New Entropy and Distance Measures of Intuitionistic Fuzzy Sets

Jinfang Huang
School of Software
Yunnan University
Kunming, China
E-mail: Miss_huang1399@163.com

Xin Jin
School of Software
Yunnan University
Kunming, China
E-mail: xinxin_jin@163.com

Dianwu Fang
School of Information
Yunnan University
Kunming, China
E-mail: fangdianwu@ynu.edu.cn

Shin-Jye Lee
Institute of Technology Management
National Chiao Tung University
Hsinchu, Taiwan
E-mail: camhero@gmail.com

Qian Jiang*
School of Software
Yunnan University
Kunming, China
E-mail: jiangqian_1221@163.com
(Corresponding author)

Shaowen Yao
School of Software
Yunnan University
Kunming, China
E-mail: yaosw@ynu.edu.cn

Abstract—In fuzzy set theory, the distance and entropy measure of intuitionistic fuzzy sets (IFSs) have received extensive concern because of the capability for handling imprecise or uncertain problems. However, most of the existing modeling methods for distance and entropy measure are imperfect in teams of intelligibility and performance. In this work, we proposed a new geometric modeling method that can be simultaneously used for distance and fuzzy entropy modeling of IFSs. We used rigorously mathematical derivation to prove that the proposed distance and fuzzy entropy measures satisfy the properties of the definitions. In the experiments, we applied the proposed distance and fuzzy entropy measure into pattern recognition, medical diagnosis, and multi-attribute decision making to examine the usability of the two measures in practical situations.

Keywords—entropy, distance measure, intuitionistic fuzzy set, multi-attribute decision making, medical diagnosis, pattern recognition.

I. INTRODUCTION

Intuitionistic fuzzy set theory (IFS) is proved as an outstanding theory for processing the imprecise or uncertain information in the real world. As an important theoretical branch of IFSs, the fuzzy distance and entropy measure have been widely-studied and applied into many areas, such as pattern recognition, multi-attribute decision making, and image processing [1].

In 1995, Chen [2] first defined a set of distance measures between IFSs. In 1999, Hong and Kim [3] studied the distance measure proposed by Chen and pointed out that the measure may produce unreasonable results in some situations; thus the authors improved the distance measure to propose a new distance measure. Since then, the research on distance measure of IFSs has been increased substantially. In 2002, Dengfeng and Chuntian [4] defined a distance measure that was applied to pattern recognition to verify the performance. In 2003, Michell [5] pointed out the shortcomings of Dengfeng and Chuntian's method to introduce a distance measure method that was expected to overcome the shortcomings. In 2004, Szmidt and Kacprzyk [6] proposed a distance measure that was applied to support medical diagnostic reasoning. In 2007, Vlachos [7] proposed a distance measure based on the concept of cross entropy. Subsequently, Hung [8], Ye [9], Boran [10], Song [11], Chen [12], Hwang [13], and Jiang[14] proposed a series of distance measures that were applied to pattern recognition, decision-making, clustering, and image processing [1]. The most existing distance measure methods are introduced though mathematical modeling that is usually abstract and difficult to understand. Obviously, the modeling methods based on geometric model are easier to understand.

As an representation to measure the fuzziness of IFS, fuzzy entropy also can be used for the problems of decision-making and image processing [13][14]. In 1996, Burillo and Bustince [15] first proposed a set of entropy for IFS and interval value fuzzy set (IVFS) based on hesitancy degree. In 2001, Szmidt and Kacprzyk [16] proposed fuzzy entropy model of IFS based on geometry. In 2006, Zeng and Li [17] proposed the entropy measure method of IVFS, and this method can be transformed into the entropy model of IFS. Hung and Yang [18] defined an IFS based on probability and proposed two entropy measure methods. In 2007, Vlachos and Sergiadis [7] used the cross entropy of IFS to proposed a fuzzy entropy model. Subsequently, Zhang [19], Ye [20], and Jiang[21] proposed a series of fuzzy entropy measures from different viewpoints, and these methods were applied to pattern recognition, multi-attribute decision making, and image processing [21][22][45]. Most of the existing measure methods may produce unreasonable results in practical problems, therefore, more works are needed to improve the performance of entropy measure theory.

In this work, we proposed an understandable modeling method that can be applied to establish the distance and entropy measure of IFSs. The proposed method has better interpretability than the most of previous measure methods. The effectiveness and rationality of the two proposed measures are first demonstrated by numerical experiments. Meanwhile, the two measures were also applied into pattern recognition, medical diagnosis, and multi-attribute decision-making. The experiments verified the usability of the two proposed methods.

The rest of this paper is organized as follows. In section II, the related basic knowledge are introduced. In section III, we show the proposed distance and entropy measure model, and the effectiveness of the two measures are verified by experiments. In Section IV, the conclusions are given.

II. PROELIMINARIES

In this section, the involved preliminaries are provided.

A. Intuitionistic Fuzzy Set and Interval value Fuzzy Set

In 1986, Atanassov [22] introduced a hesitation degree on the basis of conventional fuzzy sets to propose IFS that is defined as follows:

Suppose A as an IFS in universe of discourse $U=\{x_1,x_2,\dots,x_n\}$ that is expressed as:

$$A=\{\langle x_i,\mu_A(x_i),\nu_A(x_i)\rangle | x_i\in U\},$$

where $\mu_A(x_i)$ and $\nu_A(x_i)$ represent the membership degree and non-membership degree of x_i belonging to IFS A. The introduced index is $\pi_A(x_i)$ that can be obtained by $\pi_A(x_i)=1-\mu_A(x_i)-\nu_A(x_i)$, and it represents the degree of uncertainty of x_i belonging to IFS A. IFS A can be transformed to IVFS A, and IVFS A can be expressed as: $A=[\mu_A(x_i),1-\nu_A(x_i)]$.

B. *The Properties of Intuitionistic Fuzzy Sets*

The following relationship between IFSs $A=\{\langle x_i,\mu_A(x_i),\nu_A(x_i)\rangle | x_i\in U\}$ and $B=\{\langle x_i,\mu_B(x_i),\nu_B(x_i)\rangle | x_i\in U\}$ in a given universe of discourse $U=\{x_1,x_2,\dots,x_n\}$ that are expressed as [47][48]:

1) $A=B \Leftrightarrow \forall x_i\in U, \mu_A(x_i)=\mu_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$;
2) $A\le B \Leftrightarrow \forall x_i\in U, \mu_A(x_i)\le\mu_B(x_i)$ and $\nu_A(x_i)\ge\nu_B(x_i)$;
3) $A\ge B \Leftrightarrow \forall x_i\in U, \mu_A(x_i)\ge\mu_B(x_i)$ and $\nu_A(x_i)\le\nu_B(x_i)$;
4) $A^C=\{\langle x,\nu_A(x_i),\mu_A(x_i)\rangle | x_i\in U\}$, A^C is the complementary set of A;
5) $A+B=\{\langle x,\mu_A(x_i)+\mu_B(x_i)-\mu_A(x_i)\mu_B(x_i),\nu_A(x_i)\nu_B(x_i),\rangle | x_i\in U\}$.

III. Our Proposed Methods

In this section, we introduce the new method based on geometric modeling, which has the following two advantages. First, the interpretability of our method is better than most of the mathematical modeling methods. Second, this method can be applied into distance measure and entropy measure of IFS, simultaneously. Besides, we verified the applicability of our proposed distance and fuzzy entropy measures in some practical situations including pattern recognition, medical diagnosis, and multi-attribute decision-making.

A. *Distance measure of intuitionistic fuzzy set*

We first introduce the properties of distance measure between IFSs; and then the proposed measure is reported in detail; at last, two kinds of experiments are performed to verify the validity of our theory.

1) *The Properties of Distance Measure*

Distance is used to measure the difference between two IFSs, and the properties are shown as following [14][46]:

D_1: $0\le D(A,B)\le 1$;

D_2: $D(A,B)=0$ if and only if $A=B$;

D_3: $D(A,B)=D(B,A)$, for all $A,B\in U$ are established;

D_4: For $A,B,C\in U$, if $A\subseteq B\subseteq C$, then $D(A,B)\le D(A,C)$, $D(B,C)\le D(A,C)$.

2) *Our New Distance Measure*

Suppose A and B are two IFSs in $U=\{x_1,x_2,\dots,x_n\}$, which are transformed into IVFS and expressed as:

$$A=[\mu_A(x_i),1-\nu_A(x_i)],\ B=[\mu_B(x_i),1-\nu_B(x_i)].$$

For convenience, (A_1,A_2) and (B_1,B_2) are respectively used to represent the transformed intuitionistic fuzzy number $(\mu_A(x_i),1-\nu_A(x_i))$ and $(\mu_B(x_i),1-\nu_B(x_i))$, where $0\le A_1\le A_2\le 1$ and $0\le B_1\le B_2\le 1$. As shown in Fig 1, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are respectively transformed from the IFSs A and B. Point O_1 is the intersection of A_1B_1 and the diagonal of the unit square, and point O_2 is the intersection of A_2B_2 and the diagonal of the unit square. Line A_1M_1, A_2M_2, B_1N_1, and B_2N_2 are perpendicular to the diagonal.

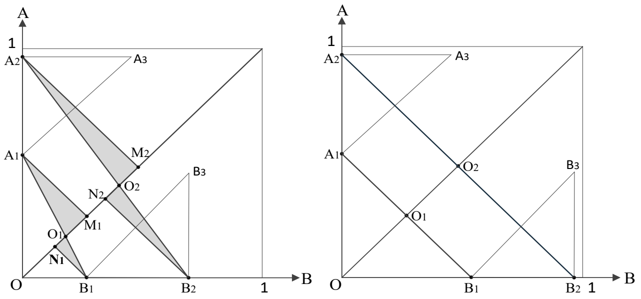

Fig 1. The distance measure model.

A_1B_1 can expressed as: $\mu_A * x+\mu_B * y-\mu_A\mu_B=0$;

A_2B_2 can expressed as: $(\nu_A-1)*x+(\nu_B-1)*y-(\nu_A-1)(\nu_B-1)=0$;

Point O_1 is expressed as: $\left(\frac{\mu_A\mu_B}{\mu_A+\mu_B},\frac{\mu_A\mu_B}{\mu_A+\mu_B}\right)$,

Point O_2 is expressed as: $\left(\frac{(\nu_A-1)(1-\nu_B)}{\nu_A+\nu_B-2},\frac{(\nu_A-1)(1-\nu_B)}{\nu_A+\nu_B-2}\right)$;

We can get: $\overline{OO_1}=\sqrt{2}\times\frac{\mu_A\mu_B}{\mu_A+\mu_B}$, $\overline{OO_2}=\sqrt{2}\times\frac{(\nu_A-1)(1-\nu_B)}{\nu_A+\nu_B-2}$;

From Fig 1, we can get the following:

$$\overline{A_1M_1}=\overline{OM_1}=\frac{\sqrt{2}}{2}\times\overline{OA_1}=\frac{\sqrt{2}}{2}\mu_A,\ \overline{A_2M_2}=\overline{OM_2}=\frac{\sqrt{2}}{2}\times\overline{OA_2}=\frac{\sqrt{2}}{2}(1-\nu_A),$$

$$\overline{B_1N_1}=\overline{ON_1}=\frac{\sqrt{2}}{2}\times\overline{OB_1}=\frac{\sqrt{2}}{2}\mu_B,\ \overline{B_2N_2}=\overline{ON_2}=\frac{\sqrt{2}}{2}\times\overline{OB_2}=\frac{\sqrt{2}}{2}(1-\nu_B).$$

And we can further get the following:

$$\overline{O_1M_1}=\left|\overline{OO_1}-\overline{OM_1}\right|=\frac{\sqrt{2}}{2}\times\left|\frac{\mu_A\mu_B-\mu_A^2}{\mu_A+\mu_B}\right|,\ \overline{O_2M_2}=\left|\overline{OO_2}-\overline{OM_2}\right|=\frac{\sqrt{2}}{2}\times\left|\frac{\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A}{\nu_A+\nu_B-2}\right|;$$

$$\overline{O_1N_1}=\left|\overline{ON_1}-\overline{OO_1}\right|=\frac{\sqrt{2}}{2}\times\left|\frac{\mu_B^2-\mu_A\mu_B}{\mu_A+\mu_B}\right|,\ \overline{O_2N_2}=\left|\overline{OO_2}-\overline{ON_2}\right|=\frac{\sqrt{2}}{2}\times\left|\frac{-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A}{\nu_A+\nu_B-2}\right|.$$

So $S_{\Delta A_1M_1O_1}=S_1=\frac{1}{2}\times\overline{A_1M_1}\times\overline{O_1M_1}=\frac{1}{4}\times\left|\frac{\mu_A^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|$,

$$S_{\Delta B_1N_1O_1}=S_2=\frac{1}{2}\times\overline{B_1N_1}\times\overline{ON_1}=\frac{1}{4}\times\left|\frac{\mu_B^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|,$$

$$S_{\Delta A_2M_2O_2}=S_3=\frac{1}{2}\times\overline{A_2M_2}\times\overline{O_2M_2}=\frac{1}{4}\times\left|\frac{(1-\nu_A)(\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|,$$

$$S_{\Delta B_2N_2O_2}=S_4=\frac{1}{2}\times\overline{B_2N_2}\times\overline{O_2N_2}=\frac{1}{4}\times\left|\frac{(1-\nu_B)(-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|.$$

The proposed distance measure $D(A,B)$ can be expressed as:

$$D(A,B)=\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_B^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|\right| +\left|\left|\frac{(1-\nu_A)(\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_B)(-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|\right|\right) \quad (1)$$

The proof is introduced as follows:

Property: D_1: $0\le D(A,B)\le 1$

Proof.

$D(A,B)=2\times(|S_1-S_2|+|S_3-S_4|)$, so $D(A,B)$ takes the minimum value when $S_1=S_2, S_3=S_4$.

First, $S_1=S_2$, $\frac{1}{4}\times\left|\frac{\mu_A^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|=\frac{1}{4}\times\left|\frac{\mu_B^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|\Rightarrow\mu_A^2=\mu_B^2$, because $\mu_A\geq 0$ and $\mu_B\geq 0$, we can obtained $\mu_A=\mu_B$.

Second, $S_3=S_4$,

$$\Rightarrow(1-\nu_A)(\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A)=(1-\nu_B)(-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A)$$
$$\Rightarrow 2\nu_A^2-\nu_A^3+\nu_A^2\nu_B-4\nu_A\nu_B+2\nu_B^2-\nu_B^3+\nu_A\nu_B^2=0$$
$$\Rightarrow(\nu_A-\nu_B)^2(2-\nu_A-\nu_B)=0$$

We can get $\nu_A=\nu_B$ or $\nu_A=\nu_B=1$.

In summary, when $\mu_A=\mu_B$ and $\nu_A=\nu_B$, $D_{\min}(A,B)=0$.

Due to Eq.(1), we known that:

$$D_{\max}(A,B)=2\times\left(|S_1-S_2|_{\max}+|S_3-S_4|_{\max}\right)=\frac{1}{2}\times\left[(\mu_A-\mu_B)^2+(\nu_A-\nu_B)^2\right]$$

When $\begin{cases}\mu_A=1,\nu_A=0\\ \mu_B=0,\nu_B=1\end{cases}$ or $\begin{cases}\mu_A=0,\nu_A=1\\ \mu_B=1,\nu_B=0\end{cases}$, $D_{\max}(A,B)=1$.

In summary, we can get: $0\leq D(A,B)\leq 1$.

The proof is completed.

■

Property: D_2: $D(A,B)=0\Leftrightarrow A=B$

Proof.

$D(A,B)=0\Rightarrow A=B$

$$D(A,B)=0\Rightarrow\begin{cases}\left|\left|\frac{\mu_A^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_B^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|\right|=0 & (2)\\ \left|\left|\frac{(1-\nu_A)(\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_B)(-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|\right|=0 & (3)\end{cases}$$

From (2), we can get:

$$(\mu_A^2-\mu_B^2)\times\left|\frac{\mu_B-\mu_A}{\mu_A+\mu_B}\right|=0\Rightarrow\mu_A^2-\mu_B^2=0 \text{ or } \left|\frac{\mu_B-\mu_A}{\mu_A+\mu_B}\right|=0\Rightarrow\mu_A=\mu_B,$$

From (3), we can get:

$$\left|\frac{(\nu_B-\nu_A)(\nu_A-1)^2}{\nu_A+\nu_B-2}\right|-\left|\frac{(\nu_B-\nu_A)(\nu_B-1)^2}{\nu_A+\nu_B-2}\right|=0\Rightarrow\left|\frac{(\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|\times\left[(\nu_A-1)^2-(\nu_B-1)^2\right]=0$$
$$\Rightarrow\left|\frac{(\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|=0 \text{ or } (\nu_A-1)^2-(\nu_B-1)^2=0\Rightarrow\nu_B=\nu_A$$

So, $\mu_A=\mu_B,\nu_A=\nu_B\Rightarrow A=B$, $D(A,B)=0\Rightarrow A=B$.

$A=B\Rightarrow D(A,B)=0$

$A=B\Rightarrow\mu_A=\mu_B,\nu_A=\nu_B$, take $\mu_A=\mu_B,\nu_A=\nu_B$ into Eq. (1), and we can get $D(A,B)=0$.

To sum up, we can obtain $D(A,B)=0\Leftrightarrow A=B$.

The proof is completed.

■

Property: D_3: $D(A,B)=D(B,A)$

Proof.

$$D(B,A)=\frac{1}{2}\times\left(\left|\left|\frac{\mu_B^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_A^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|\right|+\left|\left|\frac{(1-\nu_B)(\nu_B^2-\nu_A\nu_B+\nu_A-\nu_B)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_A)(-\nu_A^2+\nu_A\nu_B+\nu_A-\nu_B)}{\nu_A+\nu_B-2}\right|\right|\right)$$

$$=\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_B^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|\right|+\left|\left|\frac{(1-\nu_A)(-\nu_A^2-\nu_A\nu_B+\nu_A-\nu_B)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_B)(\nu_B^2-\nu_A\nu_B+\nu_A-\nu_B)}{\nu_A+\nu_B-2}\right|\right|\right)$$

$$=\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_B^2(\mu_A-\mu_B)}{\mu_A+\mu_B}\right|\right|+\left|\left|\frac{(1-\nu_A)(\nu_A^2+\nu_A\nu_B-\nu_A+\nu_B)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_B)(-\nu_B^2+\nu_A\nu_B-\nu_A+\nu_B)}{\nu_A+\nu_B-2}\right|\right|\right)$$
$$=D(A,B)$$

The proof is completed.

■

Property:

D_4: if $A\subseteq B\subseteq C$, $D(A,B)\leq D(A,C)$ and $D(B,C)\leq D(A,C)$

Proof.

If $A\subseteq B\subseteq C$, we can get the following: $\begin{cases}0\leq\mu_A\leq\mu_B\leq\mu_C\leq 1\\ 0\leq\nu_C\leq\nu_B\leq\nu_A\leq 1\end{cases}$.

$$D(A,C)=\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\mu_C-\mu_A)}{\mu_A+\mu_C}\right|-\left|\frac{\mu_C^2(\mu_C-\mu_A)}{\mu_A+\mu_C}\right|\right|+\left|\left|\frac{(1-\nu_A)(\nu_A^2-\nu_A\nu_C+\nu_C-\nu_A)}{\nu_A+\nu_C-2}\right|-\left|\frac{(1-\nu_C)(-\nu_C^2+\nu_A\nu_C+\nu_C-\nu_A)}{\nu_A+\nu_C-2}\right|\right|\right)$$

$$=\frac{1}{2}\times\left(\left|\frac{(\mu_A^2-\mu_C^2)(\mu_C-\mu_A)}{\mu_A+\mu_C}\right|+\left|\frac{(\nu_C-\nu_A)(1-\nu_A)^2-(\nu_C-\nu_A)(\nu_C-1)^2}{\nu_A+\nu_C-2}\right|\right)$$

$$=\frac{1}{2}\times\left[(\mu_C-\mu_A)^2+(\nu_A-\nu_C)^2\right]$$

$$D(A,B)=\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|-\left|\frac{\mu_B^2(\mu_B-\mu_A)}{\mu_A+\mu_B}\right|\right|+\left|\left|\frac{(1-\nu_A)(\nu_A^2-\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|-\left|\frac{(1-\nu_B)(-\nu_B^2+\nu_A\nu_B+\nu_B-\nu_A)}{\nu_A+\nu_B-2}\right|\right|\right)$$
$$=\frac{1}{2}\times\left[(\mu_B-\mu_A)^2+(\nu_A-\nu_B)^2\right]$$

So, $D(A,C)-D(A,B)=\frac{1}{2}\times\left[(\mu_C-\mu_A)^2+(\nu_A-\nu_C)^2-(\mu_B-\mu_A)^2-(\nu_A-\nu_B)^2\right]$

$$=\frac{1}{2}\times\left[(\mu_C-2\mu_A+\mu_B)(\mu_C-\mu_B)+(\nu_A-2\nu_C+\nu_A)(\nu_B-\nu_C)\right]\Rightarrow D(A,C)-D(A,B)\geq 0$$

Correspondingly, $D(A,C)\geq D(B,C)$ can be proved in the same way.

The proof is completed.

■

3) Applications for Pattern Recognition

Example 1 [23] Given three patterns S_1, S_2, and S_3, which are expressed by three IFSs A_1, A_2 and A_3 in the universe of discourse $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$, as:

$A_1=\{\langle x_1,0.94,0.0\rangle,\langle x_2,0.88,0.0\rangle,\langle x_3,0.82,0.0\rangle,\langle x_4,0.78,0.02\rangle,\langle x_5,0.75,0.05\rangle,\langle x_6,0.72,0.08\rangle\}$

$A_2=\{\langle x_1,0.86,0.07\rangle,\langle x_2,0.92,0.04\rangle,\langle x_3,0.98,0.01\rangle,\langle x_4,0.98,0\rangle,\langle x_5,0.95,0\rangle,\langle x_6,0.92,0\rangle\}$

$A_3=\{\langle x_1,0.66,0.14\rangle,\langle x_2,0.72,0.08\rangle,\langle x_3,0.78,0.02\rangle,\langle x_4,0.84,0\rangle,\langle x_5,0.9,0\rangle,\langle x_6,0.96,0\rangle\}$

The test pattern is denoted by B in $U=\{x_1,x_2,x_3,x_4,x_5,x_6\}$, as:

$B=\{\langle x_1,0.53,0.27\rangle,\langle x_2,0.56,0.24\rangle,\langle x_3,0.59,0.21\rangle,\langle x_4,0.64,0.18\rangle,\langle x_5,0.7,0.15\rangle,\langle x_6,0.76,0.12\rangle\}$

The results of the proposed method are contrasted with these of other distance measure methods as shown in **TABLE I.** Only one method (D_V) cannot classify B into the given patterns because the problem of dividing by zero. The other methods including the proposed method can make a decision that B is classified into S_3.

TABLE I Comparison of pattern recognition results.

Method	$S_1(A_1,B)$	$S_2(A_2,B)$	$S_3(A_3,B)$	result
S_B[10]	0.8181	0.7600	0.8325	S$_3$
S_C [2]	0.8225	0.7600	0.8325	S$_3$
S_{CCL} [12]	0.8169	0.7432	0.8292	S$_3$
S_{DC} [4]	0.8225	0.7600	0.8325	S$_3$
S_{HK} [3]	0.8158	0.7600	0.8325	S$_3$
d_{HY} [30]	0.7900	0.6950	0.8200	S$_3$
S_{HY}^1 [30]	0.7105	0.5877	0.7399	S$_3$
S_{HY}^2 [30]	0.6708	0.5383	0.6958	S$_3$
S_{Lp}^l [8]	0.8158	0.7600	0.8325	S$_3$
S_{Lp}^e [8]	0.6620	0.5634	0.6715	S$_3$
S_{Lp}^c [8]	0.6239	0.5192	0.6247	S$_3$
S_H^1 [31]	0.6568	0.5785	0.6668	S$_3$
S_H^2 [31]	0.8158	0.7600	0.8325	S$_3$
S_H^{pk1} [31]	0.6568	0.5785	0.0668	S$_3$
S_H^{pk2} [31]	0.7801	0.7299	0.7995	S$_3$
S_{JJ} [14]	0.8202	0.7596	0.8313	S$_3$
S_{LS}^1 [32]	0.8158	0.7600	0.8325	S$_3$
S_{LS}^2 [32]	0.8192	0.7600	0.8325	S$_3$
S_{LS}^3 [32]	0.8686	0.8183	0.8842	S$_3$
S_S [33]	0.9204	0.9154	0.9431	S$_3$
S_M [5]	0.8158	0.7600	0.8325	S$_3$
D_W [34]	0.8029	0.7275	0.8263	S$_3$
S_L [35]	0.8087	0.7346	0.8307	S$_3$
S_Y [9]	0.9517	0.9557	0.9669	S$_3$
D_V [7]	N/A	N/A	N/A	NAN
S_{HJF}	0.9518	0.9298	0.9675	S$_3$

Note: The meaning of N/A is: due to the problem of dividing by 0, it is impossible to calculate. NAN means cannot be determined.

4) Applications for Medical Diagnosis

Example 2 [23][24][25] Suppose a set of diagnosis as $A=\{A_1,A_2,A_3,A_4,A_5\}$, and the five elements are represented by IFSs S_1,S_2,S_3,S_4 and S_5 in the universe of discourse $U=\{x_1,x_2,x_3,x_4,x_5\}$ where x_1 means *Temperature*, x_2 means *Headache*, x_3 means *StomachPain*, x_4 means *Cough*, and x_5 means *ChestPain*. The results are shown in **TABLE II**. The IFSs S_1,S_2,S_3,S_4, and S_5 are expressed as follows:

$$S_1(\textit{Viral fever})=\{\langle x_1,0.4,0.0\rangle,\langle x_2,0.3,0.5\rangle,\langle x_3,0.1,0.7\rangle,\langle x_4,0.4,0.3\rangle,\langle x_5,0.1,0.7\rangle\}$$

$$S_2(\textit{Malaria})=\{\langle x_1,0.7,0.0\rangle,\langle x_2,0.2,0.6\rangle,\langle x_3,0.0,0.9\rangle,\langle x_4,0.7,0.0\rangle,\langle x_5,0.1,0.8\rangle\}$$

$$S_3(\text{Typhoid})=\{\langle x_1,0.3,0.3\rangle,\langle x_2,0.6,0.1\rangle,\langle x_3,0.2,0.7\rangle,\langle x_4,0.2,0.6\rangle,\langle x_5,0.1,0.9\rangle\}$$

$$S_4(\text{Stomach problem})=\{\langle x_1,0.1,0.7\rangle,\langle x_2,0.2,0.4\rangle,\langle x_3,0.8,0.0\rangle,\langle x_4,0.2,0.7\rangle,\langle x_5,0.2,0.7\rangle\}$$

$$S_5(\textit{Chest problem})=\{\langle x_1,0.1,0.8\rangle,\langle x_2,0.0,0.8\rangle,\langle x_3,0.2,0.8\rangle,\langle x_4,0.2,0.8\rangle,\langle x_5,0.8,0.1\rangle\}$$

Suppose the diagnosis of a patient is unknown, but the patient's symptoms are expressed by IFS B, as follows:

$$B(\textit{Patient})=\{\langle x_1,0.8,0.1\rangle,\langle x_2,0.6,0.1\rangle,\langle x_3,0.2,0.8\rangle,\langle x_4,0.6,0.1\rangle,\langle x_5,0.1,0.6\rangle\}$$

The diagnostic results are shown in **TABLE II**, D_V cannot make a decision because the problem of dividing by zero. S_C, S_{DC}, S_{LS}^2, S_Y, S_S, and S_{CCL} classify B into S$_1$; however, other methods classify B into S$_2$, and these methods are S_{LS}^1, S_{LS}^3, S_M, d_{HY}, S_{HY}^1, S_{HY}^2, S_{Lp}^l, S_{Lp}^e, S_{Lp}^c, S_L, D_W, S_H^1, S_H^2, S_H^{pk1}, S_H^{pk2}, S_B, S_{JJ}, and the proposed method S_{HJF}.

TABLE II. Comparison of pattern recognition results.

Method	$S_1(A_1,B)$	$S_2(A_2,B)$	$S_3(A_3,B)$	$S_4(A_4,B)$	$S_5(A_5,B)$	result
S_B [10]	0.8191	0.8236	0.8000	0.5492	0.5000	S$_2$
S_C [2]	0.8500	0.8400	0.8000	0.5600	0.5000	S$_1$
S_{CCL} [12]	0.8417	0.8360	0.8067	0.5593	0.5047	S$_1$
S_{DC} [4]	0.8500	0.8400	0.8000	0.5600	0.5000	S$_1$
d_{HY} [30]	0.7600	0.7800	0.7200	0.4800	0.4400	S$_2$
S_{HY}^1 [30]	0.6738	0.7006	0.6389	0.3893	0.3629	S$_2$
S_{HY}^2 [30]	0.6320	0.6606	0.6047	0.3569	0.3412	S$_2$
S_{Lp}^l [8]	0.8100	0.8200	0.8000	0.5400	0.5000	S$_2$
S_{Lp}^e [8]	0.6524	0.6770	0.6643	0.3529	0.3415	S$_2$
S_{Lp}^c [8]	0.6136	0.6387	0.6379	0.3319	0.3347	S$_2$
S_H^1 [31]	0.6077	0.6566	0.6427	0.3134	0.3107	S$_2$
S_H^2 [31]	0.8100	0.8200	0.8000	0.5400	0.5000	S$_2$
S_H^{pk1} [31]	0.6077	0.6566	0.6427	0.3134	0.3107	S$_2$
S_H^{pk2} [31]	0.7370	0.7698	0.7408	0.4338	0.3917	S$_2$
S_{JJ} [14]	0.8314	0.8350	0.7979	0.5413	0.4968	S$_2$
S_{LS}^1 [32]	0.8100	0.8200	0.8000	0.5400	0.5000	S$_2$
S_{LS}^2 [32]	0.8400	0.8300	0.8000	0.5500	0.5000	S$_1$
S_{LS}^3 [32]	0.7533	0.7600	0.7400	0.6600	0.6467	S$_2$
S_S [33]	0.9347	0.9228	0.9223	0.7673	0.7490	S$_1$
S_M [5]	0.8100	0.8200	0.8000	0.5400	0.5000	S$_2$
D_W [34]	0.7850	0.8000	0.7600	0.5100	0.4700	S$_2$
S_L [35]	0.7416	0.7791	0.7412	0.5120	0.4823	S$_2$
S_Y [9]	0.9046	0.8602	0.8510	0.5033	0.4542	S$_1$
D_V [7]	N/A	N/A	0.8987	N/A	N/A	NAN
S_{HJF}	0.9414	0.9458	0.9060	0.7030	0.7130	S$_2$

Note: The meaning of N/A is: due to the problem of dividing by 0, it is impossible to calculate. NAN means cannot be determined.

Example 3 [14] Suppose a set of diagnoses as $A=\{A_1,A_2,A_3,A_4,A_5\}$, the four elements are represented by the IFSs S_1, S_2, S_3 and S_4 in the universe of discourse $U=\{x_1,x_2,x_3,x_4,x_5\}$ where x_1 means *characterofstool*, x_2 means *bellyache*, x_3 means *ictussileus*, x_4 means *chronicsileus*, and x_5 means *anemia*. IFSs $S_1,S_2,S_3,$ and S_4 are represented as:

$$S_1(\textit{metastasis})=\{\langle x_1,0.4,0.4\rangle,\langle x_2,0.3,0.3\rangle,\langle x_3,0.5,0.1\rangle,\langle x_4,0.5,0.2\rangle,\langle x_5,0.6,0.2\rangle\}$$

$$S_2(\textit{recurrence})=\{\langle x_1,0.2,0.6\rangle,\langle x_2,0.3,0.5\rangle,\langle x_3,0.2,0.3\rangle,\langle x_4,0.7,0.1\rangle,\langle x_5,0.8,0.0\rangle\}$$

$$S_3(\textit{bad})=\{\langle x_1,0.1,0.9\rangle,\langle x_2,0.0,0.1\rangle,\langle x_3,0.2,0.7\rangle,\langle x_4,0.1,0.8\rangle,\langle x_5,0.2,0.8\rangle\}$$

$$S_4(\textit{well})=\{\langle x_1,0.8,0.2\rangle,\langle x_2,0.9,0.0\rangle,\langle x_3,1.0,0.0\rangle,\langle x_4,0.7,0.2\rangle,\langle x_5,0.6,0.4\rangle\}$$

Given an unknown diagnosis, the symptoms are expressed by IFS B as follows:

$$B(\textit{Patient})=\{\langle x_1,0.3,0.5\rangle,\langle x_2,0.4,0.4\rangle,\langle x_3,0.6,0.2\rangle,\langle x_4,0.5,0.1\rangle,\langle x_5,0.9,0.0\rangle\}$$

The diagnostic results are shown in **TABLE III.** S_{HK}, S_{LS}^1, S_M, S_{Lp}^l, S_{Lp}^e, S_{Lp}^c and S_H^2 cannot make reasonable decision because $S_1(A_1,B)=S_2(A_2,B)>S_3(A_3,B)$ and $S_1(A_1,B)=S_2(A_2,B)>S_4(A_4,B)$, D_V cannot make a diagnosis because the denominator is zero. The proposed method and other methods and can make a diagnosis and classify the patient as S_1.

B. *Entropy of intuitionistic fuzzy set*

In this sub-section, the properties of entropy of IFS are first introduced; and then the proposed measure is reported in detail; moreover, the numerical experiments and applications for multi-attribute decision-making are performed to show the rationality of our method.

TABLE III. Comparison of pattern recognition results.

Method	$S_1(A_1,B)$	$S_2(A_2,B)$	$S_3(A_3,B)$	$S_4(A_4,B)$	result
S_B [10]	0.9029	0.8800	0.5800	0.6755	S_1
S_C [2]	0.9200	0.8800	0.5800	0.6900	S_1
S_{CCL} [12]	0.9053	0.8720	0.5833	0.6750	S_1
S_{DC} [4]	0.9200	0.8800	0.5800	0.6900	S_1
S_{HK} [3]	**0.8800**	**0.8800**	0.5200	0.6700	NAN
d_{HY} [30]	0.8600	0.8200	0.4400	0.6000	S_1
S_{HY}^1 [30]	0.7976	0.7480	0.3334	0.4850	S_1
S_{HY}^2 [30]	0.7622	0.7100	0.2956	0.4381	S_1
S_{Lp}^l [8]	**0.8800**	**0.8800**	0.5200	0.6700	NAN
S_{Lp}^e [8]	**0.7612**	**0.7612**	0.3079	0.4546	NAN
S_{Lp}^c [8]	**0.7227**	**0.7227**	0.2847	0.4193	NAN
S_H^1 [31]	0.7361	0.7278	0.2408	0.4641	S_1
S_H^2 [31]	**0.8800**	**0.8800**	0.5200	0.6700	NAN
S_H^{pk1} [31]	0.7361	0.7278	0.2408	0.4641	S_1
S_H^{pk2} [31]	0.8436	0.8327	0.3673	0.6249	S_1
S_{HW} [13]	0.7515	0.8344	0.4289	0.2633	S_1
S_{JJ} [14]	0.9029	0.8779	0.5528	0.6894	S_1
S_{LS}^1 [32]	**0.8800**	**0.8800**	0.5200	0.6700	NAN
S_{LS}^2 [32]	0.9000	0.8800	0.5550	0.6850	S_1
S_{LS}^3 [32]	0.9133	0.9000	0.6533	0.7567	S_1
S_S [33]	0.9669	0.9841	0.7309	0.8372	S_1
S_M [5]	**0.8800**	**0.8800**	0.5200	0.6700	NAN
D_W [34]	0.8700	0.8500	0.4800	0.6350	S_1
S_L [35]	0.8378	0.8279	0.4448	0.6269	S_1
S_Y [9]	0.9789	0.9468	0.5479	0.8383	S_1
D_V [7]	N/A	N/A	N/A	N/A	NAN
S_{HJF}	0.9836	0.9691	0.7925	0.8759	S_1

Note: The meaning of N/A is: due to the problem of dividing by 0, it is impossible to calculate. NAN means cannot be determined.

1) *The Properties of Entropy of IFS*

Entropy is used to measure the fuzzy degree of an IFS, and the properties are shown as following [15][16]:

E_1: $E(A)=0$ if and only if A is crisp set;

E_2: $E(A)=1$ if and only if $\mu(x_i)=\nu(x_i)$ for each $x_i \in U$;

E_3: $E(A)=E(A^C)$, A^C is the complementary set of A;

E_4: $E(A)\le E(B)$, for any $x_i \in U$;

When $\mu_B(x)\ge \nu_B(x)$, there are $\mu_A(x)\ge \mu_B(x), \nu_B \ge \nu_A(x)$;

or when $\mu_B(x)\le \nu_B(x)$, there are $\mu_A(x)\le \mu_B(x), \nu_B \le \nu_A(x)$.

2) *Our New Fuzzy Entropy*

According to the same modeling method shown in Fig. 1, we introduce the proposed fuzzy entropy in this sub-section.

Suppose A is an IFS in $U=\{x_1,x_2,\cdots$ that is expressed as: $A=[\mu_A(x_i),1-\nu_A(x_i)]$. Because A^C is the complementary set of A, we can know that A^C can be expressed as: $A^C=[\nu_A(x_i),1-\mu_A(x_i)]$. As shown in Fig 2, the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are respectively transformed from the IFSs A and B. The proposed fuzzy entropy can be defined as follows:

$$E(A)=1-2\times\left(\left|S_1-S_2\right|+\left|S_3-S_4\right|\right)$$

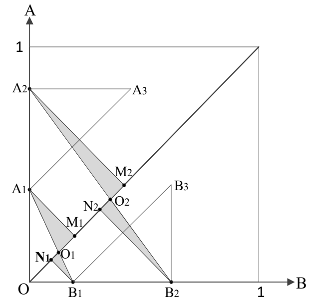

Fig 2. The fuzzy entropy model.

$$=1-\frac{1}{2}\times\left(\left|\left|\frac{\mu_A^2(\nu_A-\mu_A)}{\mu_A+\nu_A}\right|-\left|\frac{\nu_A^2(\nu_A-\mu_A)}{\mu_A+\nu_A}\right|\right| + \left|\left|\frac{(1-\nu_A)(\nu_A^2-\mu_A\nu_A+\mu_A-\nu_A)}{\nu_A+\mu_A-2}\right|-\left|\frac{(1-\mu_A)(-\mu_A^2+\mu_A\nu_A+\mu_A-\nu_A)}{\nu_A+\mu_A-2}\right|\right|\right) \quad (4)$$

The Eq. (4) can be simplified as $E(A)=1-(\mu_A-\nu_A)^2$.

To prove the rationality of the proposed entropy measure, the four properties of fuzzy entropy are verified as follows:

Property: E_1: $E(A)=0$ if and only if A is crisp set.

Proof.

$E(A)=0 \Rightarrow A$ is crisp set.

When $E(A)=0$,

$$E(A)=1-(\mu_A-\nu_A)^2=0 \Rightarrow \mu_A-\nu_A=\pm 1 \Rightarrow \begin{cases}\mu_A=1\\ \nu_A=0\end{cases} \text{or} \begin{cases}\mu_A=0\\ \nu_A=1\end{cases} \Rightarrow \text{A is crisp set.}$$

A is crisp set $\Rightarrow E(A)=0$

Because A is crisp set, as $\mu_A=1,\nu_A=0$ $\mu_A=1,\nu_A=0$ or $\mu_A=0,\nu_A=1$, we can get $E(A)=0$ according to Eq.(4).

Thus we can get $E(A)=0$ if and only if A is crisp set.

The proof is completed.

■

Property: E_2: $E(A)=1$ if and only if $\mu(x)=\nu(x)$ for each $x\in U$.

Proof.

$E(A)=1 \Rightarrow \mu_A(x)=\nu_A(x)$.

$E(A)=1-(\mu_A-\nu_A)^2=1 \Rightarrow (\mu_A-\nu_A)^2=0 \Rightarrow \mu_A=\nu_A$.

$\mu_A(x)=\nu_A(x) \Rightarrow E(A)=1$.

$E(A)=1-(\mu_A-\nu_A)^2=1$.

In summary, we can get: $E(A)=1$ if and only if $\mu(x)=\nu(x)$ for each $x\in U$.

The proof is completed.

■

Property: E_3: $E(A)=E(A^C)$, A^C is the complementary set of A.

Proof.

Because A^C is the complementary set of A, we can can the following:

$$E(A)=1-(\mu_A-\nu_A)^2=1-(\nu_{A^C}-\mu_{A^C})^2=1-(\mu_{A^C}-\nu_{A^C})^2=E(A^C)$$

The proof is completed.

∎

Property: E_4 : $E(A) \le E(B)$, for any $x_i \in U$;

When $\mu_B(x) \ge \nu_B(x)$, there are $\mu_A(x) \ge \mu_B(x), \nu_B \ge \nu_A(x)$;

or when $\mu_B(x) \le \nu_B(x)$, there are $\mu_A(x) \le \mu_B(x), \nu_B \le \nu_A(x)$.

TABLE IV. Comparison of numerical experimental results

Method	α_1	α_2	α_3	α_4
E_{B1} [15]	0.0000	0.2000	0.4000	0.6000
E_{B2} [15]	0.0000	0.3600	0.6400	0.8400
E_{B3} [15]	0.0000	0.0229	0.1049	0.2712
E_{B4} [15]	0.0000	0.2392	0.5146	0.7649
E_{ZJ} [19]	0.4286	0.3333	0.2000	0.0000
E_{ZL} [17]	0.6000	0.6000	0.6000	0.6000
E_{SK} [16]	0.4286	0.5000	0.5556	0.6000
E_{V1} [36]	0.8813	0.8490	0.7900	N/A
E_{V2} [7]	0.7241	0.6364	0.6190	0.6923
E_{V3} [7]	0.4286	0.4286	0.4286	0.4286
E_{XX} [37]	0.8426	0.8525	0.8612	0.8689
E_{JW} [21]	0.5676	0.6667	0.7377	0.7895
E_{Z1} [38]	0.1500	0.2400	0.3500	0.4800
E_Z [26]	0.6000	0.7528	0.8343	0.8789
E_{Y1} [20]	0.8329	0.8329	0.8329	0.8329
E_{Y2} [20]	0.8329	0.8329	0.8329	0.8329
E_{HC} [18]	0.4200	0.5600	0.5800	0.4800
E_S [18]	0.6109	0.9503	0.9433	0.6731
E_{ZA} [39]	0.6000	0.5528	0.4343	0.2789
E_{ZB} [39]	0.6000	0.6000	0.6000	0.4000
E_{ZC} [39]	0.6000	0.4000	0.2000	0.0000
E_{ZD} [39]	0.6000	0.4000	0.2000	0.0000
E_{ZE} [39]	0.6000	0.5000	0.4000	0.2000
E_{HJF}	0.8400	0.8976	0.9424	0.9744

Note: N/A means cannot be calculated.

Proof.

When $\nu_A \le \nu_B \le \mu_B \le \mu_A$,

$$E(A) - E(B) = \left[1-\left(\mu_A-\nu_A\right)^2\right]-\left[1-\left(\mu_B-\nu_B\right)^2\right]$$

$$=\left(\mu_B-\nu_B\right)^2-\left(\mu_A-\nu_A\right)^2=\left(\mu_B-\nu_A+\mu_A-\nu_B\right)\left(\mu_B-\mu_A+\nu_A-\nu_B\right)\le 0$$

When $\nu_A \ge \nu_B \ge \mu_B \ge \mu_A$,

$$E(A) - E(B) = \left[1-\left(\mu_A-\nu_A\right)^2\right]-\left[1-\left(\mu_B-\nu_B\right)^2\right]$$

$$=\left(\mu_B-\nu_B\right)^2-\left(\mu_A-\nu_A\right)^2=\left(\mu_B-\nu_A+\mu_A-\nu_B\right)\left(\mu_B-\mu_A+\nu_A-\nu_B\right)\le 0$$

Therefore, we can infer $E(A) \le E(B)$, for any $x_i \in U$;

When $\mu_B(x) \ge \nu_B(x)$, there are $\mu_A(x) \ge \mu_B(x), \nu_B \ge \nu_A(x)$;

or when $\mu_B(x) \le \nu_B(x)$, there are $\mu_A(x) \le \mu_B(x), \nu_B \le \nu_A(x)$.

From the above analysis, we can get that the proposed fuzzy entropy satisfies the fourth property.

The proof is completed.

∎

3) Numerical Experiments

In this sub-section, we performed some popular numerical experiments to compare the performance of the proposed entropy model with the existing fuzzy entropy models.

Example 4 [26][27][28] Suppose $\alpha_i\,(i=1,2,3,4)$ indicates four houses with intention to buy: $\alpha_1=(0.7,0.3)$, $\alpha_2=(0.6,0.2)$, $\alpha_3=(0.5,0.1)$, $\alpha_4=(0.4,0)$.

In extreme cases, for α_1, the purchase intention value is 70%, and the non-purchase intention value is 30%. However, for α_4 , the purchase intention value is 40%, and the uncertain intention value is 60%. Intuitively, the intention to buy house α_1 is greater than that of the intention to buy house α_4 . From a mathematical point view, we can consider that the order of the entropy values of the four house purchase intentions is:

$$E(\alpha_1) < E(\alpha_2) < E(\alpha_3) < E(\alpha_4) \tag{5}$$

In this sub-section, we chose a variety of fuzzy entropy measures to compare with the proposed fuzzy entropy measure. The results are shown in the **TABLE IV.** As **TABLE IV**, we find $E_B, E_{SK}, E_{XX}, E_{JW}, E_{Z1}, E_Z$ and the proposed fuzzy entropy measure method E_{HJF} satisfy (5), and other fuzzy entropy measures have a different order:

$E_{ZJ}(\alpha_1) > E_{ZJ}(\alpha_2) > E_{ZJ}(\alpha_3) > E_{ZJ}(\alpha_4)$, $E_{ZL}(\alpha_1) = E_{ZL}(\alpha_2) = E_{ZL}(\alpha_3) = E_{ZL}(\alpha_4)$

$E_{V2}(\alpha_3) < E_{V2}(\alpha_2) < E_{V2}(\alpha_4) < E_{V2}(\alpha_1)$, $E_{V3}(\alpha_1) = E_{V3}(\alpha_2) = E_{V3}(\alpha_3) = E_{V3}(\alpha_4)$

$E_Y(\alpha_1) = E_Y(\alpha_2) = E_Y(\alpha_3) = E_Y(\alpha_4)$, $E_{hc}(\alpha_1) < E_{hc}(\alpha_4) < E_{hc}(\alpha_2) < E_{hc}(\alpha_3)$

$E_S(\alpha_1) < E_S(\alpha_4) < E_S(\alpha_3) < E_S(\alpha_2)$, $E_{ZA}(\alpha_4) < E_{ZA}(\alpha_3) < E_{ZA}(\alpha_2) < E_{ZA}(\alpha_1)$

$E_{ZB}(\alpha_4) < E_{ZB}(\alpha_1) = E_{ZB}(\alpha_2) = E_{ZB}(\alpha_3)$, $E_{ZC}(\alpha_4) < E_{ZC}(\alpha_3) < E_{ZC}(\alpha_2) < E_{ZC}(\alpha_1)$

$E_{ZD}(\alpha_4) < E_{ZD}(\alpha_3) < E_{ZD}(\alpha_2) < E_{ZD}(\alpha_1)$, $E_{ZE}(\alpha_4) < E_{ZE}(\alpha_3) < E_{ZE}(\alpha_2) < E_{ZE}(\alpha_1)$

According to the above analysis, $E_B, E_{SK}, E_{XX}, E_{JW}, E_{Z1}, E_Z$ and the proposed method E_{HJF} can get a reasonable sequence of fuzzy entropy measures. Therefore, we can infer that the performance of the above mentioned methods are good, and better than that of other contrast fuzzy entropy measures.

Example 5 [14][29] Suppose $A_i\,(i=1,2,3)$ indicates three IFSs in the universe of discourse $U=\{x_1,x_2,...,x_n\}$ and $A_1=\{\langle x,0.4,0.6\rangle | x\in U\}$, $A_2=\{\langle x,0.3,0.5\rangle | x\in U\}$, $A_3=\{\langle x,0.1,0.3\rangle | x\in U\}$.

The experimental results obtained by different fuzzy entropy measures are shown in **TABLE V**. The existing literatures did not indicate a completely reasonable entropy ranking. However, based on the ranking results obtained by most fuzzy entropy methods and the proposed entropy method, the entropy ranking can be considered as follows:

$$E(A_1) < E(A_2) < E(A_3) \tag{6}$$

Therefore, we can infer the proposed method is effective.

4) Applications for Multi-Attribute Decision-making

Example 6 [43] [44] A college questionnaire surveyed the readers' satisfaction with the library. The first questionnaire is A_1, 258 questionnaires were distributed, and 250 were returned. The second questionnaire is A_2, 210 questionnaires were distributed, and 199 were returned. The third questionnaire is A_3, 260 questionnaires were distributed, and 250 were returned. The decision matrix obtained from the three questionnaire results is shown a set A . C_1 is work efficiency, C_2 is professional quality, C_3 is knowledge structure, C_4 is skill structure, C_5 is the number of documents,

C_6 is document quality, C_7 is document management, C_8 is complete equipment, C_9 is equipment maintenance, C_{10} is the layout in library and C_{11} is the health in library.

$$A=\left(\begin{array}{llllll} & C_1 & C_2 & C_3 & C_4 & C_5 \\ A_1 & \langle 0.38,0.52\rangle & \langle 0.54,0.43\rangle & \langle 0.66,0.32\rangle & \langle 0.43,0.50\rangle & \langle 0.38,0.10\rangle \\ A_2 & \langle 0.39,0.48\rangle & \langle 0.38,0.42\rangle & \langle 0.44,0.52\rangle & \langle 0.38,0.52\rangle & \langle 0.50,0.42\rangle \\ A_3 & \langle 0.38,0.52\rangle & \langle 0.46,0.52\rangle & \langle 0.38,0.42\rangle & \langle 0.41,0.56\rangle & \langle 0.78,0.12\rangle \end{array}\right.$$
$$\left.\begin{array}{lllllll} & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} \\ A_1 & \langle 0.6,0.2\rangle & \langle 0.38,0.52\rangle & \langle 0.38,0.52\rangle & \langle 0.38,0.5\rangle & \langle 0.36,0.52\rangle & \langle 0.38,0.46\rangle \\ A_2 & \langle 0.38,0.55\rangle & \langle 0.42,0.52\rangle & \langle 0.38,0.4\rangle & \langle 0.48,0.5\rangle & \langle 0.38,0.52\rangle & \langle 0.44,0.34\rangle \\ A_3 & \langle 0.5,0.43\rangle & \langle 0.48,0.5\rangle & \langle 0.49,0.32\rangle & \langle 0.58,0.22\rangle & \langle 0.5,0.33\rangle & \langle 0.6,0.22\rangle \end{array}\right)$$

TABLE V. Comparison of numerical experimental results

Case	A_1	A_2	A_3	Ranking
E_{B1}[15]	0.0000	0.2000	0.6000	$E(A_1)<E(A_2)<E(A_3)$
E_{B2}[15]	0.0000	0.3600	0.8400	$E(A_1)<E(A_2)<E(A_3)$
E_{B3}[15]	0.0000	0.0229	0.2712	$E(A_1)<E(A_2)<E(A_3)$
E_{B4}[15]	0.0000	0.2392	0.7649	$E(A_1)<E(A_2)<E(A_3)$
E_{ZJ}[19]	0.6667	0.6000	0.3333	$E(A_3)<E(A_2)<E(A_1)$
E_{ZL}[17]	0.8000	0.8000	0.8000	$E(A_1)=E(A_2)=E(A_3)$
E_{SK}[16]	0.6667	0.7143	0.7778	$E(A_1)<E(A_2)<E(A_3)$
E_{V1}[36]	0.9710	0.9635	0.9245	$E(A_3)<E(A_2)<E(A_1)$
E_{V2}[7]	0.9231	0.8947	0.9130	$E(A_2)<E(A_3)<E(A_1)$
E_{V3}[7]	0.6667	0.6000	0.3333	$E(A_3)<E(A_2)<E(A_1)$
E_{XX}[37]	0.9607	0.9632	0.9673	$E(A_1)<E(A_2)<E(A_3)$
E_{JW}[21]	0.8571	0.8974	0.9403	$E(A_1)<E(A_2)<E(A_3)$
E_{Z1}[38]	0.2000	0.3000	0.5600	$E(A_1)<E(A_2)<E(A_3)$
E_{Z}[26]	0.8000	0.9172	0.9675	$E(A_1)<E(A_2)<E(A_3)$
E_{Y1}[20]	0.9580	0.9580	0.9580	$E(A_1)=E(A_2)=E(A_3)$
E_{Y2}[20]	0.9580	0.9580	0.9580	$E(A_1)=E(A_2)=E(A_3)$
E_{HC}[18]	0.4800	0.6200	0.5400	$E(A_1)<E(A_3)<E(A_2)$
E_{S}[18]	N/A	N/A	N/A	NAN
E_{ZA}[39]	0.8000	0.7172	0.3675	$E(A_3)<E(A_2)<E(A_1)$
E_{ZB}[39]	0.8000	0.8000	0.4000	$E(A_3)<E(A_2)=E(A_1)$
E_{ZC}[39]	0.8000	0.6000	0.2000	$E(A_3)<E(A_2)<E(A_1)$
E_{ZD}[39]	0.8000	0.6000	0.2000	$E(A_3)<E(A_2)<E(A_1)$
E_{ZE}[39]	0.8000	0.7000	0.3000	$E(A_3)<E(A_2)<E(A_1)$
E_{HJF}	0.9600	0.9744	0.9936	$E(A_1)<E(A_2)<E(A_3)$

Note: N/A means cannot be calculated, NAN means cannot be sorted.

According to the multi-attribute decision-making algorithm used in [43], we can get the following steps:

Step1: Find the j-th entropy in the decision matrix A, and we can get:
$E_1=0.9804$, $E_2=0.9964$, $E_3=0.9984$, $E_4=0.9775$, $E_5=0.5644$, $E_6=0.9951$, $E_7=0.9996$, $E_8=0.9711$, $E_9=0.8704$, $E_{10}=0.9711$, $E_{11}=0.8556$.

Step2: Calculate the j-th entropy weight, and we can get:
$W_1=0.0239$, $W_2=0.0044$, $W_3=0.002$, $W_4=0.0274$, $W_5=0.5312$, $W_6=0.006$, $W_7=0.0005$, $W_8=0.0352$, $W_9=0.158$, $W_{10}=0.0352$, $W_{11}=0.1761$.

Step3: The decision matrix is weighted according to the following formula to obtain the matrix R:

$$R=\left(R_{ij}\right)_{m\times n}=\left(W_j A_{ij}\right)_{m\times n}.$$

$R_{11}=(0.0114,0.9845)$ $R_{12}=(0.0034,0.9963)$ $R_{13}=(0.0021,0.9978)$
$R_{14}=(0.0153,0.9812)$ $R_{15}=(0.2243,0.2943)$ $R_{16}=(0.0055,0.9904)$
$R_{17}=(0.0002,0.9997)$ $R_{18}=(0.0167,0.9772)$ $R_{19}=(0.0728,0.8962)$
$R_{110}=(0.0156,0.9772)$ $R_{111}=(0.0807,0.8722)$ $R_{21}=(0.0114,0.9845)$
$R_{22}=(0.0034,0.9963)$ $R_{23}=(0.0021,0.9978)$ $R_{24}=(0.0153,0.9812)$
$R_{25}=(0.2243,0.2943)$ $R_{26}=(0.0055,0.9904)$ $R_{27}=(0.0002,0.9997)$
$R_{28}=(0.0167,0.9772)$ $R_{29}=(0.0728,0.8962)$ $R_{210}=(0.0156,0.9772)$
$R_{211}=(0.0807,0.8722)$ $R_{31}=(0.0114,0.9845)$ $R_{32}=(0.0034,0.9963)$
$R_{33}=(0.0021,0.9978)$ $R_{34}=(0.0153,0.9812)$ $R_{35}=(0.2243,0.2943)$
$R_{36}=(0.0055,0.9904)$ $R_{37}=(0.0002,0.9997)$ $R_{38}=(0.0167,0.9772)$
$R_{39}=(0.0728,0.8962)$ $R_{310}=(0.0156,0.9772)$ $R_{311}=(0.0807,0.8722)$

Step4: By using the formula (A+B) to calculate the matrix R in rows, the comprehensive evaluation value of the described information by intuitionistic fuzzy set is obtained.
$A_1=[0.383887,0.208862]$ $A_2=[0.471995,0.423093]$ $A_3=[0.694242,0.173256]$.

Step5: The score function $S(A_i)$ and the accurate function $H(A_i)$ of A_i (i=1, 2, 3) are calculated by the formula that is used in [44]. Thus, we can get:
$S(A_1)=0.21567$, $S(A_2)=0.051641$, $S(A_3)=0.31859]$
$H(A_1)=0.7304$, $H(A_2)=0.94522$, $H(A_3)=0.53049]$

We know that the results of the questionnaire are ranked as: $A_3>A_1>A_2$. This result is consistent with the results in [43] and [44]. This experiment confirmed that the proposed fuzzy entropy measure method is effective in multi-attribute decision-making.

IV. CONCLUSION

In this work, a geometric modeling methods is introduced to propose a new distance measure between IFSs. And the proposed method is interpretable and fully consistent with the property of distance. Besides, the proposed distance is verified by applying it into pattern recognition and medical diagnosis, and the experimental results show that our method has a good performance. Moreover, we also propose a new fuzzy entropy measure method using the same modeling method, and the rationality of the fuzzy entropy model are verified by mathematical proof and experiments; besides, we also applies the fuzzy entropy measure into multi-attribute decision making. In addition, it is worth noting that the proposed distance and fuzzy entropy measures in this paper can be generalized to interval-valued intuitionistic fuzzy sets.

ACKNOWLEDGMENT

This study is funded by the National Natural Science Foundation of China (61863036), China Postdoctoral Science Foundation (2019M653507), Postdoctoral Science Foundation of Yunnan Province in China, and Yunnan University's Research Innovation Fund for Graduate Students (No. 2019164, 2019166).

REFERENCES

[1] G. Beliakov, M. Pagola and T. Wilkin, “Vector valued similarity measures for Atanassov’s intuitionistic fuzzy sets,” Information Sciences, vol. 280, pp.352-367, 2014.

[2] S. M. Chen, “Measures of Similarity between Vague Sets,” Fuzzy Sets and Systems, vol. 74, pp. 217-223, 1995.

[3] D. H. Hong and C. Kim, “A Note on Similarity Measures between Vague Sets and between Elements,” Information Sciences, vol. 115, pp. 83-96, 1999.

[4] D. F. Li and C. T. Cheng, "New Similarity Measures of Intuitionistic Fu zzy Sets and Application to Pattern Recognitions," Pattern Recognition Letters, vol. 23, pp. 221-225, 2002.

[5] H. B. Mitchell, "On the Dengfeng–Chuntian Similarity Measure and Its Application to Pattern Recognition," Pattern Recognition Letters, vol. 24, pp. 3101-3104, 2003.

[6] E. Szmidt and J. Kacprzyk, "A similarity measure for intuitionistic fuzzy sets and its application in supporting medical diagnostic reasoning," Lecture Notes in Artificial Intelligence, vol. 3070, pp. 388-393, 2004.

[7] I. K. Vlachos and G. D. Sergiadis, "Intuitionistic fuzzy information-applications to pattern recognition," Pattern Recognition Letters, vol. 28, pp. 197-206, 2007.

[8] W. L. Hung and M. S. Yang, "Similarity Measures of Intuitionistic Fuzzy Sets Based on Lp Metric,"International Journal of Approximate Reasoning, vol. 46, pp. 120-136, 2007.

[9] J. Ye, "Cosine Similarity Measures for Intuitionistic Fuzzy Sets and Their Applications," Mathematical and Computer Modelling,vol. 53, pp. 91-97, 2011.

[10] F. E. Boran and D. Akay, "A Biparametric Similarity Measure on Intuitionistic Fuzzy Sets with Applications to Pattern Recognition," Information Sciences, vol. 255, pp. 45-57, 2014.

[11] Y. Song, X. Wang, L. Lei, et al. "A new similarity measure between intuitionistic fuzzy sets and its application to pattern recognition," Abstract and Applied Analysis, pp. 1-11, 2014.

[12] S. M. Chen, S. H. Cheng, T. C. et al. "A novel similarity measure between intuitionistic fuzzy sets based on transformation techniques with applications to pattern recognition," Information sciences, vol. 343, pp. 15-40, 2016.

[13] C. M. Hwang, M. S. Yang and W. L. Hung, "New Similarity Measures of Intuitionistic Fuzzy Sets Based on the Jaccard Index with Its Application to Clustering," International Journal of Intelligent Systems, pp. 1672-1688, 2018.

[14] Q. Jiang, X Jin , Shin-Jye L , et al. "A new similarity/distance measure between intuitionistic fuzzy sets based on the transformed isosceles triangles and its applications to pattern recognition," Expert Systems with Applications, vol. 116, pp. 439-453, 2018.

[15] P. Burillo and H. Bustince, "Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets," Fuzzy Sets and Systems, vol. 78, pp. 305-316, 1996.

[16] E. Szmidt and J, Kacprzyk, "Entropy for intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 118, pp. 467-477, 2001.

[17] W. Zeng and H. Li, "Relationship between similarity measure and entropy of interval valued fuzzy sets," Fuzzy Sets and Systems, vol. 157, pp. 1477-1484, 2006.

[18] W. L. Hung and M. S. Yang, "Fuzzy entropy on intuitionistic fuzzy sets," International journal of intelligent systems, vol. 21, pp. 443-451, 2006.

[19] Q. S. Zhang and S. Y. Jiang, "A note on information entropy measuresfor vague sets and its applications," Information Sciences, vol. 178, pp. 1334-1342, 2008.

[20] J. Ye, "Two effective measures of intuitionistic fuzzy entropy," Computing, vol. 87, pp. 55-62, 2010.

[21] D. Y. Jiang and Y. X. Wang, "A new Entropy and its properties based on the improved axiomatic Definition of Intuitionistic Fuzzy Entropy," Mathematical Problems in Engineering, vol. 12, pp. 1-6, 2018.

[22] K. Atanassov, "Intuitionistic Fuzzy Sets," Fuzzy Sets and Systems, vol. 20, pp. 87-96, 1986.

[23] I. Iancu, "Intuitionistic Fuzzy Similarity Measures Based on Frank T-norms Family," Pattern Recognition Letter, vol. 42, pp. 128-136, 2014.

[24] G. Beliakov, M. Pagola and T. Wilkin, "Vector Valued Similarity Measures for Atanassov's Intuitionistic Fuzzy Sets," Information Sciences, vol. 280, pp. 352-367, 2014.

[25] C. Zhang and H. Fu, "Similarity Measures on Three Kinds of Fuzzy sets," Pattern Recognition Letter, vol. 27, pp. 1307-1317, 2006.

[26] H. M. Zhang, "Entropy for intuitionistic fuzzy sets based on distance and intuitionistic index," International Journal of Uncertainty Fuzziness and Knowledge-Based Systems, vol. 21, pp. 139-155, 2013.

[27] E. Szmidt and J. Kacprzyk, "Some problems with entropy measures for the Atanassov intuitionistic fuzzy sets," Lecture Notes in Computer Science, pp. 291-297, 2007.

[28] M. M. Xia and Z. S. Xu, "Entropy/cross entropy-based group decision making under intuitionistic fuzzy environment," Information Fusion, vol. 13, pp. 31-47, 2012.

[29] X. Liang, C. Wei and M. Xia, "New entropy, similarity measure of intuitionistic fuzzy sets and their applications in group decision making," International Journal of Computational Intelligence Systems, vol. 6, pp. 987-1001, 2013.

[30] W. L. Hung and M. S. Yang, "Similarity Measures of Intuitionistic Fuzzy Sets Based on Hausdorff Distance," Pattern Recognition Letters, vol. 25, pp. 1603-1611, 2004.

[31] W. L. Hung and M. S. Yang, "On Similarity Measures between Intuitionistic Fuzzy Sets," International Journal of Intelligent systems, vol. 23, pp. 364-38, 2010.

[32] Z. Z. Liang, and P. F. Shi, "Similarity Measures on Intuitionistic Fuzzy Sets," Pattern Recognition Letters, vol. 24, pp. 2687-2693, 2003.

[33] Y. F. Song, X. D. Wang and H. L. Zhang, "A Distance Measure between Intuitionistic Fuzzy Belief Functions," Knowledge-Based Systems, vol. 86, pp. 288-298, 2015.

[34] W. Wang and X. Xin, "Distance Measure between Intuitionistic Fuzzy Sets," Pattern Recognition Letters, vol. 26, pp. 2063-2069, 2005.

[35] H. W. Liu, "New Similarity Measures between Intuitionistic Fuzzy Sets and between Elements,". Mathematical and Computer Modelling, vol. 42, pp. 61-70, 2005.

[36] I. K. Vlachos, G. D. Sergiadis, "Inner product based entropy in the intuitionistic fuzzy setting," International journal of uncertainty fuzziness and knowledge-based systems, vol. 14, pp. 351-366, 2006.

[37] Z. S. Xu and M. M. Xia, "Hesitant fuzzy entropy and cross-entropy and their use in multiattribute decision-making," International Journal of Intelligent Systems, vol. 27, pp. 799-822, 2012.

[38] Y. J. Zhu, B. Li and F. Deng, "A new definition and formula of entropy for intuitionistic fuzzy set," Journal of Intelligent & Fuzzy Systems, vol. 30, pp. 3057-30669, 2016.

[39] H. Zhang, W. Zhang and C. Mei, "Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure," Knowledge-Based Systems, vol. 22, pp. 449-454, 2009

[40] J. H. Yuan and X. G. Luo, "Approach for multi-attribute decision making based on novel intuitionistic fuzzy entropy and evidential reasoning," Computers & Industrial Engineering, vol. 135, pp. 643–654, 2019.

[41] G. Wang, J. Zhang, Y. F. Song and Q. Li, "An Entropy-Based Knowledge Measure for Atanassov's Intuitionistic Fuzzy Sets and Its Application to Multiple Attribute Decision Making," Entropy, vol. 20, pp. 981–997, 2018

[42] G. W. Wei, "Maximizing deviation method for multiple attribute decision making in intuitionistic fuzzy setting," Knowledge-Based Systems, vol. 21, pp. 833–836, 2008.

[43] Y. Y. Jia, "Comprehensive Evaluation of University Library Reader Satisfaction Based on Intuitionistic Fuzzy Information," Journal of Information, vol. 7,pp. 142-145.

[44] N. N. Sun, "Fuzzy entropy of intuitionistic fuzzy sets and its application in multi-attribute decision-making," Yunnan University of Finance and Economics, 2013.

[45] B. Pekala, U. Bentkowska, J. Fernandez and H. Bustince, "Equivalence measures for Atanassov intuitionistic fuzzy setting used to algorithm of image processing," IEEE World Congress on Computational IntelligenceI, pp. 1-6, 2019.

[46] S. Cherif, N. Baklouti, V. Snasel and A. M. Alimi, "New Fuzzy Similarity Measures: from Intuitionistic to Type-2 Fuzzy Sets," IEEE World Congress on Computational IntelligenceI, pp. 1-6, 2017.

[47] L. Baccour and A. M. Alimi, "Distance Measures for Intuitionistic Fuzzy Sets and Interval Valued Intuitionistic Fuzzy Sets," IEEE World Congress on Computational IntelligenceI, pp. 1-6, 2019.

[48] B. Fares, L. Baccour and A. M. Alimi, "Distance Measures between Interval Valued Intuitionistic Fuzzy Sets and Application in Multi-Criteria Decision Making," IEEE World Congress on Computational IntelligenceI, pp. 1-6, 2019.