# The use of concave and convex functions to optimize the feed-rate of numerically controlled machine tools

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***Abstract*—Optimality and computational efficiency are two very desirable but also competitive attributes of optimal feed planning. A well-designed algorithm can vastly increase machining productivity by reducing tool positioning time subject to limits of the machine tool. The nonlinear optimization problem aims to achieve the highest possible feed along the tool path, while limiting the speed of the actuator level, acceleration and Jerk profiles. Methods proposed in the literature either use rather complex nonlinear optimization solvers, such as Sequential Quadratic Programming, use iterative heuristics that extends computation time, or use conventional assumptions that reduce computation time but lead to slower tool motion.**

**The problem of optimal feed-rate planning along a curved tool path for multi-axis CNC machines with a Jerk limit for each axis is addressed. However, the use of Jerk (rate of change of acceleration) into the feed-rate scheduling problem causes generating both, computationally efficient solutions and simultaneously guaranteeing optimality, is a challenging problem. To solve this problem, we propose the approach of modifying a Jerk parameter, through the use of a pair of convex and concave functions, including aggregation functions. In this technique, the suggested algorithm indicates the points at which the increasing convex/concave function and the adequate dual decreasing function modeling the Jerk parameter is used.**

***Index Terms*—CNC machine tools; Feed-rate; Jerk parameter (Jerk); Convex/Concave functions; Aggregation functions.**

## I. Introduction

NC multi-task machines belong to the most modern group and are designed mainly as machining centers. Structures of this type arose as a result of the demand for machining using several numerical axes controlled simultaneously and achieving very high accuracy of machined parts.

Multi-axis machine tools provide excellent ability to achieve excellent machined surface quality and better tool motion flexibility for complex parts machining. In the last decade, many researchers had focused their research on the parametric interpolation [1], kinematics and geometrical error modeling [2], [3], efficient tool-path generation [4], real time contour error estimation [5] and constrained feed-rate scheduling [6], [7]. Among them, the feed-rate scheduling, which serves as a kernel factor that determines the machining productivity and dimensional accuracy, is recently gaining significant attention from the NC manufacturers industry and research community. To achieve the maximum productivity, the tool axis traverse rates and spindle speeds are typically required to be as large as possible [8].

The feed-rate optimization along curved tool paths is an important problem in CNC machining. In the feed-rate planning, the acceleration on each axis of the machine must be constrained, because the torque capabilities of the axes drives are limited. Therefore, the problem is how to identify the feed-rate along a given path such that the machining time is minimal without exceeding the capabilities of the actuators. The problem of the optimal feed-rate planning along a given parametric tool path has received a significant amount of attention in the CNC machining literature due to its ability to increase the productivity of CNC machining by using the full ability of the machines. In [9] and [10] algorithms to determine the minimum time motion for a robot manipulator along a specific path (at least a smooth curve) with acceleration bounds on $X, Y, Z$ axes are proposed. Authors in [11], [12] and [13] considered the feed-rate for CNC machining, also with acceleration bounds on $X, Y, Z$ axes, and gave a piecewise analytic expression of the optimal velocity planning function. However, the acceleration profile obtained with the above methods has discontinuities, since the acceleration may change from the maximum to the minimum instantly. These discontinuities correspond to step changes in the force output demanded of the drive, cause of vibrations and some contouring errors. One method to minimize the residual vibration is introducing Jerk constraints along each axis to the original problem. Then we will obtain a feed-rate planning with continuous acceleration. Jerk ramps the acceleration to smooth the velocity. As a parameter belongs to the aspect of achieving better performance, one that may work with other performance, enhancing control features.

Several techniques, like the classical look-ahead algorithm [14], [15], sine-curve [16] and s-shape curve [17], [18], have been developed to generate a Jerk-limited feed profile. However, these methods suffer from lacking of sufficient consideration of the "hard limits" of machine tool driving servo actuators physical capability, typically due to an over-simplification of the assumptions in the earlier study efforts.

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This could lead into early tool wear or even possible damage to the mechanical structure of the machine too.
To handle this issue, dichotomy iteration algorithm [19], bidirectional scanning algorithm [20] and curve evolution algorithm [21] have been presented to solve the problems of feed-rate interpolation with confined axis Jerk constraints. However, solving such constrained nonlinear problem is generally computation intensive and time-consuming. Different methods, like the Sequential Quadratic Programming algorithm [22] and the Greedy algorithm [23], have been used to realize the optimality. Moreover, in the paper [24] authors presented a quintic spline trajectory generation algorithm that produces continuous position, velocity, and acceleration profiles with confined tangential acceleration and Jerk. In [25] an online method to obtain smooth, Jerk-bounded trajectories with fifth order polynomials for industrial applications have been developed and implemented. This method is near time optimal with confined tangential Jerk and acceleration. The paper [18] proposed a dynamics-based interpolator with real-time look-ahead algorithm to generate a smooth and tangential Jerk-confined acceleration feed-rate profile. Similar considerations were made in [26] for the NURBS curves.

In this paper, a new method of the construction of Jerk by use of convex and concave functions to feed-rate optimization is presented for multi-axis machining satisfying both the servo drive physical limits and the tool motion constraints. The remainder of this paper is organized as follows. Section II presents information on the principles of CNC machine operation, including the impact on its performance different kind of parameters. Section III formulates the problem of feed-rate optimization using the Jerk parameter. The necessary concepts related to convexity and concavity for the real functions and their use in the proposed approach along with the proper implementation are shown in Section IV. Experiments and results are discussed in Section V, and Section VI provides the concluding remarks.

## II. The use of CNC machines

A CNC machine is an automatic, computer controlled machine. The microcomputer connected to the system is to intercept the machine's regulation and control functions. With five axes, the CNC machine operates in five directions in which the spindle can move. In comparison to conventional machines having only three axes, the tool allows to set different angles to the surface, so that the same treatment is faster and more accurate. What is more, the service life of machines is extended.

Computer-controlled devices, such as machines with five axes, contain a microcomputer that can be variously programmed. Information on activity, processing parameters, geometry or the sequence of motions are alphanumerically coded. The use of this type of solution enables the production of various shapes in a quick and precise manner.

In NC, there are three basic types of motion control system:

- Point-to-point (PTP) - to position the tool form one point to another within coordinate system. Each tool axis is controlled independently, and therefore; the programmed motion always is in rapid travers. Once the tool reaches the desired location, the machining operation is performed at that position (machining can only take place after positioning is completed).
- Straight-cut control systems - capable of moving the cutting tool parallel to one of the major axes at a controlled rate suitable for machining. This feed control is shared by all the programmable axes of the NC machine;
- Contouring - the most complex and flexible type of machine tool control. It is capable of performing both PTP and straight-cut operations. In addition, the distinguishing feature of contouring NC systems is their capacity for simultaneous control of more than one axis movement of the machine tool. Contouring system generates a continuously controlled tool path by the capability of computing the points of the path (interpolating). For this reason, it is also called continuous-path NC system.

### A. Linear and rotary axes

The work of CNC machines can be visualised in the Cartesian coordinate system, where the control takes place. With its help it is possible to indicate the positions of the spindle and the processed object. The coordinates have to be parallel to the guides of the machine tool [27].

The CNC machine tools with five axes are equipped with three linear axes, indicated with the letters X, Y, Z and with two of the three rotational axes, which are associated with the linear ones and marked with the letters A, B, C (see Fig. 1). The rotational axes can be used for one or both of the following orientation (a) of the work-part to present different surfaces for machining; (b) of the tool or work-head at some angle relative to the part. These additional axes permit machining of complex work-part geometries.

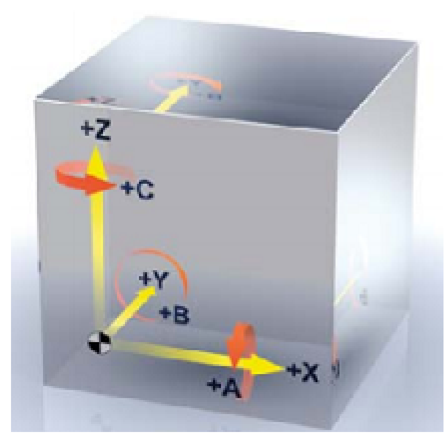


Fig. 1. The visualisation of the linear axes X, Y, Z and the rotational A, B, C [27].

The axis A is connected with the linear axis X and enables its rotation, the axes B is connected with linear axis Y and enables its rotation, and finally axis C makes it possible to rotate around the linear axis Z. However, most of CNC systems do not require all six axes, only the five most common. The simultaneous control of five axes enables creating more sophisticated shapes and increases the limits of usage of the CNC machine tools. As a result, the execution time of a given element is shorter, its accuracy is greater, and fewer people are needed to the control of the machine, since the processed object is not transferred between the machines.

Different combinations can be synthesized to configure the five-axis machine tool by changing the location and order of the rotary drives. In [28] there were classified the configurations of five-axis machine tool as three common types: two rotary drives on the table; two rotary drives are attached to the spindle head; and a hybrid combination between the spindle and table.

Figure 2 presents the solution for the head-head type in which the table is a processed object and does not move in this case.

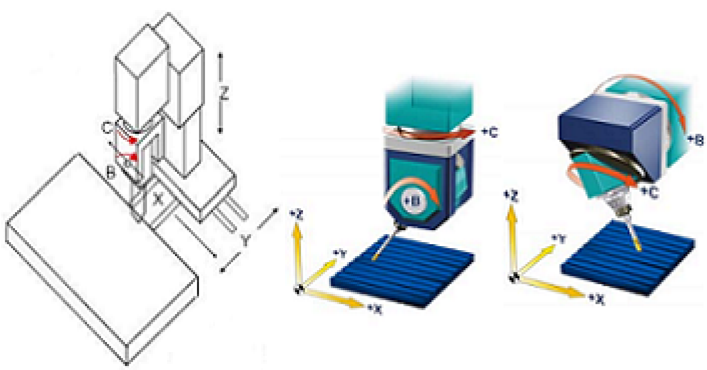


Fig. 2. The kinematic solution of head-head type in CNC machine tools with five axes [29], [27].

For the table-table type, the tool does not rotate, but the table around the C and B axes (see Fig. 3).

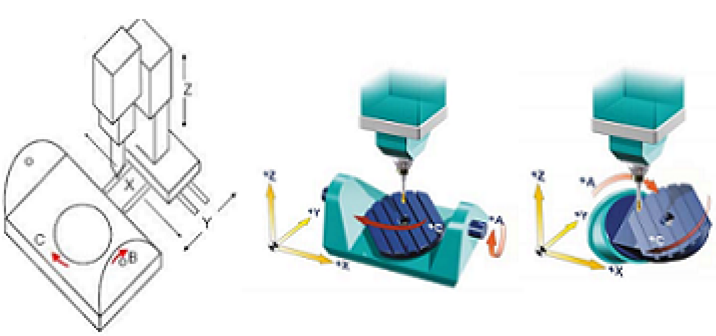


Fig. 3. The kinematic solution of table-table type in CNC machine tools with five axes [29], [27].

In head-table type the first rotate axis is in the head and the second is in the table (see Fig. 4).

Some of the CNC machine tools are of the kinematic type, where the rotary axis is inclined [29]. The solutions of such cases are presented in Fig. 5.

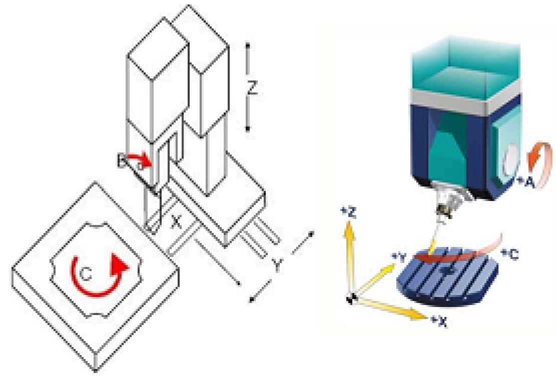


Fig. 4. The kinematic solution of head-table type in CNC machine tools with five axes [29], [27].

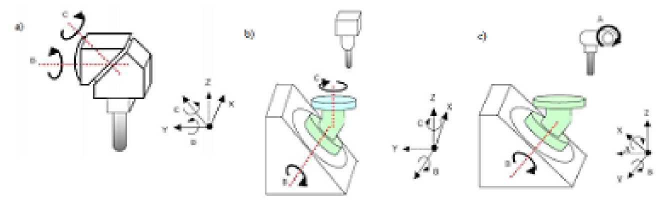

Fig. 5. The kinematic solution in CNC machine tools with an inclined rotary axis (a) head-head type (b) table-table type (c) head-table type [29], [27].

### *B. The influence of the Jerk parameter on the work of the machine*

Modern CNCs do not directly take advantage of a dynamical model of the machine axes, but the reference trajectories are assumed to be made physically achievable by the means of path planning stage. From the available parameters, it is known that the maximum Jerk value (per axis) can limit the oscillatory behaviour of the load. Indeed, the Jerk value represents the rate of change of acceleration, and for this reason it makes it possible to act on the smoothness degree of the movement.

The aim of Jerk is to optimize the speed of the spindle's acceleration. It means that the movement of the tool from a given point to another point is smoother and softer. With this feature, the machine tool and all its elements are protected, which improves the quality of produced goods and prolongs the usage of the machine. However, there are some disadvantages such as the longer processing time for an item [27].

The Jerk parameter $j$ can be calculated with the formula:

$$\vec{j}(t) = \frac{\mathrm{d}a}{\mathrm{d}t} = \dot{\vec{a}}(t) = \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = \ddot{\vec{v}}(t) = \frac{\mathrm{d}^3 r}{\mathrm{d}t^3} = \dddot{\vec{r}}(t), \quad (1)$$

where:

- a- acceleration,
- v- velocity (often used as speed),
- r- movement (relocation),
- t- time.

## III. The formulation of the research problem

As mentioned earlier, the key issue in research on CNC machine tools is the combination of several seemingly contradictory parameters, namely the time of performing the task of machining the element, the smoothness of the task, which affects the service life of the machine components (servomechanisms of particular linear and rotary axes etc.) and precision, which should satisfy the manufacturer.

To ensure the manufacturer the appropriate quality, recurrence and the time of production, a rather sophisticated set of parameters is used, which is associated with the kinematics of the machine, especially the kinematics of particular linear and rotary axes. Depending on the type of controller used in the machine, there are dozens of parameters for each axis that are strictly connected to each other and regardless of the axis, whether linear or rotary, affect each other. An appropriate or less accurate choice of all the parameters results in greater or smaller accuracy of performing a given element.

The maximum speed of particular axis, the maximum acceleration of the axes and Jerk- a parameter characterizing the rate of change of acceleration given for each linear and rotary axis are specified. The last parameter is usually selected from values or shapes previously implemented by the manufacturer.

Currently, there are four Jerk shapes to choose from: linear (constant value), trapezoidal, triangular and curve (nonlinear) selected experimentally by the controller manufacturer (see e.g. Figs. 6-7).

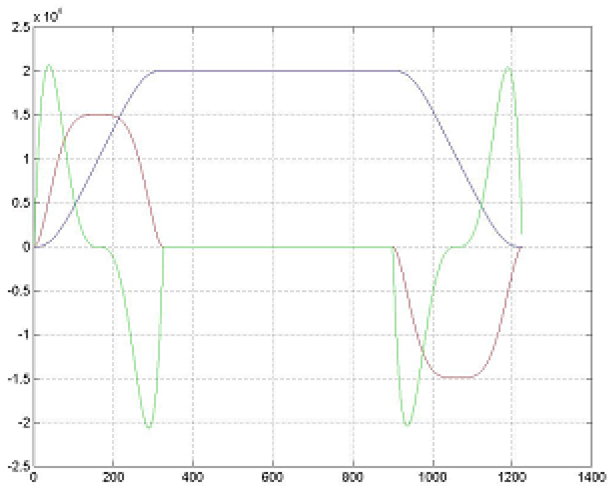

Fig. 6. Accelerating and braking CNC machines- nonlinear Jerk.

The most common move profiles for linear motion systems are trapezoidal and triangular. In a trapezoidal move profile, the system accelerates from zero to its maximum speed, travels at that speed for a specified time (or distance), and then decelerates to zero. Conversely, the triangular move profile accelerates from zero to maximum speed and then immediately decelerates back to zero, with no constant velocity (i.e. all the move time is spent accelerating or decelerating).

But in reality, neither of these move profiles is particularly ideal for motion systems - especially those that require smooth travel, high positioning accuracy, or stability at the end of the move. The reason is that the acceleration and deceleration process leads to a jerk (a yank), which can not be confused with the Jerk parameter (Jerk).

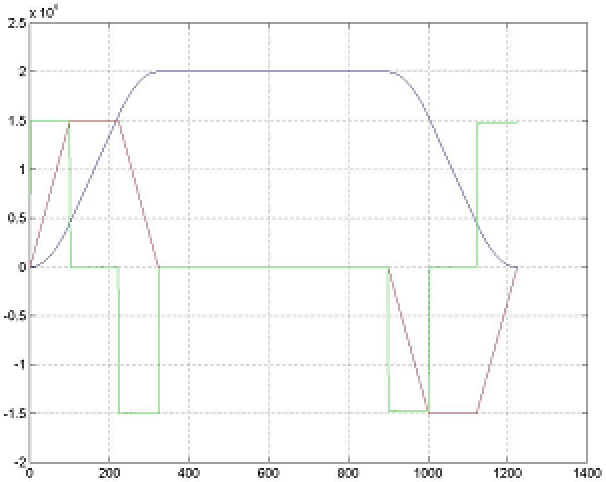

Fig. 7. Accelerating and braking CNC machines- trapezoidal Jerk.

Just as acceleration is the rate of change of velocity, Jerk is the rate of change of acceleration. In other words, Jerk is the rate at which acceleration is increasing or decreasing. Jerk is generally undesirable because it creates a rapid movement. In industrial applications such as machine tools or dispensing systems, a rapid change in acceleration, i.e. jerk, causes the system to vibrate. The higher the jerk, the stronger the vibrations. Moreover, vibrations decrease positioning accuracy while increasing settling time.

The way to avoid jerk is to reduce the rate of acceleration or deceleration. In motion control systems, this is done by using an S-curve motion profile [30], instead of the trapezoidal profile. In a trapezoidal move profile, acceleration occurs instantly and Jerk is infinite. To reduce the amount of jerk generated during the move, the transitions at the beginning and end of acceleration and deceleration are smoothed into an "S" shape. The resulting profile is referred to an S-curve move profile.

The trade-off of using an S-curve versus a trapezoidal move profile is that the overall time for the move is longer with an S-curve profile. The reason is that ramping acceleration (and deceleration) takes longer than the instantaneous acceleration of a trapezoidal move. However, the time advantage gained by using a trapezoidal move profile may be negated by a longer settling time, due to vibrations induced by high levels of jerk. And because jerk puts extensive strain on mechanical components, even if a trapezoidal move is used as the basis, some amount of smoothing is typically applied to the acceleration and deceleration phases, making the move profile more S-shaped.

Choosing one of Jerk's features causes the machine to work. It accelerates faster or slower and tends to achieve a certain speed in every line of G-code (also RS-274, the most widely used numerical control programming language (NC)). However, the machine is not always able to reach the feed rate specified and not at every point in the G-code. The main and impassable restriction is that the machine in any time of performing the program (G-Code) cannot exceed the given feed-rate, but it can move with a lower speed than specified.

The tool may not always give a specific geometric shape. This inaccuracy occurs due to the geometric complexity of the manufactured object and the speed at which the head of machine (spindle) moves. It is assumed that the tool has been properly selected for the given processed item.

Moreover, return to the machine's reference position (reversion or return) is a very complicated and demanding operation for a CNC machine, in particular for machine servo-drives, since the selected axis must stop with a predetermined dynamics at a point specified in the currently processed G-cod line and start in the opposite direction also with the dynamics previously set. The phenomenon of the reversion on one or many axes occurs very often during the normal industrial operation of the CNC machine. However, it is often hidden because the shape of the element performed does not seem to require a turn, but after the tool implementation and its dimensions i.e. length, diameter and operating mode, it occurs that one or more axes must stop and turn back. This process requires shifting the machine operating point from the spindle axis to the end of the given tool, or more precisely the conversion of coordinates generated in any CAM type program and shifting them by the given tool parameters. Typically, such a process is carried out many times during the execution of the selected element on the CNC machine, because the machine performs certain technological operations with various tools. The number of these exchanges depends on the type of machine, exactly the type of so-called magazine tool, tool tray and degree of product technological complexity.

As can be deduced from the above description, currently used Jerk optimization methods are still not good enough, as they have both advantages and disadvantages, in particular when it concerns feed with the reversion. Our proposal (referring to the feed-rate profiles with S-curve acc/dec) to use pairs of concave-convex functions meets the expectations of their further improvement, in addition, importantly, taking into account the recurrence.

## IV. The proposed methodology

As announced, our considerations will include pairs of convex and concave functions.

**Definition 1.** Let $X$ be a convex set in a real vector space and let $f : X \to \mathbb{R}$ be a function.
$f$ is called convex if

$$\forall x_1, x_2 \in X, \forall t \in [0,1] :$$
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2),$$

$f$ is called concave if

$$\forall x_1, x_2 \in X, \forall t \in [0,1] :$$
$$f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2).$$

Based on the papers [31], [32], we know that we can aggregate convex or concave functions and, using a weighted average we preserve convexity or concavity, respectively.

**Theorem 1.** *If $w_1, \ldots, w_n \geq 0$ and $f_1, \ldots, f_n$ are all convex (concave), then so is*

$$w_1 f_1 + \cdots + w_n f_n.$$

*In particular, the sum of two convex (concave) functions is convex (concave).*

In order to optimize the machine's operation, i.e. to maximize its work speed while avoiding jerks that shorten the proper functioning of the machine, we will use the following algorithm for the consecutive assumptions for each axis:

$$\mathbf{V}(x_k) \to \max \quad for \;\; all \;\; x_k \in \{z_n, z_n + 1\}$$

and

$$\mathbf{V}(x_k) \leq \max_{V(x_k)},$$

where $\max_{V(x_k)}$ means the maximum speed (velocity) that the machine can reach at a given point $x_k$ and $V(x_k)$ is the current predicted speed (velocity), and $\mathbf{V}(x_k)$ is the new increased speed (velocity) in the algorithm.
The limitations of the maximum permissible values also apply acceleration and Jerk in all axes of the machine.
An important element of the propose algorithm in this paper is determining a point that changes the assignment of an increasing convex function to a decreasing (dual) convex function. The idea of the algorithm is based on the use of the convexity or concavity function to assign it to the Jerk parameter to model the feed-rate. Accordingly, we analyze each pair of neighboring geometric points and beginning from the starting point, we assign to successive Jerk intermediate points with a given function of "increasing" convexity or concavity (increasing the speed). Then, depending on the speed set for the environment of the next geometric point and the new speed generated using the above mentioned function, we set the point (mentioned earlier) at which we change the function characterizing Jerk to dual, respectively changing the convexity or concavity.

## V. Results of the preliminary experiments

In order to check the validity and usability of the proposed Jerk modification method, preliminary tests were carried out on a five-axis CNC machine manufactured by Fanum for the case in which it performs a recurrence. The Fanum company is a Polish producer and constructor of various types of CNC machines intended for woodworking, various types of composites or nesting. The LAMBDA ST machining center is a large-size machine with a classic design - a movable portal moving along the work table. This machine is distinguished from other similar constructions available on the market in that the portal not only has servo drives with gears on both sides, but also has a double rail guide system.

The guides are placed one above the other at a distance ensuring very high stability of operation. The dimensions of the working fields are X axis 2100 mm, Y axis 3100 mm, Z axis 1000 mm, rotary axis A $-115^0/+115^0$, axis C $-360^0/+360^0$ (see Fig. 8).

**Algorithm 1:** Jerk Modeling Algorithm

Selecting the machine's operating mode.

**Input Data:**

• characteristics of $t$ geometrical points - change of the geometry (location $X, Y, Z$, the maximum permissible speed $\max_{V(z_n)}$, the actual suggested speed $V(z_n)$),

$$\{z_n\}, \quad n = 1, ..., t, \; t \in N$$

• characteristics of $m$ intermediate points between each par of geometric points (location $X, Y, Z$, the maximum permissible speed $\max_{V(x_k)}$, the actual suggested speed $V(x_k)$),

$$\{x_k\}, \quad k = 1, ..., m, \; m \in N.$$

**Result:** characteristics of the intermediate points between the geometric points machine work optimization: achieving the maximum speed at the next geometric point with simultaneous smooth transition of the machine (without jerks).

**Step 1.** Analysis of m subsequent intermediate points $\{x_1, ..., x_m\}$ in search of the next geometric point $z_t$

**For** $k = 1, ..., m$ we calculate

$$d := |V(x_k) - V(x_{k+1})|.$$

**If**

$$|V(x_k) - V(x_{k+1})| > \mathcal{M},$$

//expected high speed change

**then**

$$x_{k+1} = z_t.$$

**Step 2.** Starting from the points $x_1$, we determine Jerk using the convex increasing functions $f_k$, which we aggregate (see //* below) and we increase the speed of the machine to the point $x_l$, starting from which the dual decreasing functions convex $f_k^d$ works to $z_t$.

The method for selecting the Jerk function change point is as follows:

**For** $h = 1...k$

**if** $\mathbf{V}(x_h) = max_{V(x_h)}$ **then**

$x_h := x_l$

**else**

**if** $|\mathbf{V}(x_k) - V(x_h)| = d$ **then**

$x_h := x_l$

**else**

$h := h + 1;$

**end**

**end**

**Step 3.** After crossing the point $z_t$, we again determine Jerk using the convex increasing function and increase the speed of the machine, i.e. we return to Step 1 until the geometric points of the figure are exhausted.

//* The selection of convex functions $f$ increasing and decreasing $f^d$ in point 2 is done by applying to k functions $f_k$ and $f_k^d$ aggregated by means of a weighted average function, so that the highest precision reaches the point $z_t$ at the highest possible speed, $k \in \{1, ..., 10\}$.

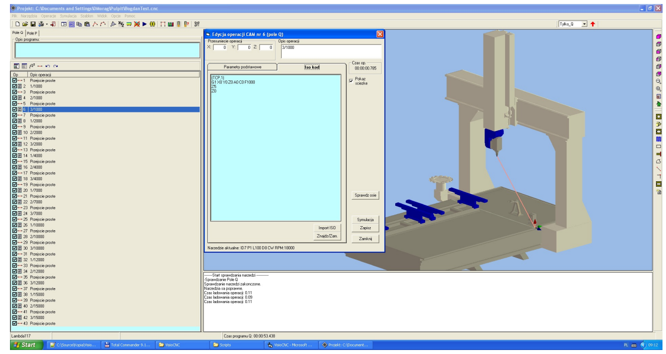

Fig. 8. Lambda ST machine on which tests were performed; Fanum company https://www.fanum.pl/en/.

The first stage of the tests was to determine parameters related to the machine dynamics, i.e. acceleration and maximum speeds for individual axes. To simplify calculations during tests, the machine head only moved in the one Z axis. This axis was chosen because it is usually the slowest one (with the lowest maximum speed value) compared to other linear X and Y axes. Parameters adopted for experiments are: maximum axis speed $20000\frac{mm}{min}$ and acceleration $30000\frac{mm}{s^2}$.

The second part of the tests consisted of the so-called zeroing the machine, i.e. setting the absolute zero for all machine axes and choosing the tool (its geometrical and working parameters) that will perform the tasks entered into the machine controller, and determining the coordinates of the point that will be the starting point for working operations.

The experiments were carried out for G-Code in the case in which the machine makes a turn (very important issue from the point of view of machine operation), i.e. after reaching the given geometric point, it returns to the starting point, i.e. to absolute zero (during the test 239,523 mm for Z axis). In the presented case, the machine moves 5 mm in a straight line, brakes and returns to the starting point each time for different feed-rates. The tests were carried out without the functions modifying the Jerk parameter (with the fixed value) and also using functions announced in Section IV (Algorithm 1 and presented on the left). The results obtained are summarized in Table I, where $\mathbf{Z_F}$ means ideal, standard geometric coordinates of the point after moving by a given distance; $\mathbf{Z_G}$ real coordinates of the point reached by the machine without using an algorithm; $\mathbf{\Delta Z_{GF}}$ the absolute value of the difference between $Z_G$ and $Z_F$; $\mathbf{Z_H}$ - the actual coordinates of the point reached by the machine using the algorithm; $\mathbf{\Delta Z_{HF}}$- the absolute value of the difference between $Z_H$ and $Z_F$.

The results presented in the above table show that after applying the proposed modifying function of Jerk, the error between $\Delta Z_{GF}$ and $\Delta Z_{HF}$ was reduced more than a half on average.

The following families of convex functions have been used for pre-testing due to the different values of the $\alpha$ and $\beta$ parameters:

$$f_k(x) = d * \beta * \alpha^x,$$

TABLE I
THE RESULTS OF EXPERIMENTS.

| speed | $\mathbf{Z_F}$ | $\mathbf{Z_G}$ | $\mathbf{\Delta Z_{GF}}$ | $\mathbf{\Delta t_{GF}}$ | $\mathbf{Z_H}$ | $\mathbf{\Delta Z_{HF}}$ | $\mathbf{\Delta t_{HF}}$ |
|---|---|---|---|---|---|---|---|
| mm/min | mm | mm | mm | s | mm | mm | s |
| 1000 | 239,523 | 239,512 | 0,011 | 0,34 | 239,519 | 0,004 | 0,38 |
| 2000 | 239,523 | 239,512 | 0,011 | 0,21 | 239,519 | 0,004 | 0,26 |
| 4000 | 239,523 | 239,513 | 0,01 | 0,17 | 239,519 | 0,004 | 0,23 |
| 7000 | 239,523 | 239,515 | 0,008 | 0,17 | 239,519 | 0,004 | 0,23 |
| 10000 | 239,523 | 239,515 | 0,008 | 0,17 | 239,519 | 0,004 | 0,23 |
| 12000 | 239,523 | 239,503 | 0,02 | 0,18 | 239,519 | 0,004 | 0,23 |
| 15000 | 239,523 | 239,51 | 0,013 | 0,17 | 239,519 | 0,004 | 0,23 |
| 20000 | 239,523 | 239,505 | 0,018 | 0,17 | 239,519 | 0,004 | 0,23 |

$$f_k^d(x) = 2 * d * \beta * (\frac{1}{\alpha})^x,$$

where the values $\beta$ were randomly generated value from $[0, 1]$ for the individual $f_k$ for given $\alpha$ (presented results are made for $\alpha = 2$ for which we obtained the optimal result from ten randomly chosen values $\alpha > 1$), $d$ was designated in the Algorithm 1 and obtained k functions are aggregated by the arithmetic mean.

An analysis of the results obtained indicates that, the total feed time, for example, at a set speed of $1000 \frac{mm}{min}$ was $2 * 0.38 = 0.76 sec$. Thus, the machining time was slightly longer compared to the constant Jerk. However, despite this, the algorithm has a positive effect on other important factors, including increased machining accuracy, which at 5 mm is significantly difficult to achieve.

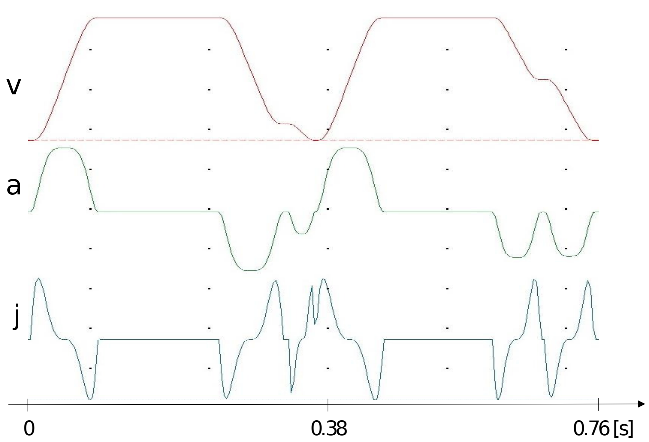


Fig. 9. Image from the oscilloscope of the absolute values of Jerk (j), acceleration (a) and velocity (v). The result of experiment based on the Algorithm 1 for a feed with the reversion at a distance of $2 * 5$ mm during 0.76 seconds.

Most importantly, experimental results visualized in Fig. 9 reveal the effectiveness of this approach in improving two aspects of the machine's work, i.e. both the increase in the accuracy of hitting the geometric points (very close to the assumed) and the machine's efficiency by increasing the smoothness of work while maintaining the optimal speed.

## VI. CONCLUSIONS

This paper presents the CNC feed-rate optimization algorithm based on the modification of the Jerk parameter using a pair of concave-convex functions. Several experiments were achieved using the paid application CAD/CAM software. The proposed method handles velocity, acceleration and Jerk constraints imposed by the machines feed drives. The use of the proposed methodology provides both improvements in the accuracy of hitting the geometric points and the machine's performance.

Considering the results obtained it should be noted that, they relate to cases in which the machine moves in a straight line along one linear axis, the Z axis in our case. Therefore, it is planned to perform further experiments for a more geometrically complicated research object and for cases in which the spindle moves along various trajectories involving all linear axes simultaneously. In addition, cases for A and C rotary axes will also be investigated.
Testing other pairs of convex-concave functions modifying the Jerk parameter is another important direction of further research. Hence, we will also make a comparative analysis both between the various functions proposed and other methods existing in the literature that effectively optimize the feed-rate and its quality.

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