On ordinal sums of t-norms and t-conorms on bounded posets

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***Abstract*—This paper continues and generalizes the line of research on ordinal sum of t-norms and t-conorms on bounded lattices. We introduce a new ordinal sum construction on bounded posets based on interior and closure operators. Our proposed method provides a simple tool to introduce new classes of t-norms and t-conorms. Several necessary and sufficient conditions are presented for ensuring whether our generalized ordinal sum on a bounded posets of arbitrary t-norms is, in fact, a t-norm. We show that in this general setting the existence of our ordinal sum for t-norms requires that the respective interior operators are t-norm preserving.**

***Index Terms*—t-norms, t-conorms, bounded posets, ordinal sum**

I. Introduction

Fuzzy sets and systems form an important technique in applications, e.g., in multicriteria decision-making, fuzzy control, image processing, etc. The operations of t-norms and t-conorms on the unit interval [13], [20] serve as natural interpretations of operations of conjunction and disjunction, respectively. Nowadays, fuzzy logic started to use more general structures of truth values, following the seminal work of Goguen [10]. A paradigmatic example of such general structures are bounded lattices. Therefore, it was quite natural to begin to study t-norms and t-conorms on bounded lattices [1], [5]. In the general setting of bounded posets t-norms were studied e.g. in [21].

The paper continues and generalizes the line of research represented by [3], [8], [9], [15], [18], [19] and [16]. We follow in particular the approach started in [8] based on a new and more general definition of an ordinal sum of t-norms and t-conorms on bounded lattices. The authors used the concept of lattice interior and closure operators [17] to pick a sublattice (or sublattices) of a given bounded lattice appropriate for their construction. The importance of ordinal sums follows from the fact that a t-norm on the unit interval is continuous if and only if it is uniquely representable as an ordinal sum of continuous Archimedean t-norms [13].

In this paper, as a continuation of our previous work [8], we introduce a new construction of an ordinal sum of t-norms and t-conorms on bounded posets. For this aim we use the very general concept of interior and closure operators on bounded posets. Since our construction generalizes that one of [8] it also covers as a special case constructions from [3], [7] and [9]. A natural motivation for studying t-norms and t-conorms on bounded posets can be, inter alia, t-norms or t-conorms on non-linearly ordered sets, as is the case of intuitionistic fuzzy t-norms or fuzzy t-conorms, that are used in interval-valued fuzzy set theory [6].

The structure of this paper is as follows. In Section II we recall some basic notions and properties related to posets, and interior operators and t-norms on bounded posets. Section III is devoted to the main result of our paper. We propose an alternative definition of ordinal sum of t-norms on a bounded poset using interior operators and prove that it really is a t-norm. We show that in this general setting the existence of our ordinal sum for t-norms requires that the respective interior operator is t-norm preserving and that the product of two non-unital elements is in the image of this interior operator. Moreover, we show that this interior operator is, in some sense, uniquely determined.

Similarly, in Section IV we introduce an alternative definition of ordinal sum of t-conorms on a bounded poset using closure operators and prove respective results for t-conorms. In Section V we exemplify our proposed ordinal sum construction of t-norms on a bounded poset using simple illustrative examples. Finally, in Section VI, we draw some conclusions and identify topics for further research.

II. Preliminaries

We say that a poset $(L, \leq)$ [11] with the corresponding order $\leq$ is bounded [12], if there exist two elements $0_L, 1_L \in L$ such

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that for all $x \in L$ it holds that $0_L \leq x \leq 1_L$. We call 0_L and 1_L the bottom and the top element, respectively, and write this bounded poset as $(L, \leq, 0_L, 1_L)$.

Now we define intervals on bounded posets.

Definition 2.1: Let $(L, \leq, 0_L, 1_L)$ be a bounded poset. Let $a, b \in L$ be such that $a \leq b$. The closed subinterval $[a, b]$ of L is a subposet of L defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Similarly, open subinterval (a, b) of L is defined as $(a, b) = \{x \in L \mid a < x < b\}$. Definitions of semi-open subintervals $(a, b]$ and $[a, b)$ are obvious.

Next, let us provide the definition of an interior operator on a bounded poset.

Definition 2.2: Let L be a bounded poset. A map $h : L \to L$ is said to be an *interior operator on* L if, for all $x, y \in L$,

(i) $h(1_L) = 1_L$,
(ii) $h(h(x)) = h(x)$,
(iii) $h(x) \leq h(y)$ if $x \leq y$,
(iv) $h(x) \leq x$.

Obviously, each interior operator is a homomorphism of a bounded poset to itself (i.e., a $0, 1$ and order-preserving endomorphism). That is, $h(x) \leq h(y)$ whenever $x \leq y$ for all $x, y \in L$, which is exactly condition (iii) from Definition 2.2 and $h(0_L) = 0_L, h(1_L) = 1_L$. It is easy to see that the identity map id_L is an interior operator on L.

Example 2.3: [8] Let L be a bounded lattice, and let $a, b \in L$ be arbitrary such that $a \leq b$. Then, the map $h_{a,b} : L \to L$ defined by

$$h_{a,b}(x) = \begin{cases} x, & x \geq b, \\ x \wedge a, & \text{otherwise}, \end{cases} \tag{1}$$

for any $x \in L$, is an interior operator on L.

The definition of a t-norm and a t-conorm on a bounded poset is as follows (for lattices see [18, Definition 3.1]).

Definition 2.4: An operation $\cdot : L^2 \to L$ on a bounded poset $(L, \leq, 0_L, 1_L)$ is a *t-norm* (*t-conorm*) if it is commutative, associative, non-decreasing with respect to both variables and 1_L (0_L) is its neutral element. We also say that $(L, \leq, \cdot, 0_L, 1_L)$ is a *commutative integral partially ordered monoid* if $\cdot$ is a t-norm.

Remark 2.5: As a simple consequence of the monotonicity of a t-norm (t-conorm) and the fact that 1_L (0_L) is the neutral element, we find that $x \cdot y \leq x, y$ ($x \cdot y \geq x, y$) for any $x, y \in L$.

Note also that $\cdot$ is a t-norm on a bounded poset $(L, \leq, 0_L, 1_L)$ if and only if $\cdot$ is a t-conorm on the dual bounded poset $(L, \geq, 1_L, 0_L)$.

III. Ordinal sums of t-norms on bounded posets

The ordinal sums construction can be traced back to Birkhoff [2] for the case of partially ordered sets and also to Clifford [4] for the case of semigroups. The basic idea is that we have a family of pairwise disjoint sets, where each of them is equipped either with an order (for posets) or with an associative operation (for semigroups). This family is indexed by a linearly ordered index set. The ordinal sum is then the union of these sets equipped either with an appropriate partial order (for posets) or an appropriate associative operation (for semigroups). For more details on these ordinal sum constructions, see [18, Section 2].

The following theorem provides a construction of t-norms on posets with the help of an interior operator similarly to a construction of t-norms on meet semilattices (see [8, Theorem 3.1.]).

Theorem 3.1: Let L be a bounded poset, and let $h : L \to L$ be an interior operator on L. Let M denote the image of L under h, i.e., $h(L) = M$. Then,

(i) M is a sub-poset of L with the bottom element 0_L and the top element 1_L,
(ii) if V is a t-norm on M, then there exists its extension to a t-norm $\cdot$ on L as follows:

$$x \cdot y = \begin{cases} V(h(x), h(y)), & x, y \in L \setminus \{1_L\}, \\ x, & y = 1, \\ y, & x = 1. \end{cases}$$

Proof. To prove that M is a bounded sub-poset of L, it is sufficient to show that $1_L \in M$ and $0_L \in M$. From (i) of Definition 2.2 we have that $h(1_L) = 1_L$, hence $1_L \in M$, and from (iv) of the same definition it follows that $h(0_L) = 0_L$, hence $0_L \in M$.

To prove the second statement, one has verify that the map $\cdot$ satisfies the axioms of t-norms (Definition 2.4). But this can be done in the same manner as in [8, Theorem 3.1.] so we omit the proof. ■

The following theorem introduces the ordinal sum of t-norms on bounded posets that can be partitioned into a chain of subintervals. For meet semilattices the claim of this theorem follows from [8, Theorem 3.2.].

Theorem 3.2: Let L be a bounded poset, and let us assume that there exist $b_1, \ldots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 < \cdots < b_n$ and $L = \bigcup_{i=0}^{n} [b_i, b_{i+1}]$, where $b_0 = 0_L$ and $b_{n+1} = 1_L$. Then,

(i) $[b_i, b_{i+1}]$, $i = 0, \ldots, n$, is a bounded poset which is sub-poset of L,
(ii) if V_i is a t-norm on $[b_i, b_{i+1}]$ for $i = 0, \ldots, n$, then the *ordinal sum of t-norms* $V_1, \ldots, V_n$ *on* L defined as follows:

$$x \cdot y = \begin{cases} V_i(x, y), & x, y \in [b_i, b_{i+1}), \\ x \wedge y, & \text{otherwise}, \end{cases} \tag{2}$$

is a t-norm on L.
(iii) if $x \in L$ and $x \leq b_i$, $i = 0, \ldots, n$, then $x \cdot b_i = x$.

Proof. Since the proof of the first two parts of our theorem mimics the proof of [8, Theorem 3.2.] we will omit it. Note only that definition of $\cdot$ in (2) is correct. Namely if $x, y \in [b_i, b_{i+1})$ then $V_i(x, y) \in [b_i, b_{i+1})$. If $x \in [b_i, b_{i+1})$, $y \in [b_j, b_{j+1})$ and $i \neq j$ then either $i < j$ in which case $x \leq b_{i+1} \leq y$ and hence $x \wedge y = x$ or $j < i$ in which case $y \leq b_{j+1} \leq x$ and hence $x \wedge y = y$.

Let us prove (iii). Assume $x \in L$ and $x \leq b_i$, $i = 0, \dots, n$. If $x = b_i$ then $x \cdot b_i = V_i(b_i, b_i) = b_i = x$ since V_i is a t-norm on $[b_i, b_{i+1}]$. Suppose now that $x < b_i$. Then $x \cdot b_i = x \wedge b_i = x$. ∎

By Theorem 3.2 we can introduce a t-norm on L by the ordinal sum of t-norms on a bounded poset $L = [b_0, b_1] \cup \dots \cup [b_n, b_{n+1}]$, where $0_L = b_0 < b_1 < \dots < b_n < b_{n+1} = 1_L$. In what follows, we extend this construction to a more general bounded posets, where $[b_0, b_1] \cup \dots \cup [b_n, b_{n+1}] \subseteq L$.

Theorem 3.3: Let L be a bounded poset, and let us assume that there exist $b_1, \dots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 < \dots < b_n$ and $M = \bigcup_{i=0}^{n} [b_i, b_{i+1}] \subseteq L$, where $b_0 = 0_L$ and $b_{n+1} = 1_L$. If h is an interior operator on L such that $h(L) \subseteq M$, b_i is a fixed point of h and V_i is a t-norm on $J_{i+} = h(L) \cap [b_i, b_{i+1}]$ for $i = 0, \dots, n$, then

$$x \cdot y = \begin{cases} V_i(h(x), h(y)), & (h(x), h(y)) \in {J_i}^2, \\ h(x) \wedge h(y), & (h(x), h(y)) \in J_i \times J_j, \text{ for } i \neq j, \\ x \wedge y, & \text{otherwise,} \end{cases} \tag{3}$$

for any $x, y \in L$, where $J_i = h(L) \cap [b_i, b_{i+1})$, is a t-norm on L, which is called the *h-ordinal sum of t-norms $V_0, \dots, V_n$ on L.*

Proof. Since the proof uses the same method as the proof of [8, Theorem 3.3.] we will omit it. ∎

The following proposition determines a set of necessary structural conditions on the interior operator h which are required for our t-norm to be an h-ordinal sum of t-norms. In particular, h has to be a homomorphism with respect to the obtained h-ordinal sum of t-norms, i.e., h has to be *t-norm preserving*.

Proposition 3.4: Let L be a bounded poset and let us assume that there exist $b_1, \dots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 < \dots < b_n$ and $M = \bigcup_{i=0}^{n} [b_i, b_{i+1}] \subseteq L$, where $b_0 = 0_L$ and $b_{n+1} = 1_L$. Let h be an interior operator on L such that $J = h(L) \subseteq M$, b_i is a fixed point of h and V_i is a t-norm on $J_{i+} = h(L) \cap [b_i, b_{i+1}]$ for $i = 0, \dots, n$. Let $\cdot$ be the h-ordinal sum of t-norms $V_0, \dots, V_n$ on L and $x, y \in L$. Then

(a) If $h(x) \in [b_i, b_{i+1})$ for $i = 0, \dots, n-1$ then $h(x) = x \cdot b_j$ for $j = i+1, \dots, n$. In particular, if $x \notin [b_n, b_{n+1}]$ then $h(x) = x \cdot b_n < b_n$ and $\bigcup_{i=0}^{n-1} J_{i+} = \{x \cdot b_n \mid x \leq b_n\}$.
(b) If $h(x), h(y) \in [b_i, b_{i+1})$ for $i = 0, \dots, n$ then $h(x) \cdot h(y) = x \cdot y = h(x \cdot y)$.
(c) If $h(x) \in [b_i, b_{i+1})$ for $i = 0, \dots, n-1$ and $h(y) \in [b_j, b_{j+1})$ for $j = 1, \dots, n$, $i < j$ then $h(x) \cdot h(y) = x \cdot y = h(x \cdot y)$, $h(y) \cdot h(x) = y \cdot x = h(y \cdot x)$ and $x \cdot y = y \cdot x$.
(d) $(M, \cdot, \leq, 0_L, 1_L)$, $(J, \cdot, \leq, 0_L, 1_L)$ and $([b_n, b_{n+1}], \cdot, \leq, b_n, 1_L)$ are partially ordered monoids, and the mappings $h \colon L \to L$, $h \colon L \to J$ and the restrictions $\tilde{h} = h \restriction [b_n, b_{n+1}] \colon [b_n, b_{n+1}] \to [b_n, b_{n+1}]$ and $\overline{h} = h \restriction M \colon M \to J$ are homomorphisms of partially ordered monoids.

Proof. (a) Assume that $h(x) \in [b_i, b_{i+1})$ for $i = 0, \dots, n-1$. Then $b_{i+1} \notin J_i$, $b_{i+1} \in J_{i+1}$ and $h(x) < b_{i+1}$. Hence $h(x) < b_{i+1} = h(b_{i+1})$ and

$$x \cdot b_{i+1} = h(x) \wedge h(b_{i+1}) = h(x).$$

Let $i + 1 < k \leq n$. Then $b_{i+1} \cdot b_k = b_{i+1}$ and

$$\begin{aligned} x \cdot b_k &= h(x) \wedge b_k = h(x) \wedge b_{i+1} \wedge b_k = (x \cdot b_{i+1}) \wedge b_k \\ &= h(x \cdot b_{i+1}) \wedge b_k = (x \cdot b_{i+1}) \cdot b_k \\ &= x \cdot (b_{i+1} \cdot b_k) = x \cdot b_{i+1} = h(x). \end{aligned}$$

(b) Let $h(x), h(y) \in [b_i, b_{i+1})$ for $i = 0, \dots, n$. Then $h(x) = h(h(x)), h(y) = h(h(y)) \in J_i$ and

$$\begin{aligned} h(x) \cdot h(y) &= V_i(h(h(x)), h(h(y))) \\ &= V_i(h(x), h(y)) = x \cdot y \in J_i. \end{aligned}$$

Hence also $h(x) \cdot h(y) = h(x \cdot y)$.
(c) Let $h(x) \in [b_i, b_{i+1})$ for $i = 0, \dots, n-1$ and $h(y) \in [b_j, b_{j+1})$ for $j = 1, \dots, n$, $i < j$. Then $h(x) = h(h(x)) \in J_i, h(y) = h(h(y)) \in J_j$, $h(x) < h(y)$ and

$$\begin{aligned} h(x) \cdot h(y) &= h(h(x)) \wedge h(h(y)) = h(x) \wedge h(y) \\ &= x \cdot y = h(x) \in J_i. \end{aligned}$$

Therefore $h(x) \cdot h(y) = h(x \cdot y)$. Similarly, we obtain $h(y) \cdot h(x) = y \cdot x = h(y \cdot x)$ and $x \cdot y = y \cdot x$.
(d) Clearly, $h(0_L) = 0_L = 0_M = 0_J$, $h(1_L) = 1_L = 1_M = 1_J$ and $\cdot(M \times M) \subseteq M$, $\cdot(J \times J) \subseteq J$ and $\cdot([b_{n-1}, b_n] \times [b_{n-1}, b_n]) \subseteq [b_{n-1}, b_n]$. The remaining part follows from (b) and (c). ∎

In what follows, if we say that $\cdot$ is an h-ordinal sum of t-norms $V_0, \dots, V_n$ on L, we assume that the assumptions of Theorem 3.3 are satisfied for certain $b_1, \dots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 < \dots < b_n$ and $\cdot$ is a t-norm constructed by formula (3). This theorem provides a general construction of a t-norm on a bounded poset L (see Fig. 1) as an ordinal sum of t-norms defined on closed subintervals of bounded sub-posets from L with the help of an interior operator. One can see there that to determine $x \cdot y$, first, the elements x and y are sent to sub-posets J_{n-1} and J_0, respectively, by the interior operator h, and then the meet operation is applied to these images $h(x)$ and $h(y)$ (the second line of (3)), because the sub-poset J_{n-1} is different from the sub-poset J_0. If images of x as well as of z belong to the same bounded sub-poset J_{n-1}, then the t-norm V_{n-1} defined on J_{n-1} is applied instead of the meet (the first line of (3)). It should be noted that in the assumptions of Theorem 3.3 we require that any b_i ($i = 1, \dots, n$) has to be a fixed point of the interior operator h to guarantee the following condition:

$$b_i \cdot b_j = b_i \wedge b_j, \quad i, j = 0, \dots, n+1, \tag{4}$$

which seems to be quite natural for ordinal sum constructions of t-norms defined on bounded sub-posets of L.

The following interesting question naturally arises from previous considerations. Let $\cdot$ be any t-norm on a bounded poset L, for which there exists a finite chain $0_L = b_0 < b_1 < \dots < b_n < b_{n+1} = 1_L$ in L such that $b_i \cdot b_i = b_i$, and the restriction of $\cdot$ to a bounded sub-poset J_{i+} of $[b_i, b_{i+1}]$ with

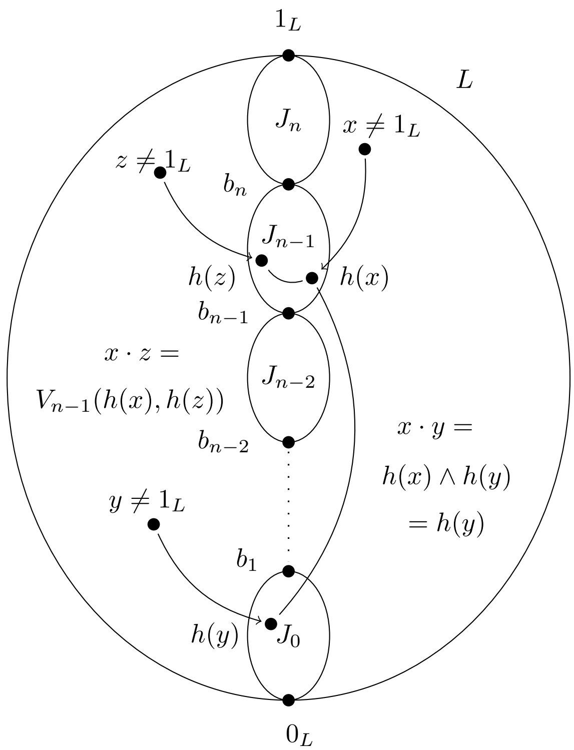

Fig. 1. A general construction of the h-ordinal sum of t-norms $V_0, \ldots, V_n$ defined on the bounded sub-posets $J_0, \ldots, J_n$, respectively. Ovals (e.g., J_{n-2}) indicate subposets of L. Curved lines with arrows (e.g., from y to $h(y)$) connect an element of L and its image under h. The curved line between $h(x)$ and $h(y)$ ($h(x)$ and $h(z)$, respectively) connects two elements used for the determination of $x \cdot y$ ($x \cdot z$, respectively).

the bottom element b_i and the top element b_{i+1} is a t-norm on J_{i+} for $i = 0, \ldots, n$.[1] Is it true that $\cdot$ is an h-ordinal sum of $V_0, \ldots, V_n$ for a suitable interior operator h? In other words, we are interested in the validity of a statement opposite to Theorem 3.3. Notice that if $\cdot$ is an h-ordinal sum of t-norms $V_0, \ldots, V_n$, where V_i is a t-norm on J_{i+} for $i = 0, \ldots, n$, then

$$x \cdot y \in \bigcup_{i=0}^{n} J_i, \quad x, y \in L \setminus \{1_L\}, \tag{5}$$

where $J_i = J_{i+} \setminus \{b_{i+1}\}$. Therefore, we restrict ourselves to a characterization of t-norms on L for which condition (5) holds. Moreover, an analysis of the interior operators and t-norms on L shows us that we are unable (or, we do not see a solution how) to determine interior operators from t-norms in a systematic way whenever J_{n+} is a strict subset of $[b_n, 1_L]$, i.e., there are $x \in J_{n+}$ and $y \in [b_n, 1_L] \setminus J_{n+}$ such that $x \cdot y \in J_{n+}$. Since $y \notin \bigcup_{i=0}^{n} J_i$, we should have $x \cdot y = V_n(x, h(y))$ for a suitable interior operator h on L, but the open problem for us is how to define h to ensure that $\cdot$ is an h-ordinal sum of $V_0, \ldots, V_n$. Therefore, we assume in the following theorem that $J_{n+} = [b_n, 1_L]$.

[1]Note that if the restriction of $\cdot$ to J_{i+} is a t-norm, then $\cdot$ need not be a t-norm on $[b_i, b_{i+1}]$.

Theorem 3.5: Let $\cdot$ be a t-norm on a bounded poset L, and let $0_L = b_0 < b_1 < \cdots < b_n < b_{n+1} = 1_L$ be elements of L such that $b_i \cdot b_i = b_i$ for $i = 0, \ldots, n+1$. Let

$$J = \{x \cdot y \mid x, y \in L \setminus \{1_L\}\} \cup [b_n, 1_L] \tag{6}$$

such that $J \subseteq \bigcup_{i=0}^{n} [b_i, b_{i+1}]$ and put $J_{i+} = J \cap [b_i, b_{i+1}]$ for $i = 0, \ldots, n$. Let the restriction $V_i = \cdot \restriction J_{i+}$ be a t-norm on J_{i+} for $i = 0, \ldots, n$.

(i) If $x, y \in J$, then

$$x \cdot y = \begin{cases} V_i(x, y), & (x, y) \in {J_i}^2, \\ x \wedge y, & \text{otherwise.} \end{cases} \tag{7}$$

(ii) There exists an interior operator h on L such that $\cdot$ is an h-ordinal sum of $V_0, \ldots, V_n$.

Proof. Put $B = \{b_i \mid i = 0, \ldots, n+1\}$. Then $B \subseteq J$, $J = \bigcup_{i=0}^{n} J_{i+}$ and $x, y \in J$ implies $x \cdot y \in J$. Hence $\cdot$ is a t-norm on J.

Let us verify (i). Assume that $x, y \in J$. Let $x, y \in [b_i, b_{i+1})$. Then $x, y \in J_{i+}$. Since V_i is the restriction of $\cdot$ to J_{i+} we have that $x \cdot y = V_i(x, y)$. Assume now that $x \in [b_i, b_{i+1})$, $y \in [b_j, b_{j+1})$ and $i \neq j$. Let $i < j$. Then

$$\begin{aligned} x &= x \cdot 1_L \geq x \cdot y \geq x \cdot b_{i+1} \cdot b_j \cdot y = x \cdot b_{i+1} \cdot b_j \\ &\geq x \cdot b_{i+1} \cdot b_{i+1} = x \cdot b_{i+1} = x = x \wedge y. \end{aligned}$$

The case $i > j$ can be verified similarly. Now, let $1_L \in \{x, y\}$. Then $x \cdot y = x \wedge y$. Therefore, (i) is valid.

Let us show (ii). Define the map $h : L \to L$ as

$$h(x) = \begin{cases} x, & x \in J, \\ x \cdot b_n, & \text{otherwise,} \end{cases} \tag{8}$$

for any $x \in L$. We first verify that h is an interior operator on L. Trivially $h(1_L) = 1_L$, since $1_L \in J$. Let $x \in L$ be arbitrary. If $x \in J$, then $h(h(x)) = x = h(x)$. If $x \notin J$, then $h(x) = x \cdot b_n \in J$. Hence, $h(h(x)) = h(x \cdot b_n) = x \cdot b_n = h(x)$; therefore, $h(h(x)) = h(x)$ for any $x \in L$. Further, let us show that h preserves the order on L. Let $x, y \in L$ be arbitrary such that $x \leq y$. Assume first that $x, y \in J$. Hence, we simply find that $h(x) = x \leq y = h(y)$. Let $x \in J$ and $y \notin J$. We have $b_n > x$, otherwise, $y \in J_{n+} \subseteq J$ (recall that $J_{n+} = [b_n, 1_L]$), which is a contradiction with $y \notin J$. Then we have

$$h(x) = x = x \cdot b_n \leq y \cdot b_n = h(y),$$

where Remark 2.5 is used. Let $x \notin J$ and $y \in J$. Then $h(x) = x \cdot b_n \leq x \leq y = h(y)$. Assume now $x, y \notin J$. We obtain

$$h(x) = x \cdot b_n \leq y \cdot b_n = h(y).$$

Finally, let $x \in L$ be arbitrary. Obviously, $h(x) = x$ for $x \in J$, and $h(x) = x \cdot b_n \leq x$ for $x \notin J$; therefore, $h(x) \leq x$ for any $x \in L$, which completes the proof of the claim that h is an interior operator on L.

Let V be the h-ordinal sum of $V_0, \ldots, V_n$, i.e.,

$$V(x, y) = \begin{cases} V_i(h(x), h(y)), & (h(x), h(y)) \in {J_i}^2, \\ h(x) \wedge h(y), & (h(x), h(y)) \in J_i \times J_j, i \neq j, \\ x \wedge y, & \text{otherwise,} \end{cases} \tag{9}$$

for any $x, y \in L$, where $J_i = [b_i, b_{i+1}) \cap J$. Let us prove that $\cdot = V$.

By the definition of the interior operator h, we have $h(x) = x$ for any $x \in J$. Therefore, by (i) we have that $V(x, y) = x \cdot y$ for any $x, y \in J$. Moreover, one can easily see that the restriction of V on J is an ordinal sum of $V_0, \dots, V_n$ in the sense of (2) of Theorem 3.2, i.e.

$$V(x, y) = \begin{cases} V_i(x, y), & x, y \in J_i^2, \\ x \wedge y, & \text{otherwise}, \end{cases} \tag{10}$$

for any $x, y \in J$ (recall that $h(x) = x$ for $x \in J$). From the coincidence of V and $\cdot$ on J, we obtain

$$x \cdot y = \begin{cases} V_i(x, y), & x, y \in J_i^2, \\ x \wedge y, & \text{otherwise}, \end{cases} \tag{11}$$

for any $x, y \in J$, and $\cdot$ restricted to J is an ordinal sum of $V_0, \dots, V_n$ on J. By the comparison of (9) and (11), one can find that the h-ordinal sum V of $V_0, \dots, V_n$ on L can be equivalently expressed as follows:

$$V(x, y) = \begin{cases} h(x) \cdot h(y), & x, y \in L \setminus \{1_L\}, \\ x \wedge y, & \text{otherwise}, \end{cases} \tag{12}$$

for any $x, y \in L$.

Let $x, y \in L \setminus \{1_L\}$. Since $x \cdot y \in J \subseteq \bigcup_{i=0}^{n} [b_i, b_{i+1}]$ we have that $x \cdot y \leq b_n$ or $b_n \leq x \cdot y$.

Assume first that $x \cdot y \leq b_n$. Then $x \cdot y \in J_{i+}$ for some i, $0 \leq i \leq n-1$. This implies $x \cdot y = (x \cdot y) \cdot b_{i+1} \leq (x \cdot y) \cdot b_n = x \cdot (y \cdot b_n) = (x \cdot b_n) \cdot y = (x \cdot b_n) \cdot (y \cdot b_n) \leq x \cdot y$ which yields $x \cdot y = h(x) \cdot h(y) = V(h(x), h(y)) = V(x, y)$.

Suppose now that $b_n \leq x \cdot y$. Then $b_n \leq x < 1_L$ and $b_n \leq y < 1_L$. Hence $x, y \in J$ and $x \cdot y = V(x, y)$ by (i).

The remaining case is that $1 \in \{x, y\}, x, y \in L$. But this yields $x \cdot y = x \wedge y = V(x, y)$. ■

The previous theorem shows conditions under which a t-norm on L is an h-ordinal sum of t-norms defined on bounded sub-posets of L. One can see that this theorem is, in some sense, opposite to Theorem 3.3. More precisely, each t-norm $\cdot$ on L, for which $b_i \cdot b_i = b_i$ holds for a chain $0_L = b_0 < b_1 < \dots < b_n < b_{n+1} = 1_L$ in L and $\cdot$ restricted to $[b_i, b_{i+1}] \cap J$ is a t-norm on $[b_i, b_{i+1}] \cap J$ for $i = 0, \dots, n$, where J is defined by (6), can be determined as an h-ordinal sum of t-norms for a suitable interior operator h on L.

The following theorem shows that each h-ordinal sum of t-norms is, in some sense, uniquely determined by the interior operator h.

Theorem 3.6: If $\cdot$ is an h-ordinal sum of $V_0, \dots, V_n$ and simultaneously an h'-ordinal sum of $V'_0, \dots, V'_n$, where $h(b_i) = h'(b_i) = b_i$, $h \restriction [b_n, b_{n+1}] = h' \restriction [b_n, b_{n+1}]$, V_i is a t-norm on $[b_i, b_{i+1}] \cap h(L)$ and V'_i is a t-norm on $[b_i, b_{i+1}] \cap h'(L)$ for $i = 0, \dots, n$, then $h = h'$.

Proof. Let $x \in L \setminus [b_n, b_{n+1}]$. Then $h(x) < b_n$ and $h'(x) < b_n$. From Proposition 3.4 (a) we obtain that $h(x) = x \cdot b_n = h'(x)$. ■

Recall that in the case when $[b_n, b_{n+1}] = \{b_n, b_{n+1}\}$ we obtain that each h-ordinal sum of t-norms is uniquely determined by the interior operator h.

Theorem 3.7: Let L be a bounded poset equipped with a binary operation $\cdot$. Let $0_L = b_0 < b_1 < \dots < b_n < b_{n+1} = 1_L$ be elements of L such that $b_i \cdot b_i = b_i$ for $i = 0, \dots, n+1$. Then the following conditions are equivalent:

(a) $\cdot$ is an h-ordinal sum of t-norms $V_0, \dots, V_n$ for a suitable interior operator h on L.

(b) $\cdot$ is a t-norm such that

 (i) $x \cdot b_n \neq b_n$, $y \neq 1_L$ implies $x \cdot y = x \cdot y \cdot b_n$.

 (ii) $x \cdot b_n \neq b_n$ implies $x \cdot b_n \in [b_i, b_{i+1})$ for some $i \in \{0, \dots, n-1\}$ and $x \cdot b_j = x \cdot b_n$ if $i+1 \leq j \leq n$.

Proof. (a) $\implies$ (b): Let $\cdot$ be an h-ordinal sum of t-norms $V_0, \dots, V_n$ for a suitable interior operator h on L.

(i) Let $x, y \in L$, $x \cdot b_n \neq b_n$, $y \neq 1_L$. Then $x \notin [b_n, b_{n+1}]$ and $h(x) = x \cdot b_n < b_n$. If $y \notin [b_n, b_{n+1}]$ then $h(y) = y \cdot b_n < b_n$ and from Proposition 3.4 (a), (b), (c) we obtain

$$x \cdot y = h(x) \cdot h(y) = x \cdot b_n \cdot y \cdot b_n = x \cdot y \cdot b_n.$$

Assume now that $y \in [b_n, b_{n+1}]$. Then $h(x) < b_n \leq h(y) \leq 1_L$. We obtain

$$h(x) = h(x) \wedge b_n \leq x \cdot y = h(x) \wedge h(y) \leq h(x) \wedge 1_L = h(x).$$

Hence $h(x) = x \cdot y$. Therefore also $h(x) = h(x) \cdot b_n = x \cdot y \cdot b_n$ and we have $x \cdot y \cdot b_n = x \cdot y$.

(ii) It follows immediately from Proposition 3.4 (a).

(b) $\implies$ (a): Let $\cdot$ be a t-norm satisfying (b). Let us put

$$J = \{x \cdot y \mid x, y \in L \setminus \{1_L\}\} \cup [b_n, 1_L]$$

and

$$\overline{J} = \{x \cdot b_n \mid x \in L\} \cup [b_n, 1_L].$$

Evidently, $\overline{J} \subseteq J$. Now, let $x, y \in L \setminus \{1_L\}$. If $x, y \in [b_n, b_{n+1}]$ then also $x \cdot y \in [b_n, b_{n+1}] \subseteq \overline{J}$. Assume now that $x \notin [b_n, b_{n+1}]$. Hence $x \cdot b_n \neq b_n$. We have $x \cdot y = x \cdot y \cdot b_n \in \overline{J}$ which proves that $\overline{J} = J$. From (ii) we obtain that $J = \overline{J} \subseteq \bigcup_{i=0}^{n} [b_i, b_{i+1}]$.

We now put $J_{i+} = J \cap [b_i, b_{i+1}]$ for $i = 0, \dots, n$. Since $\cdot([b_i, b_{i+1}] \times [b_i, b_{i+1}]) \subseteq [b_i, b_{i+1}]$ and $x \in J_{i+}$ for $i = 0, \dots, n-1$ implies $x = x \cdot b_n = x \cdot b_n \cdot b_n = x \cdot b_n \cdot b_{i+1} = x \cdot b_{i+1}$ we have that the restriction $V_i = \cdot \restriction J_{i+}$ is a t-norm on J_{i+} for $i = 0, \dots, n$. From Theorem 3.5 we obtain that there exists an interior operator h_c on L defined as

$$h_c(x) = \begin{cases} x, & x \in J, \\ x \cdot b_n, & \text{otherwise}, \end{cases}$$

for any $x \in L$ such that $\cdot$ is an h_c-ordinal sum of $V_0, \dots, V_n$. ■

We will call the above defined interior operator h_c on L a *canonical interior operator*.

Corollary 3.8: Let L be a bounded poset equipped with a binary operation $\cdot$. Let $0_L = b_0 < b_1 < \dots < b_n < b_{n+1} = 1_L$ be elements of L such that $b_i \cdot b_i = b_i$ for $i = 0, \dots, n+1$. Then the following conditions are equivalent:

(a) $\cdot$ is an h-ordinal sum of t-norms $V_0, \ldots, V_n$ for a suitable interior operator h on L.
(b) $\cdot$ is an h_c-ordinal sum of t-norms $V_0, \ldots, V_n$ for the canonical interior operator h_c on L.

IV. Ordinal sums of t-conorms on bounded posets

The investigation of ordinal sums of t-conorms on bounded posets is dual to our previous considerations. By Remark 2.5 we immediately obtain the following series of theorems.

We start with the definition of a closure operator, which is conceptually dual to the interior operator intensively used in our ordinal sum constructions of t-norms.

Definition 4.1: Let L be a bounded poset. A map $g : L \to L$ is said to be a *closure operator on* L if, for all $x, y \in L$,

1) $g(0_L) = 0_L$,
2) $g(g(x)) = g(x)$,
3) $g(x) \le g(y)$ if $x \le y$,
4) $x \le g(x)$.

It is easy to see that the identity map 1_L is a closure operator on L.

The following theorem provides a construction for t-conorms on the bounded posets with the help of a closure operator, dually to the construction of t-norms in Theorem 3.1.

Theorem 4.2: Let L be a bounded poset, and let $g : L \to L$ be a closure operator on L. Let M denote the image of L under g, i.e., $g(L) = M$. Then,

(i) M is a bounded sub-poset of L with the bottom element 0_L and the top element 1_L,
(ii) if W is a t-conorm on M, then there exists its extension to a t-conorm $\cdot$ on L as follows:

$$x \cdot y = \begin{cases} W(g(x), g(y)), & x, y \in L \setminus \{0_L\}, \\ x \vee y, & \text{otherwise.} \end{cases}$$

In the next theorem, the ordinal sum of t-conorms on bounded posets that can be partitioned into a chain of subintervals is introduced.

Theorem 4.3: Let L be a bounded poset, and let us assume that there exist $b_1, \ldots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 > \cdots > b_n$ and $L = \bigcup_{i=0}^{n} [b_{i+1}, b_i]$, where $b_{n+1} = 0_L$ and $b_0 = 1_L$. Then,

(i) $[b_{i+1}, b_i]$, $i = 0, \ldots, n+1$, is a bounded sub-poset of L,
(ii) if W_i is a t-conorm on $[b_{i+1}, b_i]$ for $i = 0, \ldots, n$, then the ordinal sum of t-conorms $W_1, \ldots, W_n$ defined as follows:

$$x \cdot y = \begin{cases} W_i(x, y), & x, y \in (b_{i+1}, b_i], \\ x \vee y, & \text{otherwise,} \end{cases}$$

is a t-conorm on L.
(iii) if $x \in L$ and $x \ge b_i$, $i = 0, \ldots, n$, then $x \cdot b_i = x$.

The construction introduced in the previous theorem will be now extended by means of a closure operator to more general bounded posets.

Theorem 4.4: Let L be a bounded poset, and let us assume that there exist $b_1, \ldots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 > \cdots > b_n$ and $M = \bigcup_{i=0}^{n} [b_{i+1}, b_i] \subset L$, where $b_{n+1} = 0_L$ and $b_0 = 1_L$. If g is a closure operator on L such that $g(L) \subseteq M$, b_i is a fixed point of g and W_i is a t-conorm on $J_{i+} = g(L) \cap [b_{i+1}, b_i]$ for $i = 0, \ldots, n$, then

$$x \cdot y = \begin{cases} W_i(g(x), g(y)), & (g(x), g(y)) \in {J_i}^2, \\ g(x) \vee g(y), & (g(x), g(y)) \in J_i \times J_j, i \neq j, \\ x \vee y, & \text{otherwise,} \end{cases} \tag{13}$$

for any $x, y \in L$, where $J_i = g(L) \cap (b_{i+1}, b_i]$, is a t-conorm on L, which is called the *g-ordinal sum of t-conorms* $W_0, \ldots, W_n$ on L.

In what follows, if we say that $\cdot$ is a g-ordinal sum of t-conorms $W_0, \ldots, W_n$ on L, we assume that the assumptions of Theorem 4.4 are satisfied for certain $b_1, \ldots, b_n \in L \setminus \{0_L, 1_L\}$ such that $b_1 > \cdots > b_n$ and $\cdot$ is a t-conorm constructed by formula (13).

The following theorem shows that each g-ordinal sum of t-conorms is, in some sense, uniquely determined by the closure operator g.

Theorem 4.5: If S is a g-ordinal sum of $W_0, \ldots, W_n$ and simultaneously an g'-ordinal sum of $W'_0, \ldots, W'_n$, where $g(b_i) = g'(b_i) = b_i$, $g \upharpoonright [b_{n+1}, b_n] = g' \upharpoonright [b_{n+1}, b_n]$, W_i is a t-conorm on $[b_{i+1}, b_i] \cap g(L)$ and W'_i is a t-conorm on $[b_{i+1}, b_i] \cap g'(L)$ for $i = 0, \ldots, n$, then $g = g'$.

Finally, the result dual to Theorem 3.5 is provided, stating that, under certain conditions, each t-conorm on a bounded poset L is a g-ordinal sum of t-conorms $W_0, \ldots, W_n$ for a suitable closure operator g.

Theorem 4.6: Let $\cdot$ be a t-conorm on a bounded poset L, and let $0_L = b_{n+1} < b_n < \cdots < b_1 < b_0 = 1_L$ be elements of L such that $b_i \cdot b_i = b_i$ for $i = 0, \ldots, n+1$. Let

$$J = \{x \cdot y \mid x, y \in L \setminus \{0_L\}\} \cup [0_L, b_n]$$

and put $J_{i+} = J \cap [b_{i+1}, b_i]$ for $i = 0, \ldots, n$. If J is a bounded sub-poset of L such that $J \subseteq \bigcup_{i=0}^{n} [b_{i+1}, b_i]$ and the restriction $W_i = \cdot \upharpoonright J_{i+}$ is a t-conorm on J_{i+} for $i = 0, \ldots, n$, then there exists a closure operator g on L such that $\cdot$ is a g-ordinal sum of $W_0, \ldots, W_n$.

V. Examples

In this section, we first present an example of a t-norm on a complete lattice due to Zhang, Ouyang and De Baets [16, Example 4.1] which was introduced as an illustration of their construction. We show that it can be obtained also with our proposed ordinal sums construction. Second, we show an example of a t-norm on a bounded poset which is not a lattice obtained by our generalized ordinal sums construction.

Example 5.1: Consider the complete lattice L with Hasse diagram shown in Figure 2. Let us put $b_0 = 0_L$, $b_1 = a$, $b_2 = f$, $b_3 = h$, $b_4 = 1_L$. Let $J = L \setminus \{i, j, k\}$. We define a map $H\colon L \to L$ by $H(x) = x$ for $x \in J$, $H(i) = f$, $H(j) = g$ and $H(k) = 0_L$. Evidently, H is an interior operator on L, $H(L) \subseteq [0_L, a] \cup [a, f] \cup [f, h] \cup [h, 1_L]$, $H(0_L) = 0_L$, $H(a) = a$, $H(f) = f$, $H(h) = h$ and $H(1_L) = 1_L$. It is routine to verify that the following operations V_i on $[b_i, b_{i+1}]$ are t-norms for any $i = 0, \ldots, 3$.

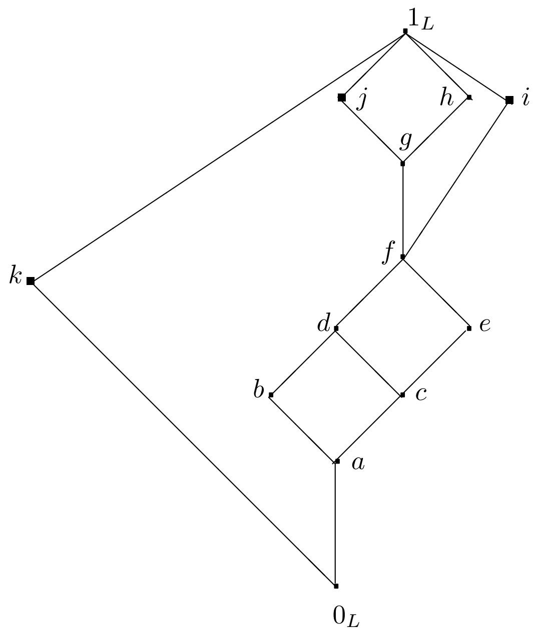

Fig. 2. Hasse diagram of the lattice L in Example 5.1.

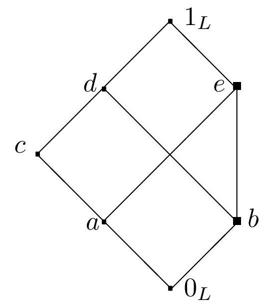

Fig. 3. Hasse diagram of the bounded poset L in Example 5.2.

V_0	0_L	a
0_L	0_L	0_L
a	0_L	a

V_1	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	b	a	b
c	a	a	c	c	c	c
d	a	b	c	d	c	d
e	a	a	c	c	c	e
f	a	b	c	d	e	f

V_2	f	g	h
f	f	f	f
g	f	f	g
h	f	g	h

V_3	h	1_L
h	h	h
1_L	h	1_L

Hence by Theorem 3.3 we obtain a t-norm T which coincides with the t-norm introduced in [16, Example 4.1] and is shown below.

T	0_L	a	b	c	d	e	f	g	h	i	j	k	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	a	a	a	a	a	a	a	a	a	0_L	a
b	0_L	a	a	a	b	a	b	b	b	b	b	0_L	b
c	0_L	a	a	c	c	c	c	c	c	c	c	0_L	c
d	0_L	a	b	c	d	c	d	d	d	d	d	0_L	d
e	0_L	a	a	c	c	c	e	e	e	e	e	0_L	e
f	0_L	a	b	c	d	e	f	f	f	f	f	0_L	f
g	0_L	a	b	c	d	e	f	f	g	f	f	0_L	g
h	0_L	a	b	c	d	e	f	g	h	f	g	0_L	h
i	0_L	a	b	c	d	e	f	f	f	f	f	0_L	i
j	0_L	a	b	c	d	e	f	f	g	f	f	0_L	j
k	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L	k
1_L	0_L	a	b	c	d	e	f	g	h	i	j	k	1_L

Example 5.2: Consider the bounded poset L (which is not a lattice) with Hasse diagram shown in Figure 3. Let us put $b_0 = 0_L$, $b_1 = c$, $b_2 = 1_L$. Let $J = L \setminus \{b, e\}$. We define a map $h\colon L \to L$ by $h(x) = x$ for $x \in J$, $h(b) = 0_L$ and $h(e) = a$. Evidently, h is an interior operator on L, $h(L) \subseteq [0_L, c] \cup [c, 1_L]$, $h(0_L) = 0_L$, $h(c) = c$ and $h(1_L) = 1_L$.

The following three-valued operations V_i on $[b_i, b_{i+1}]$ are t-norms for any $i = 0, 1$. Namely, V_0 is the Gdel t-norm and V_1 is the Łukasiewicz t-norm.

V_0	0_L	a	c
0_L	0_L	0_L	0_L
a	0_L	a	a
c	0_L	a	c

V_1	c	d	1_L
c	c	c	c
d	c	c	d
1_L	c	d	1_L

Hence by Theorem 3.3 we obtain a t-norm $\cdot$ shown below.

$\cdot$	0_L	a	b	c	d	e	1_L
0_L	0_L	0_L	0_L	0_L	0_L	0_L	0_L
a	0_L	a	0_L	a	a	a	a
b	0_L	0_L	0_L	0_L	0_L	0_L	b
c	0_L	a	0_L	c	c	a	c
d	0_L	a	0_L	c	c	a	d
e	0_L	a	0_L	a	a	a	e
1_L	0_L	a	b	c	d	e	1_L

VI. Conclusion

This paper provides an alternative definition of genealized ordinal sum construction. In our approach we assume the existence of an interior operator h on a bounded poset and a finite chain of subintervals of a bounded subposet of fixed points of h, on which t-norms are defined. Analogous results for t-conorms were presented (without proofs). For future work, there are two directions to be followed. First, we would like to study our generalized ordinal sum construction in the context of uninorms. Second, we plan to study the possibility to use an infinite chain of subintervals in our construction.

Also, it would be interesting to present particular results for the case of discrete chains $M = \{a_0 < a_1 < \cdots < a_n\}$ similarly to [14].

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