix} 0 & 0 \\ C_i(x) & C_i(x) \end{bmatrix}$, $\bar{D}_i(\bar{x}) = \begin{bmatrix} 0 & 0 \\ D_i(x) & D_i(x) \end{bmatrix}$, $\bar{B}_i(\bar{x}) = \begin{bmatrix} 0, & B_i^T(x) \end{bmatrix}^T$, $\bar{K}_j(\bar{x}) = \begin{bmatrix} K_j^r(x)\ , & K_j^e(x) \end{bmatrix}$.

By the augmented SFPS in (9), the robust stochastic H_∞ reference tracking control performance in (8) can be rewritten as:

$$J_\infty(u) = \sup_{\bar{v} \in \mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n_x})} \frac{\begin{array}{l} E\{\int_0^{t_f}\bar{x}^T\bar{Q}\bar{x} + u^TRudt \\ -V(\bar{x}(0))\} \end{array}}{E\{\int_0^{t_f}\bar{v}^T\bar{v}dt\}} \tag{10}$$

where $\bar{Q} = diag\{0, Q\}$.

Before achieving our main result, two useful lemmas are proposed to help the design of robust H_∞ tracking control for SPFS.

Lemma 1 [20, Lemma 2,]: Let X, Y be two matrices with appropriate dimension, the following inequality holds:

$$X^TY + Y^TX \leq \rho X^TX + \frac{1}{\rho}Y^TY$$

for any $\rho > 0$.

Lemma 2 [18, Th. 1.16]: With the twice-differentiable Lyapunov function $V(\bar{x})$ satisfied with $V(0) = 0, V(\cdot) \geq 0$ for the augmented SPFS in (9) , the Itô-Lévy formula for $dV(\bar{x})$ is given as:

$$\begin{aligned} dV(\bar{x}) = \sum_{i=1}^{r}\sum_{j=1}^{r} & \mu_j(z)\mu_i(z)\{[(\tfrac{\partial V(\bar{x})}{\partial \bar{x}})^T(\bar{A}_i(\bar{x})\bar{x} + \bar{B}_i(\bar{x}) \\ & \times \bar{K}_j(\bar{x})\bar{x} + \bar{B}_{v,i}(\bar{x})\bar{v}) \ + \tfrac{1}{2}\bar{x}^T\bar{C}_i^T(\bar{x})\tfrac{\partial^2 V(\bar{x})}{\partial \bar{x}^2}\bar{C}_j(\bar{x})\bar{x}]dt \\ & + (\tfrac{\partial V(\bar{x})}{\partial \bar{x}})^T\bar{C}_i(\bar{x})\bar{x}dw + [V(\Upsilon_n(\bar{x}) + \bar{x}) - V(\bar{x})]dn\} \end{aligned} \tag{11}$$

where $\Upsilon_n(\bar{x}) = \sum_{n=1}^{r}\mu_n(z)\bar{D}_n(\bar{x})\bar{x}$.

Based on the above lemmas, the robust stochastic H_∞ reference tracking control design with polynomial Lyapunov function for SPFS in (3) is given as follows:

Theorem 1: Consider the SPFS in (3) and reference model in (4). If there exist a positive non-singular symmetric polynomial matrix $P(\bar{x}(t))$, $\rho > 0$ and a set of polynomial controller gains $\{\bar{K}_j(\bar{x})\}_{j=1}^r$ satisfy the following HJIs:

$$\begin{aligned} & E\{\bar{x}^T[\bar{Q} + \bar{K}_j^T(\bar{x})R\bar{K}_j(\bar{x}) + P(\bar{x})\bar{A}_i(\bar{x}) + \bar{A}_i^T(\bar{x})P(\bar{x}) \\ & + P(\bar{x})\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) + \bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P(\bar{x}) + \tfrac{1}{2}\bar{C}_i^T(\bar{x}) \\ & \times \tfrac{\partial^2\bar{x}^TP(\bar{x})\bar{x}}{\partial \bar{x}^2}\bar{C}_i(\bar{x}) + \lambda(P(\Upsilon_n(\bar{x}) + \bar{x}) - P(\bar{x}) \\ & + \bar{D}_i^T(\bar{x})P(\Upsilon_v(\bar{x}) + \bar{x}) + P(\Upsilon_s(\bar{x}) + \bar{x})\bar{D}_i(\bar{x}) \\ & + \bar{D}_i^T(\bar{x})P(\Upsilon_d(\bar{x}) + \bar{x})\bar{D}_i(\bar{x})) + \sum_{l=1}^{2n_x}\tfrac{\partial P(\bar{x})}{\partial \bar{x}_l}(\bar{A}_i^l(\bar{x})\bar{x} \\ & + \bar{B}_i^l(\bar{x})\bar{K}_j(\bar{x})\bar{x} + \tfrac{4n_x}{\rho}\bar{x}\bar{B}_{v,i}^l(\bar{x})(\bar{B}_{v,i}^l(\bar{x}))^T\bar{x}^T\tfrac{\partial P(\bar{x})}{\partial \bar{x}_l}^T) \\ & + \tfrac{2}{\rho}P(\bar{x})\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})P(\bar{x})]\bar{x}\} < 0 \\ & for\ i, j = 1, \cdots, r, \end{aligned} \tag{12}$$

where $\bar{x}_l$ denotes the lth component in $\bar{x}$, $\bar{A}_i^l(\bar{x})$, $\bar{B}_i^l(\bar{x})$, $\bar{B}_{v,i}^l(\bar{x})$ are the lth row vector in polynomial matrices $\bar{A}_i(\bar{x})$,

$\bar{B}_i(\bar{x})$, $\bar{B}_{v,i}(\bar{x})$, respectively, then the robust stochastic H_∞ reference tracking control performance in (10) can be achieved with a disturbance attenuation level $\rho > 0$. Moreover, if the disturbance $\bar{v}$ is vanished, the augmented SPFS in (9) is asymptotically stable in probability.

Proof: Consider the numerator part of robust H_∞ tracking control performance in (10) with polynomial Lyapunov function $V(\bar{x}) = \bar{x}^T P(\bar{x})\bar{x}$, then we have

$$\begin{aligned} &E\{\int_0^{t_f} \bar{x}^T\bar{Q}\bar{x} + u^T Rudt - V(\bar{x}(0))\} \\ &= E\{\int_0^{t_f} \bar{x}^T\bar{Q}\bar{x} + u^T Rudt + dV(\bar{x})\} \\ &-E\{V(\bar{x}(0))\} - E\{V(\bar{x}(t_f))\} + E\{V(\bar{x}(0))\}. \end{aligned} \tag{13}$$

By using the Itô-Lévy formula in (11) with the fact $E\{dw\} = 0$ and $E\{dn\} = \lambda dt$, (13) can be written as:

$$\begin{aligned} &E\{\int_0^{t_f} \bar{x}^T\bar{Q}\bar{x} + u^T Rudt - V(\bar{x}(0))\} \\ &= E\{\int_0^{t_f} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_j(z)\mu_i(z)\{\bar{x}^T[\bar{Q} + \bar{K}_i^T(\bar{x})R\bar{K}_j(\bar{x}) \\ &+P(\bar{x})\bar{A}_i(\bar{x}) + \bar{A}_i^T(\bar{x})P(\bar{x}) + P(\bar{x})\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) \\ &+\bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P(\bar{x}) + \tfrac{1}{2}\bar{C}_i^T(\bar{x})\tfrac{\partial^2\bar{x}^T P(\bar{x})\bar{x}}{\partial\bar{x}^2}\bar{C}_j(\bar{x}) \\ &+\lambda(P(\Upsilon_n(\bar{x}) + \bar{x}) - P(\bar{x}) + \bar{D}_i^T(\bar{x})P(\Upsilon_v(\bar{x}) + \bar{x}) \\ &+P(\Upsilon_s(\bar{x}) + \bar{x})\bar{D}_i(\bar{x}) + \bar{D}_i^T(\bar{x})P(\Upsilon_d(\bar{x}) + \bar{x})\bar{D}_j(\bar{x}))]\bar{x} \\ &+\bar{x}^T P(\bar{x})\bar{B}_{v,i}(\bar{x})\bar{v}(t) + \bar{v}^T(t)\bar{B}_{v,i}^T(\bar{x})P(\bar{x})\bar{x} \\ &+\bar{x}^T\sum_{l=1}^{2n_x}\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}(\bar{A}_i^l(\bar{x})\bar{x} + \bar{B}_i^l(\bar{x})\bar{K}_j(\bar{x})\bar{x} + \bar{B}_{v,i}^l(\bar{x})\bar{v})\bar{x}\}dt\} \\ &-E\{V(\bar{x}(t_f))\} \end{aligned} \tag{14}$$

where $\Upsilon_n(\bar{x}) = \sum_{n=1}^{r}\mu_n(z)\bar{D}_n(\bar{x})\bar{x}$. It is worth to point out that $\bar{B}_{v,i}^l(\bar{x})\bar{v}$ is a scalar, i.e., $\bar{B}_{v,i}^l(\bar{x})\bar{v} \in \mathbb{R}^1$.

By using Lemma 1, the following inequalities are held:

$$\begin{aligned} &\bar{x}^T P(\bar{x})\bar{B}_{v,i}(\bar{x})\bar{v} + \bar{v}^T\bar{B}_{v,i}^T(\bar{x})P(\bar{x})\bar{x} \\ &\leq \tfrac{2}{\rho}\bar{x}^T P(\bar{x})\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})P(\bar{x})\bar{x} + \tfrac{\rho}{2}\bar{v}^T\bar{v} \end{aligned} \tag{15}$$

$$\begin{aligned} &\bar{x}^T\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}\bar{B}_{v,i}^l(\bar{x})\bar{v}\bar{x} \\ &= \bar{x}^T\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}\bar{x}\bar{B}_{v,i}^l(\bar{x})\bar{v} \\ &\leq \tfrac{4n_x}{\rho}\bar{x}^T\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}\bar{x}\bar{B}_{v,i}^l(\bar{x})(\bar{B}_{v,i}^l(\bar{x}))^T\bar{x}^T\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}^T\bar{x} + \tfrac{\rho}{4n_x}\bar{v}^T\bar{v} \end{aligned} \tag{16}$$

$for\ i = 1, \cdots, r,\ l = 1, \cdots, 2n_x,\ \rho > 0.$

By substituting (15)–(16) into (14), we immediately have

$$\begin{aligned} &E\{\int_0^{t_f} \bar{x}^T\bar{Q}\bar{x} + u^T Rudt - V(\bar{x}(0))\} \\ &\leq E\{\int_0^{t_f} \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_j(z)\mu_i(z)[\bar{x}^T[\bar{Q} + \bar{K}_j^T(\bar{x})R\bar{K}_j(\bar{x}) \\ &+P(\bar{x})\bar{A}_i(\bar{x}) + \bar{A}_i^T(\bar{x})P(\bar{x}) + P(\bar{x})\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) \\ &+\bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P(\bar{x}) + \tfrac{1}{2}\bar{C}_i^T(\bar{x})\tfrac{\partial^2\bar{x}^T P(\bar{x})\bar{x}}{\partial\bar{x}^2}\bar{C}_i(\bar{x}) \\ &+\lambda(P(\Upsilon_n(\bar{x}) + \bar{x}) - P(\bar{x}) + \bar{D}_i^T(\bar{x})P(\Upsilon_v(\bar{x}) + \bar{x}) \\ &+P(\Upsilon_s(\bar{x}) + \bar{x})\bar{D}_i(\bar{x}) + \bar{D}_i^T(\bar{x})P(\Upsilon_d(\bar{x}) + \bar{x})\bar{D}_i(\bar{x})) \\ &+\sum_{l=1}^{2n_x}\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}(\bar{A}_i^l(\bar{x})\bar{x} + \bar{B}_i^l(\bar{x})\bar{K}_j(\bar{x})\bar{x} + \tfrac{4n_x}{\rho}\bar{x}\bar{B}_{v,i}^l(\bar{x}) \\ &\times(\bar{B}_{v,i}^l(\bar{x}))^T\bar{x}^T\tfrac{\partial P(\bar{x})}{\partial\bar{x}_l}^T) + \tfrac{2}{\rho}P(\bar{x})\bar{B}_{v,i}(\bar{x}) \\ &\times\bar{B}_{v,i}^T(\bar{x})P(\bar{x})]\bar{x} + \rho\bar{v}^T\bar{v}dt\} \end{aligned} \tag{17}$$

If the HJIs in (12) hold, then we have following inequality:

$$\begin{aligned} &E\{\int_0^{t_f} \bar{x}^T\bar{Q}\bar{x} + u^T Rudt - V(\bar{x}(0))\} \leq \rho E\{\int_0^{t_f}\bar{v}^T\bar{v}dt\} \\ &\forall\ \bar{v} \in \mathcal{L}_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n_x}) \end{aligned} \tag{18}$$

which implies the robust H_∞ tracking control performance in (10) is achieved with a disturbance attenuation level $\rho > 0$, i.e., $J_\infty(u(t)) \leq \rho$.

On the other hand, if the disturbance $\bar{v}$ is vanished in SPFS of (9), (18) implies the following fact:

$$\begin{aligned} &E\{\int_0^{t_f}\bar{x}^T\bar{Q}\bar{x} + u^T Rudt\} \leq E\{V(\bar{x}(0))\} \\ &\forall\ t_f \in [0, \infty) \end{aligned} \tag{19}$$

. By letting $t_f \to \infty$ in (19), (19) reveals the total energy of $\bar{x}$ is bounded above by $E\{V(\bar{x}(0))\}$ and it reveals $\bar{x}$ is asymptotically stable in probability, i.e., $E\{\bar{x}\} \to 0$, as $t_f \to \infty$. $\square$

In Theorem 1, the robust stochastic H_∞ reference tracking control design for SPFS with polynomial Lyapunov function is investigated. Due to the differential compensation terms of Wiener process and Poisson counting process in Itô-Lévy formula, the terms $\frac{\partial^2\bar{x}^T P(\bar{x})\bar{x}}{\partial\bar{x}^2}$ and $P(\Upsilon_d(\bar{x}) + \bar{x}(t))$ of design variable $P(\bar{x})$ are embedded in (12). Thus, for the sufficient conditions in (12), the design problem is transformed to an equivalent HJIs-constrained problem. Although several decoupling methods have been widely investigated in the conventional control design of DPFS, there does not have any decoupling method to relax the terms of $\frac{\partial^2\bar{x}^T P(\bar{x})\bar{x}}{\partial\bar{x}^2}$ and $P(\Upsilon_d(\bar{x}) + \bar{x})$ for the control design of SPFS. To tackle with this problem, by using the quadratic Lyapunov function as $V(\bar{x}) = \bar{x}^T P\bar{x}$ with matrix $P > 0$, the following theorem is proposed to transform the robust H_∞ tracking control design into solvable SOS-constrained problem:

Theorem 2: Consider the SPFS in (3) and reference model in (4). If there exist a matrix $W > 0$, $\rho > 0$ and a set of polynomial matrices $\{M_j(\bar{x})\}_{j=1}^r$ satisfied with the following SOS conditions:

$$\begin{aligned} &-v_1^T(\left\{\begin{bmatrix} \Pi_{ij}^1(\bar{x}) & \Pi_{ij}^2(\bar{x}) \\ * & \Pi_{ij}^3(\bar{x}) \end{bmatrix} + \epsilon_{ij}((\bar{x}))I\right\})v_1\ is\ SOS \\ &,\ \forall i, j = 1, \cdots r. \end{aligned} \tag{20}$$

where $v_1 \in \mathbb{R}^{10n_x+n_u}$ is the vector that independent of $\bar{x}$, $\{\epsilon_{ij}((\bar{x}))\}_{i,j=1}^r$ are non-negative polynomials for all $\bar{x}$, $\Pi_{ij}^1(\bar{x}) = \bar{A}_i(\bar{x})W + W\bar{A}_i^T(\bar{x}) + \bar{B}_i(\bar{x}(t))M_j(\bar{x}) + M_j^T(\bar{x})\bar{B}_i^T(\bar{x}) + \lambda[W\bar{D}_i^T(\bar{x}) + \bar{D}_i(\bar{x})W]$, $\Pi_{ij}^2(\bar{x}) = [W\bar{Q}^{\frac{1}{2}},\ M_j^T(\bar{x}),\ W\bar{C}_i^T(\bar{x}),\ W\bar{D}_i^T(\bar{x}),\ \bar{B}_{v,i}(\bar{x})]$, $\Pi_{ij}^3(\bar{x}) = diag\{-I, -R^{-1}, -W, -\lambda^{-1}W, -\rho I\}$, then the polynomial controller gains can be constructed as $\bar{K}_j(\bar{x}(t)) = M_j(\bar{x})W^{-1}$, $\forall i = 1, \cdots, r$, and the robust stochastic H_∞ reference tracking control performance can be achieved with a disturbance attenuation level $\rho > 0$. Moreover, if the disturbance $\bar{v}$ is vanished, the augmented SPFS in (9) is asymptotically stable in probability.

Proof: Define the Lyapunov function as $V(\bar{x}) = \bar{x}^T P \bar{x}$ with matrix $P > 0$, (14) can be written as:

$$\begin{array}{l} E\{\int_0^{t_f} \bar{x}^T \bar{Q}\bar{x} + u^T R u dt - V(\bar{x}(0))\} \\ \leq E\{\int_0^{t_f} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_j(z)\mu_i(z)[\bar{x}^T[\bar{Q} + \bar{K}_i^T(\bar{x}) R \bar{K}_j(\bar{x}) \\ +P\bar{A}_i(\bar{x}) + \bar{A}_i^T(\bar{x})P + P\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) + \bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P \\ +\bar{C}_j^T(\bar{x})P\bar{C}_i(\bar{x}) + \lambda(\bar{D}_i^T(\bar{x})P\bar{D}_j(\bar{x}) + \bar{D}_i^T(\bar{x})P \\ +P\bar{D}_i(\bar{x})]\bar{x} + \bar{x}^T P\bar{B}_{v,i}(\bar{x})\bar{v} + \bar{v}^T\bar{B}_{v,i}^T(\bar{x})P\bar{x}dt\} \end{array} \tag{21}$$

By using Lemma 1, we have the following inequalities:

$$\begin{array}{l} \bar{x}^T P\bar{B}_{v,i}(\bar{x})\bar{v} + \bar{v}^T\bar{B}_{v,i}^T(\bar{x})P\bar{x} \\ \leq \frac{1}{\rho}\bar{x}^T P\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})P\bar{x} + \rho\bar{v}^T(t)\bar{v}(t) \\ , for\ i = 1, \cdots, r,\ \ \rho > 0 \end{array} \tag{22}$$

By utilizing (22), we have

$$\begin{array}{l} E\{\int_0^{t_f} \bar{x}^T \bar{Q}\bar{x} + u^T R u dt - V(\bar{x}(0))\} \\ \leq E\{\int_0^{t_f} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_j(z)\mu_i(z)[\bar{x}^T[\bar{Q} + \bar{K}_j^T(\bar{x}) R \bar{K}_j(\bar{x}) \\ +P\bar{A}_i(\bar{x}) + \bar{A}_i^T(\bar{x})P + P\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) \\ +\bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P + \bar{C}_i^T(\bar{x})P\bar{C}_i(\bar{x}) \\ +\lambda(\bar{D}_i^T(\bar{x})P + P\bar{D}_i(\bar{x}) + \bar{D}_i^T(\bar{x})P\bar{D}_i(\bar{x})) \\ +\frac{1}{\rho}P\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})P]\bar{x} + \rho\bar{v}^T\bar{v}dt\} \end{array} \tag{23}$$

Obviously, if the following polynomial constraints are satisfied:

$$\begin{array}{l} E\{\bar{Q} + \bar{K}_j^T(\bar{x}) R \bar{K}_j(\bar{x}) + P\bar{A}_i(\bar{x}) \\ +\bar{A}_i^T(\bar{x})P + P\bar{B}_i(\bar{x})\bar{K}_j(\bar{x}) + \bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x})P \\ +\bar{C}_i^T(\bar{x})P\bar{C}_i(\bar{x}) + \lambda(\bar{D}_i^T(\bar{x})P + P\bar{D}_i(\bar{x}) \\ +\bar{D}_i^T(\bar{x})P\bar{D}_i(\bar{x})) + \frac{1}{\rho}P\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})P\} < 0 \\ for\ i, j = 1, \cdots, r, \end{array} \tag{24}$$

then

$$\begin{array}{l} E\{\int_0^{t_f} \bar{x}^T(t)\bar{Q}\bar{x}(t) + u^T(t) R u(t) dt - V(\bar{x}(0))\} \\ \leq E\{\int_0^{t_f} \rho\bar{v}^T(t)\bar{v}(t)dt\}, \forall\ \bar{v}(t) \in L_2^{\mathcal{F}_t}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n_x}) \end{array} \tag{25}$$

, i.e., $J_\infty(u(t)) \leq \rho$.

Moreover, by pre-multiplying and post-multiplying the matrix $W = P^{-1}$ to the constraints in (24), we have

$$\begin{array}{l} E\{W\bar{Q}W + W\bar{K}_j^T(\bar{x}) R \bar{K}_j(\bar{x})W + \bar{A}_i(\bar{x})W \\ +W\bar{A}_i^T(\bar{x}) + \bar{B}_i(\bar{x})\bar{K}_j(\bar{x})W + W\bar{K}_j^T(\bar{x})\bar{B}_i^T(\bar{x}) \\ +W\bar{C}_i^T(\bar{x})W^{-1}\bar{C}_i(\bar{x})W + \lambda[W\bar{D}_i^T(\bar{x}) \\ +\bar{D}_i(\bar{x})W + W\bar{D}_i^T(\bar{x})W^{-1}\bar{D}_i(\bar{x})W] \\ +\frac{1}{\rho}\bar{B}_{v,i}(\bar{x})\bar{B}_{v,i}^T(\bar{x})\} < 0 \\ for\ i, j = 1, \cdots, r, \end{array} \tag{26}$$

By letting $M_j(\bar{x}) = \bar{K}_j(\bar{x})W$ and applying Schur complement to (26), (26) can be rewritten as:

$$\begin{array}{l} \begin{bmatrix} \Pi_{ij}^1(\bar{x}) & \Pi_{ij}^2(\bar{x}) \\ * & \Pi_{ij}^3(\bar{x}) \end{bmatrix} \leq 0 \\ ,\ \forall i, j = 1, \cdots, r. \end{array} \tag{27}$$

. Hence, the constraints in (27) hold if the SOS constraints in (20) are satisfied. Besides, from the result in (25), the asymptotic stability of augmented SPFS in (9) can be proven by similar derivation in Theorem 1. □

In Theorem 2, the robust H_∞ tracking control design for SPFS is transformed to equivalent SOS-constrained problem. With the help of third-party MATLAB SOSTOOLS toolbox and semi-definite program (SDP) solver, the feasibility of SOS constraints can be easily examined and the polynomial controller can be implemented. Even the constant matrix variable W is utilized for the fuzzy polynomial controller design in Theorem 2, (20) can be regarded as an extension of conventional stochastic T-S fuzzy tracking control design. In fact, if the polynomial matrices of SPFS in (3) are reduced to constant matrices, then the SOS constraints in (20) can be reduced to conventional LMIs constraints.

IV. Simulation Example

In this section, an investment tracking strategy design for financial system is provided as design example. Moreover, to describe the effect of environmental noise and intrinsic fluctuation, the financial system in [20] is modified as:

$$\begin{array}{l} dx_1 = (x_3 + (x_2 - a)x_1 + u_1 + v_1)dt \\ \qquad + 0.01x_1dw - 0.5x_1dn \\ dx_2 = (1 - bx_2 + x_1^2 + u_2 + v_2)dt + 0.01x_2dw - 0.5x_2dn \\ dx_3 = (-x_1 - cx_3 + u_3 + v_3)dt + 0.01x_3dw - 0.5x_3dn \end{array} \tag{28}$$

where x_1 represent the interest rate, x_2 is the investment demand, x_3 denotes the price index. The parameters $a = 1.5$, $b = 1$ and $c = 0.2$ are the saving amount, per-investment cost and elasticity coefficient of demands, respectively. $u = [u_1, u_2, u_3]^T$ and $v = [v_1, v_2, v_3]^T$ are the investment strategy and external disturbance, respectively. Since the financial system suffers from the unpredictable events such as wars and government policies in the real world, these effects should be considered as external disturbance and intrinsic fluctuations on the investment tracking strategy design. w is the 1-D standard Wiener process and n is the Poisson counting process with jump intensity $\lambda = 0.2$. Then, in (28), the terms $0.01x_idw$ and $-0.5x_idn$ are used to mimic the internal continuous and discontinuous random fluctuations on state x_i, for $i = 1, 2, 3$.

To construct the SPFS of financial system in (28), the operation domain of x_2 is defined as $[0.5, 5]$ and the function $f(x_2) = (1 - bx_2)/x_2 \in [f_{\min}, f_{\max}] = [-0.8, 1]$. Then, the financial system in (28) can be represented as the following SPFS with two IF-THEN rules: $A_1 = \begin{bmatrix} x_2 - a & 0 & 1 \\ x_1 & f_{\max} & 0 \\ -1 & 0 & -c \end{bmatrix}$, $A_2 = \begin{bmatrix} x_2 - a & 0 & 1 \\ x_1 & f_{\min} & 0 \\ -1 & 0 & -c \end{bmatrix}$, $B_1 = B_{v,1} = B_2 = B_{v,2} = diag(1, 1, 1)$, $C_1(x) = C_2(x) = diag\{0.01, 0.01, 0.01\}$, $D_1(x) = D_2(x) = diag\{-0.5, -0.5, -0.5\}$ with the grade membership functions $\mu_1(x_2) = (f(x_2) - f_{\min})/(f_{\max} - f_{\min})$, $\mu_2(x_2) = 1 - \mu_1(x_2)$.

In this example, the investor aims to regulate the stochastic financial system into a desired steady state $x_r = [3.1, 2, 1]^T$. Hence, the system matrices and reference input of reference model in (4) are given as $A_r = diag\{-1, -1, -1\}$, $B_r = diag\{1, 1, 1\}$ and $r(t) = [3.1, 2, 1]^T$. The weighting matrices of robust stochastic H_∞ tracking reference tracking control performance in (8) are given as $Q = diag\{1, 1, 1\}$ and $R = \{0.1, 0.1, 0.1\}$. The external disturbances v_1, v_2, v_3 are defined as $\{v_i = 0.2\cos t\}_{i=1}^{3}$. By applying the Theorem 2, we have disturbance attenuation level $\rho = 1$ and the fuzzy controller gains: $\bar{K}_1(\bar{x}) = [K_1^r(\bar{x})\ K_1^e(\bar{x})]$, $\bar{K}_2(\bar{x}) = [K_2^r(\bar{x})\ K_2^e(\bar{x})]$ with $K_{11}^r(\bar{x}) = [-1.88x_2 - 14.14,\ 1.17x_2 + 37.87,\ 1.07x_2 + 14.26]$, $K_{11}^e(\bar{x}) = [-1.89x_2 - 26.03,\ -1.06x_1,\ -0.12x_1^2 + 0.05x_1 - 0.14x_2^2 + 12.85x_2 + 6.75]$, $K_{12}^r(\bar{x}) = [-0.8x_1,\ 0.2x_1 - 1.69,\ 1.08x_1 - 1.07]$, $K_{12}^e = [\ -1.25x_1,\ -14.67,\ 6.98x_1 + 8.69]$, $K_{13}^r(\bar{x}) = [12.94,\ -1.1x_2^2 - 9.69,\ -21.2]$, $K_{13}^e = [\ -1.46,\ -1.02x_1,\ -2.2x_1^2 - 2.2x_2^2 + 3.05x_2 - 2.61]$, $K_{21}^r(\bar{x}) = [-0.86x_2 - 1.66,\ 1.13x_2 + 5.49,\ 1.25]$, $K_{21}^e = [-2.89x_2 - 16.75,\ -1.05x_1,\ -0.21x_1^2 - 0.12x_1 - 0.09x_2^2 + 10.89x_2 + 10.41]$, $K_{22}^r(\bar{x}) = [-0.82x_1, 1.02x_1 - 0.19, 1.01x_1]$, $K_{22}^e = [-2.31x_1, -36.08, 6.21x_1 - 10.81]$, $K_{23}^r(\bar{x}) = [13.24,\ -1.01x_2^2 - 19.12,\ -35.31]$, $K_{23}^e = [\ 4.12,\ 0.62,\ -2x_1^2 - 2.43x_2^2 + 1.39x_2^2 - 26]$, and $\{\epsilon_{ij} = 10^{-4}\}_{i,j=1}^{2}$ where $K_{ij}^r(\bar{x})$ and $K_{ij}^e(\bar{x})$ denotes the jth row in $K_i^r(\bar{x})$ and $K_1^e(\bar{x})$, respectively, for $i = 1, 2, j = 1, 2, 3$.

By using the robust fuzzy polynomial controller obtained from Theorem 2, the simulation results are shown in Figs. 1– 2. The initial states of financial system and reference model are $[2\ 3\ 1]^T$ and $[0\ 0\ 0]^T$, respectively. In Fig. 1, it can be seen that the financial system can track the corresponding reference state well. Moreover, the effect of intrinsic fluctuations and external disturbances are effectively reduced during the reference tracking process. Thus, the proposed fuzzy polynomial controller can achieve the desired robust stochastic H_∞ reference tracking performance against the effect of external disturbance and intrinsic fluctuations.

V. Conclusion

In this study, the robust stochastic H_∞ reference tracking control design is proposed for the SPFS. To effectively attenuate the effect of external disturbance during the reference tracking process, the designed robust stochastic H_∞ reference tracking control strategy aims to attenuate the effect of all possible external disturbance as well as intrinsic Wiener process and Possion counting process on the tracking error to a prescribed level. In the case of polynomial Lyapunov function, due to the differential compensation terms of stochastic processes in Itô-Lévy formula, the robust stochastic H_∞ reference tracking control design has to solve a set of HJIs. Besides, by using the quadratic Lyapunov function, the robust stochastic H_∞ reference tracking control design can be transformed to a solvable SOS-constrained problem and it can be solved by third-party MATLAB SOSTOOLS toolbox. From the simulation results, the proposed robust stochastic H_∞ reference tracking control design can achieve a great reference tracking performance for the financial market system with powerful disturbance attenuation capability. In the future, various control issues and applications on SPFS will be addressed, e.g., fuzzy polynomial filter design for SPFS.

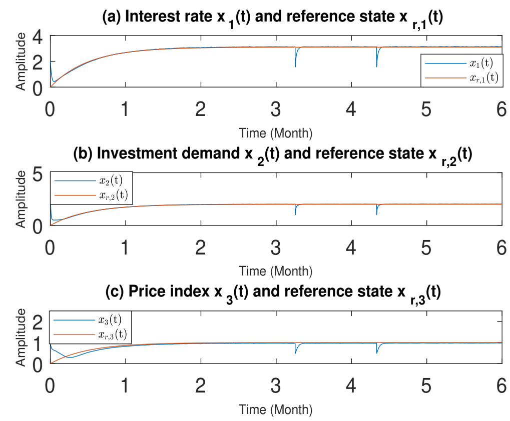

Fig. 1. The state trajectory tracking of financial system.

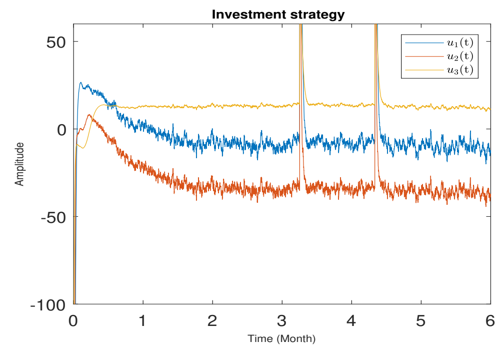

Fig. 2. The control input (investment strategy) of proposed robust stochastic H$_\infty$ reference tracking polynomial controller.

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