General Interval-valued Grouping Functions

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***Abstract*—Grouping functions are aggregation functions used in decision making based on fuzzy preference relations in order to express the measure of the amount of evidence in favor of either of the two alternatives when performing pairwise comparisons. They have been also used as a disjunction operator in some important problems, such as image thresholding and the construction of a class of implication functions for the generation of fuzzy subsethood and entropy measures. Some generalizations of this concept were recently proposed, such as n-dimensional and general grouping functions, which allowed their application in n-dimensional problems, such as fuzzy community detection. Also the concept of interval-valued overlap functions was presented, in order to deal with the uncertainty when defining membership functions. The aim of this paper is to introduce the concepts of n-dimensional interval-valued grouping functions and general interval-valued grouping functions, studying representability, characterization and construction methods.**

***Index Terms*—n-dimensional grouping functions, general grouping functions, interval-valued grouping function**

I. Introduction

Overlap functions are aggregation functions [1] that are not required to be associative, introduced by Bustince et al. in [2] to measure the degree of overlapping between two classes or objects. Grouping functions, as the dual notion of overlap function, were introduced by Bustince et al. [3] in order to express the measure of the amount of evidence in favor of either of two alternatives when performing pairwise comparisons [4] in decision making based on fuzzy preference relations [5]. In the literature, one can also find the use of grouping functions as the disjunction operator in some important problems, such as image thresholding [6] and the construction of a class of implication functions for the generation of fuzzy subsethood and entropy measures [7].

Observe that grouping functions are defined as bivariate functions. Thus, they can only be used in problems that consider just two classes or objects, since they are not required to be associative. In order to overcome this limitation, the concept of n-dimensional grouping functions was introduced by Gómez et al. [8], with an application to fuzzy community detection, and, more recently, Santos et al. [9] defined general grouping functions by relaxing the boundary conditions of n-dimensional grouping functions.

Notice that the appropriate definition of the membership functions is a crucial aspect in any fuzzy system modelling [10]. Whenever there is uncertainty in this process, usually associated with the linguistic terms [11], one may face a complex problem. In the literature, a popular way to deal with this problem is applying interval-valued fuzzy sets (IVFSs) [12]–[15], since it was shown that they can easily model both vagueness (soft class boundaries) and uncertainty (with respect to the membership function), as discussed in [16]–[21]. For that reason, IVFSs have been successfully applied in different kinds of problems such as game theory [22], decision making [23], pest control [24] and classification [25]–[27].

Interval-valued grouping functions were introduced by Qiao and Hu [28], whose definition can only be applied in problems with two classes, which is a drawback when it is necessary to deal with n-dimensional problems, as we have discussed above. Then, the objective of this paper is to define n-dimensional and general interval-valued grouping functions, studying some properties, their representation and introducing some construction methods.

The paper is organized as follows. Section II presents preliminary concepts. In Sect. III, we define n-dimensional interval-valued grouping functions, studying their representability. In Sect. IV, we introduce general interval-valued grouping functions, studying their characterization, representation and construction methods. Section V is the Conclusion.

II. Preliminaries

In this section, we recall some concepts on interval mathematics [29], grouping [3], [7], [30]–[37], n-dimensional [8],

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general [9], and interval-valued [28] grouping functions.

A. Interval Mathematics

Let $L([0,1])$ be the set of all closed subintervals of the unit interval $[0,1]$, $L([0,1]) = \{[x_1,x_2] | 0 \leq x_1 \leq x_2 \leq 1\}$. Denote $\vec{x} = (x_1,\ldots,x_n) \in [0,1]^n$ and $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$. For any $X = [x_1,x_2]$, the left and right endpoints of X are denoted, respectively, by $\underline{X}$ and $\overline{X}$, so $\underline{X} = x_1$ and $\overline{X} = x_2$. Also, denote $\underline{\vec{X}} = (\underline{X_1},\ldots,\underline{X_n})$ and $\overline{\vec{X}} = (\overline{X_1},\ldots,\overline{X_n})$, for any $\vec{X} \in L([0,1])^n$.

In the literature there are many different definitions of partial orders in $L([0,1])$. In this paper, we use the product and the inclusion orders, defined for all $X,Y \in L([0,1])$, respectively, by $X \leq_{Pr} Y \Leftrightarrow \underline{X} \leq \underline{Y} \wedge \overline{X} \leq \overline{Y}$ and $X \subseteq Y \Leftrightarrow \underline{X} \geq \underline{Y} \wedge \overline{X} \leq \overline{Y}$.

An interval-valued function $F : L([0,1])^n \rightarrow L([0,1])$ is called $\leq_{Pr}$-increasing if it is increasing with respect to the product order $\leq_{Pr}$, that is, for all $\vec{X},\vec{Y} \in L([0,1])^n$ it holds that $X_1 \leq_{Pr} Y_1,\ldots,X_n \leq_{Pr} Y_n \Rightarrow F(\vec{X}) \leq_{Pr} F(\vec{Y})$. On the other hand, F is called inclusion monotonic if, for all $\vec{X},\vec{Y} \in L([0,1])^n$, it holds that $X_1 \subseteq Y_1,\ldots,X_n \subseteq Y_n \Rightarrow F(\vec{X}) \subseteq F(\vec{Y})$.

Given an interval-valued function $F : L([0,1])^n \rightarrow L([0,1])$, we can define the projections $F^-, F^+ : [0,1]^n \rightarrow [0,1]$ of F, respectively, by:

$$F^-(x_1,\ldots,x_n) = \underline{F([x_1,x_1],\ldots,[x_n,x_n])};$$
$$F^+(x_1,\ldots,x_n) = \overline{F([x_1,x_1],\ldots,[x_n,x_n])}.$$

Given two functions $f,g : [0,1]^n \rightarrow [0,1]$ such that $f \leq g$, we define the function $\widehat{f,g} : L([0,1])^n \rightarrow L([0,1])$, for all $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$, as

$$\widehat{f,g}(\vec{X}) = [f(\underline{\vec{X}}), g(\overline{\vec{X}})].$$

An interval-valued function F is said to be Moore-continuous if it is continuous with respect to the Moore metric [29] $d_M : L([0,1])^2 \rightarrow \mathbb{R}$, defined, for all $X,Y \in L([0,1])$, by:

$$d_M(X,Y) = \max(|\underline{X} - \underline{Y}|, |\overline{X} - \overline{Y}|).$$

The Moore-metric can be extended to $L([0,1])^n$ as follows:

$$d^n_M(\vec{X},\vec{Y}) = \sqrt{d_M(X_1,Y_1)^2 + \ldots + d_M(X_n,Y_n)^2}.$$

Definition 2.1: [17] An $\leq_{Pr}$-increasing interval-valued function $F : L([0,1])^n \rightarrow L([0,1])$ is said to be representable if there exists increasing functions $f,g : [0,1]^n \rightarrow [0,1]$ such that $f \leq g$ and $F = \widehat{f,g}$.

In the context of Definition 2.1, f and g are the *representatives* of the interval-valued function F. When $F = \widehat{f,f}$, we will denote simply as $\widehat{f}$.

Proposition 2.1: [18] An $\leq_{Pr}$-increasing interval-valued function $F : L([0,1])^n \rightarrow L([0,1])$ is representable if and only if F is inclusion monotonic.

Proposition 2.2: [28] Let $F : L([0,1])^n \rightarrow L([0,1])$ be an $\leq_{Pr}$-increasing interval-valued function. Then, F is inclusion monotonic if and only if $F = \widehat{F^-,F^+}$.

Proposition 2.3: [18] If an $\leq_{Pr}$-increasing interval-valued function $F : L([0,1])^n \rightarrow L([0,1])$ is inclusion monotonic, then it holds that $\underline{F(\vec{X})} = F^-(\underline{\vec{X}})$ and $\overline{F(\vec{X})} = F^+(\overline{\vec{X}})$, for all $\vec{X} \in L([0,1])^n$.

Some interval operations that are used in this paper are defined, for all $X,Y \in L([0,1])$ as:

Infimum: $\inf(X,Y) = [\min(\underline{X},\underline{Y}), \min(\overline{X},\overline{Y})]$;
Supremum: $\sup(X,Y) = [\max(\underline{X},\underline{Y}), \max(\overline{X},\overline{Y})]$;
Sum: $X + Y = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}]$;
Limited Sum: $X \dot{+} Y = [\min(1, \underline{X} + \underline{Y}), \min(1, \overline{X} + \overline{Y})]$;
Product: $X \cdot Y = [\underline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}]$;
Exponential: $X^p = [\underline{X}^p, \overline{X}^p]$, for any $p \in \mathbb{R}$;
Subtraction: $X - Y = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}]$;
Generalized Hukuhara Division:
$X \div_H Y = [\min(\underline{X}/\underline{Y}, \overline{X}/\overline{Y}), \max(\underline{X}/\underline{Y}, \overline{X}/\overline{Y})], \underline{Y} \neq 0.$

Remark 2.1: For any $X,Y \in L([0,1])$ such that $X \leq_{Pr} Y$ one has that $X \div_H Y \in L([0,1])$.

B. General Grouping Functions and Related Concepts

Definition 2.2: [1] An aggregation function is any function $A : [0,1]^n \rightarrow [0,1]$ that is increasing in each argument and satisfies $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.

Definition 2.3: [2], [30] An overlap function is any bivariate function $O : [0,1]^2 \rightarrow [0,1]$ that satisfies the following conditions, for all $x,y \in [0,1]$: (O1) O is commutative; (O2) $O(x,y) = 0$ if and only if $x = 0$ or $y = 0$; (O3) $O(x,y) = 1$ if and only if $x = y = 1$; (O4) O is increasing; (O5) O is continuous.

Definition 2.4: [3] A grouping function is any bivariate function $G : [0,1]^2 \rightarrow [0,1]$ that satisfies the following conditions, for all $x,y \in [0,1]$: (G1) G is commutative; (G2) $G(x,y) = 0$ if and only if $x = y = 0$; (G3) $G(x,y) = 1$ if and only if $x = 1$ or $y = 1$; (G4) G is increasing; (G5) G is continuous.

Definition 2.5: [28] A function $G\colon [0,1]^2 \rightarrow [0,1]$ is a 0-grouping function if and only if the condition (G2) in Def. 2.4 is stated as follows: ($G2'$) If $x = y = 0$ then $G(x,y) = 0$. Analogously, a function $G\colon [0,1]^2 \rightarrow [0,1]$ is a 1-grouping function if and only if (G3) in Def. 2.4 is stated as follows: ($G3'$) If $x = 1$ or $y = 1$ then $G(x,y) = 1$.

Definition 2.6: [8] An n-ary function $Gn\colon [0,1]^n \rightarrow [0,1]$ is called an n-dimensional grouping function if and only if the following conditions hold: (Gn1) Gn is commutative; (Gn2) $Gn(x) = 0$ if and only if $x_i = 0$, for all $i = 1,\ldots,n$; (Gn3) $Gn(x) = 1$ if and only if there exists $i \in \{1,\ldots,n\}$ with $x_i = 1$; (Gn4) Gn is increasing; (Gn5) Gn is continuous.

Definition 2.7: [38] A function $\mathcal{GO}\colon [0,1]^n \rightarrow [0,1]$ is said to be a general overlap function if it satisfies the following conditions, for all $\vec{x} \in [0,1]^n$: ($\mathcal{GO}1$) $\mathcal{GO}$ is commutative; ($\mathcal{GO}2$) If $\prod_{i=1}^n x_i = 0$ then $\mathcal{GO}(\vec{x}) = 0$; ($\mathcal{GO}3$) If $\prod_{i=1}^n x_i = 1$ then $\mathcal{GO}(\vec{x}) = 1$; ($\mathcal{GO}4$) $\mathcal{GO}$ is increasing; ($\mathcal{GO}5$) $\mathcal{GO}$ is continuous.

Definition 2.8: [9] A function $\mathcal{GG}\colon [0,1]^n \rightarrow [0,1]$ is called a general grouping function if the following conditions hold, for all $\vec{x} \in [0,1]^n$: ($\mathcal{GG}1$) $\mathcal{GG}$ is commutative; ($\mathcal{GG}2$) If $\sum_{i=1}^n x_i = 0$ then $\mathcal{GG}(\vec{x}) = 0$; ($\mathcal{GG}3$) If there exists $i \in \{1,\dots,n\}$ such that $x_i = 1$ then $\mathcal{GG}(\vec{x}) = 1$; ($\mathcal{GG}4$) $\mathcal{GG}$ is increasing; ($\mathcal{GG}5$) $\mathcal{GG}$ is continuous.

Proposition 2.4: If $Gn\colon [0,1]^n \to [0,1]$ is an n-dimensional grouping function, 0-grouping function or 1-grouping function, then Gn is also a general grouping function.

C. Interval-valued grouping functions

Definition 2.9: [39] An interval-valued function $IA : L([0,1])^n \to L([0,1])$ is said to be an n-dimensional interval-valued aggregation function if it is $\leq_{Pr}$-increasing function and satisfies $IA([0,0],\dots,[0,0]) = [0,0]$ and $IA([1,1],\dots,[1,1]) = [1,1]$.

Definition 2.10: [18] Let $IA : L([0,1])^n \to L([0,1])$ be an n-dimensional interval-valued aggregation function. Then, IA is said to be *conjunctive* if $IA(\vec{X}) \leq_{Pr} \inf(\vec{X})$ for any $\vec{X} \in L([0,1])^n$.

Definition 2.11: [18] Let $IA : L([0,1])^n \to L([0,1])$ be an n-dimensional interval-valued aggregation function. Then, IA is said to be *disjunctive* if $IA(\vec{X}) \geq_{Pr} \sup(\vec{X})$ for any $\vec{X} \in L([0,1])^n$.

Definition 2.12: [16], [28] An interval-valued overlap function (iv-overlap function, for short) is a mapping $IO : L([0,1])^2 \to L([0,1])$ which respects the following conditions: (IO1) IO is commutative; (IO2) $IO(X,Y) = [0,0]$ if and only if $X = [0,0]$ or $Y = [0,0]$; (IO3) $IO(X,Y) = [1,1]$ if and only if $X = Y = [1,1]$; (IO4) IO is $\leq_{Pr}$-increasing in the first component: $IO(Y,X) \leq_{Pr} IO(Z,X)$ when $Y \leq_{Pr} Z$; (IO5) IO is Moore continuous.

Note that, by (IO1) and (IO4), iv-overlap functions are also monotonic in the second component.

Definition 2.13: [18] An n-dimensional iv-overlap function is a mapping $IOn : L([0,1])^n \to L([0,1])$ which respects the following conditions: (IOn1) IOn is commutative; (IOn2) $IOn(X_1,\dots,X_n) = [0,0]$ if and only if $\prod_{i=1}^n X_i = [0,0]$; (IOn3) $IOn(X_1,\dots,X_n) = [1,1]$ if and only if $\prod_{i=1}^n X_i = [1,1]$; (IOn4) IOn is $\leq_{Pr}$-increasing in the first component: $IOn(X_1,\dots,X_n) \leq_{Pr} IOn(Y,X_2,\dots,X_n)$ when $X_1 \leq_{Pr} Y$; (IOn5) IOn is Moore continuous.

Definition 2.14: [28] An interval-valued grouping function (iv-grouping function, for short) is a mapping $IG : L([0,1])^2 \to L([0,1])$ which respects the following conditions: (IG1) IG is commutative; (IG2) $IG(X,Y) = [0,0]$ if and only if $X = Y = [0,0]$; (IG3) $IG(X,Y) = [1,1]$ if and only if $X = [1,1]$ or $Y = [1,1]$; (IG4) IG is $\leq_{Pr}$-increasing in the first component: $IO(Y,X) \leq_{Pr} IO(Z,X)$ when $Y \leq_{Pr} Z$; (IG5) IG is Moore continuous.

Theorem 2.1: [28] Let $IG : L([0,1])^2 \to L([0,1])$ be an inclusion monotonic interval-valued function. Then, IG is an iv-grouping function if and only if there exist 0-grouping function G_1 and 1-grouping function G_2 such that $G_1 \leq G_2$ and $IG = \widehat{G_1, G_2}$. Also, it holds that $G_1 = IG^-$ and $G_2 = IG^+$.

Considering two grouping functions G_1 and G_2 such that $G_1 \leq G_2$, the function $\widehat{G_1, G_2}$ is a representable iv-grouping function [16]. Although, not every representable iv-grouping function has grouping functions as its representatives, as one may conclude from Theorem 2.1.

III. N-DIMENSIONAL INTERVAL-VALUED GROUPING FUNCTIONS

The concepts of iv-overlap and iv-grouping functions were both developed as bivariate functions, which limits their applicability. In order to deal with the aggregation of more than two interval-valued inputs maintaining the characteristics of an iv-grouping function, in this section we introduce the concept of n-dimensional interval-valued grouping function, studying their representation and some other properties.

Definition 3.1: An n-dimensional interval-valued grouping function is a mapping $IGn : L([0,1])^n \to L([0,1])$ that satisfies the following conditions, for all $\vec{X} \in L([0,1])^n$: (IGn1) IGn is commutative; (IGn2) $IGn(\vec{X}) = [0,0]$ if and only if $X_1 = \ldots = X_n = [0,0]$; (IGn3) $IGn(\vec{X}) = [1,1]$ if and only if there exists $i \in \{1,\dots,n\}$ with $X_i = [1,1]$; (IGn4) IGn is $\leq Pr$-increasing in the first component: $IGn(X_1,\dots,X_n) \leq_{Pr} IGn(Y,X_2,\dots,X_n)$ when $X_1 \leq_{Pr} Y$; (IGn5) IGn is Moore continuous.

It is noteworthy that $\sup(\vec{X}) = [0,0]$ if and only if $X_1 = \ldots = X_n = [0,0]$ and that $\sup(\vec{X}) = [1,1]$ if and only if there exists $i \in \{1,\dots,n\}$ with $X_i = [1,1]$.

Example 3.1: Some examples of n-dimensional iv-grouping functions are:

1. $IGn_S(\vec{X}) = \sup(\vec{X})$;
2. $IGn_p(\vec{X}) = [1,1] - \prod_{i=1}^n([1,1] - X_i^p)$, for $p > 0$;

Theorem 3.1: Let Gn_1 and Gn_2 be n-dimensional grouping functions such that $Gn_1 \leq Gn_2$. Then, the function $\widehat{Gn_1, Gn_2}$ is an n-dimensional iv-grouping function.

Proof: Conditions (IGn1)-(IGn4) are immediately obtained. Condition (IGn5) is verified by Theorem 4.2 in [40] and the fact that its extension to n-dimensional functions is trivial, as stated in [41]. ∎

Proposition 3.1: Let $IOn : L([0,1])^n \to L([0,1])$ be an n-dimensional iv-overlap function. Then, the mapping $IGn_{IOn} : L([0,1])^n \to L([0,1])$ defined by

$$IGn_{IOn}(\vec{X}) = [1,1] - IOn([1,1] - X_1,\dots,[1,1] - X_n)$$

is an n-dimensional iv-grouping function.

Proof: Let us verify if IGn_{IOn} satisfies all the conditions of Definition 3.1. Conditions (IGn1), (IGn4) and (IGn5) are verified immediately as IOn is commutative, $\leq_{Pr}$-increasing and Moore-continuous. So, the remainder conditions to be verified are:

(IGn2) $IGn_{IOn}(\vec{X}) = [0,0]$

$$\Leftrightarrow [1,1] - IOn([1,1] - X_1,\dots,[1,1] - X_n) = [0,0]$$

$$\Leftrightarrow IOn([1,1] - X_1,\dots,[1,1] - X_n) = [1,1]$$

$$\Leftrightarrow [1,1] - X_1 = \ldots = [1,1] - X_n = [1,1]$$

$$\Leftrightarrow X_1 = \ldots = X_n = [0,0];$$

(IGn3) $IGn_{IOn}(\vec{X}) = [1,1]$

$$\Leftrightarrow [1,1] - IOn([1,1]-X_1,\ldots,[1,1]-X_n) = [1,1]$$

$$\Leftrightarrow IOn([1,1]-X_1,\ldots,[1,1]-X_n) = [0,0]$$

$$\Leftrightarrow \prod_{i=1}^{n}([1,1]-X_i) = [0,0]$$

$$\Leftrightarrow \sup(X_1,\ldots,X_n) = [1,1].$$

■

It immediately follows that:

Proposition 3.2: Let $IGn : L([0,1])^n \to L([0,1])$ be an n-dimensional iv-grouping function. Then, the mapping $IOn_{Gn} : L([0,1])^n \to L([0,1])$ defined by

$$IOn_{Gn}(\vec{X}) = [1,1] - IGn([1,1]-X_1,\ldots,[1,1]-X_n)$$

is an n-dimensional iv-overlap function.

Theorem 3.2: Let $Gn : [0,1]^n \to [0,1]$ be an n-dimensional grouping function. Then, $\widehat{Gn}$ is an n-dimensional iv-grouping function.

Proof: Suppose an n-dimensional grouping function Gn an let On_{Gn} be an n-dimensional overlap function defined, for all $\vec{x} \in [0,1]^n$, as $On_{Gn}(\vec{x}) = 1 - Gn(1-x_1,\ldots,1-x_n)$. Following Proposition 3.1 we have that $IGn_{\widehat{On_{Gn}}}$ is an n-dimensional iv-grouping function. Then, we will show that $\widehat{Gn} = IGn_{\widehat{On_{Gn}}}$:

$$\begin{aligned}&IGn_{\widehat{On_{Gn}}}(\vec{X}) =\\ &\quad [1,1] - \widehat{On_{Gn}}([1,1]-X_1,\ldots,[1,1]-X_n) =\\ &\quad [1,1] - [On_{Gn}(1-\overline{X_1},\ldots,1-\overline{X_n}),\\ &\quad On_{Gn}(1-\underline{X_1},\ldots,1-\underline{X_n})] =\\ &\quad [1,1] - [1-Gn(\overline{X_1},\ldots,\overline{X_n}), 1-Gn(\underline{X_1},\ldots,\underline{X_n})] =\\ &\quad [Gn(\underline{X_1},\ldots,\underline{X_n}), Gn(\overline{X_1},\ldots,\overline{X_n})] = \widehat{Gn}(\vec{X})\end{aligned}$$

■

The following result is the adaptation of Theorem 2.1 for n-dimensional iv-grouping functions.

Theorem 3.3: Let $IGn : L([0,1])^n \to L([0,1])$ be an inclusion monotonic interval-valued function. Then, IGn is a n-dimensional iv-grouping function if and only if there exist an n-dimensional 0-grouping function Gn_1 and an n-dimensional 1-grouping function Gn_2 such that $Gn_1 \leq Gn_2$ and $IGn = \widehat{Gn_1, Gn_2}$. Also, it holds that $Gn_1 = IGn^-$ and $Gn_2 = IGn^+$.

Proof: Analogous to the proof of Theorem 3.1 in [18]. ■

Corollary 3.1: An inclusion monotonic interval-valued function $IGn : L([0,1])^n \to L([0,1])$ is a n-dimensional iv-grouping function if and only if there exist general grouping functions $\mathcal{GG}_1$ and $\mathcal{GG}_2$ such that $\mathcal{GG}_1 \leq \mathcal{GG}_2$ and $IGn = \widehat{\mathcal{GG}_1, \mathcal{GG}_2}$. In particular, one has that $\mathcal{GG}_1 = IGn^-$ and $\mathcal{GG}_2 = IGn^+$.

Proof: Immediate, as general grouping functions are a generalization of n-dimensional 0-grouping functions and n-dimensional 1-grouping functions. ■

It is clear that Corollary 3.1 also applies to bivariate iv-grouping functions, and thus, it derives from Theorem 2.1 as well. Furthermore, from Corollary 3.1 one may observe once again that not every representable n-dimensional iv-grouping function has n-dimensional grouping functions as its representatives. For that reason, we present some definitions and results regarding the representation of n-dimensional iv-grouping functions, in particular when they have n-dimensional grouping functions as both its representatives.

Definition 3.2: An n-dimensional iv-grouping function $IGn : L([0,1])^n \to L([0,1])$ is said to be g-representable if there exist n-dimensional grouping functions $Gn_1, Gn_2 : [0,1]^n \to [0,1]$, $G_1 \leq G_2$, such that $IGn = \widehat{Gn_1, Gn_2}$.

By considering Definition 3.2 for bi-variate functions ($n = 2$), we also have the same concept of g-representability for iv-grouping functions.

One may observe that Theorem 3.3 and Corollary 3.1 result from the fact that there are some n-dimensional iv-grouping functions that are inclusion monotonic but are not g-representable. So, we added conditions in which inclusion monotonic n-dimensional iv-grouping functions must satisfy in order to also be g-representable, as stated in the following theorem:

Theorem 3.4: Let $IGn : L([0,1])^n \to L([0,1])$ be a n-dimensional iv-grouping function. Then, IGn is g-representable if and only if IGn is inclusion monotonic and the following conditions are satisfied: (i) $\underline{IGn(\vec{X})} = 0 \Leftrightarrow \underline{X_1} = \ldots = \underline{X_n} = 0$; (ii) $\overline{IGn(\vec{X})} = 1 \Leftrightarrow \max(\overline{\vec{X}}) = 1$.

Proof: ($\Rightarrow$) If IGn is g-representable, then by Corollary 3.1, IGn is inclusion monotonic. Also, by Proposition 2.3 one has that $IGn^-(\underline{\vec{X}}) = \underline{IGn(\vec{X})}$ and $IGn^+(\overline{\vec{X}}) = \overline{IGn(\vec{X})}$. Furthermore, by Proposition 2.2, $IGn = \widehat{IGn^-, IGn^+}$, meaning that IGn^- and IGn^+ are both n-dimensional grouping functions. Thus,

$$\underline{IGn(\vec{X})} = 0 \Leftrightarrow IGn^-(\underline{\vec{X}}) = 0 \Leftrightarrow \underline{X_1} = \ldots = \underline{X_n} = 0$$

and

$$\overline{IGn(\vec{X})} = 1 \Leftrightarrow IGn^+(\overline{\vec{X}}) = 1 \Leftrightarrow \max(\overline{\vec{X}}) = 1.$$

($\Leftarrow$) If IGn is an n-dimensional iv-grouping function which is inclusion monotonic and satisfies conditions (i) and (ii), then, by Proposition 2.1, $\exists f, g : [0,1]^n \to [0,1]$ (both increasing) such that $IGn(\vec{X}) = [f(\underline{\vec{X}}), g(\overline{\vec{X}})]$.

By Proposition 2.3, it holds that $IGn^-(\underline{\vec{X}}) = \underline{IGn(\vec{X})} = f(\underline{\vec{X}})$ and $IGn^+(\overline{\vec{X}}) = \overline{IGn(\vec{X})} = g(\overline{\vec{X}})$.

Thus, $f = IGn^-$ and $g = IGn^+$.

Now, let's verify that IGn^- and IGn^+ satisfy the conditions in Definition 2.6:

(Gn1) It's trivial as IGn is commutative;

(Gn2) From condition (i): $IGn^-(x_1,\ldots,x_n) = 0 \Leftrightarrow \underline{IGn([x_1,x_1],\ldots,[x_n,x_n])} = 0 \Leftrightarrow \underline{[x_1,x_1]} = \ldots = \underline{[x_n,x_n]} = 0 \Leftrightarrow x_1 = \ldots = x_n = 0$;

(Gn3) From condition (ii): $IGn^+(x_1,\ldots,x_n) = 1 \Leftrightarrow \overline{IGn([x_1,x_1],\ldots,[x_n,x_n])} = 1 \Leftrightarrow \max(\overline{[x_1,x_1]},\ldots,\overline{[x_i,x_i]}) = 1 \Leftrightarrow \max(x_1,\ldots,x_n)=1$;

(Gn4) From Proposition 2.1 we have that both IGn^- and IGn^+ are increasing;

(Gn5) From Corollary 12 in [17], IGn^- and IGn^+ are continuous.

As it was proven that IGn^- and IGn^+ are n-dimensional grouping functions, then IGn is g-representable. ■

Corollary 3.2: Let $IG : L([0,1])^2 \to L([0,1])$ be an iv-grouping function. Then, IG is g-representable if and only if IG is inclusion monotonic and the following conditions are satisfied: (i) $\underline{IG(X,Y)} = 0 \Leftrightarrow \underline{X} = \underline{Y} = 0$; (ii) $\overline{IG(X,Y)} = 1 \Leftrightarrow \max(\overline{X},\overline{Y}) = 1$.

Proof: Immediate from Theorem 3.4 for $n = 2$. ■

IV. General Interval-valued Grouping Function

In this section we introduce some generalizations of n-dimensional iv-grouping functions, leading to the concept of general interval-valued grouping functions, as well as some construction methods, properties and characterization.

Definition 4.1: A function $IGn\colon L([0,1])^n \to L([0,1])$ is an n-dimensional iv-0-grouping function if and only if condition (IGn2) from Definition 3.1 is stated as follows: (IGn2') If $X_1 = \ldots = X_n = 0$ then $IGn(\vec{X}) = 0$. Analogously, a function $IGn\colon L([0,1])^n \to L([0,1])$ is an n-dimension iv-1-grouping function if and only if condition (IGn3) from Definition 3.1 is stated as follows: (IGn3') If there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$ then $IGn(\vec{X}) = [1,1]$.

Example 4.1: The n-dimensional interval-valued limited sum IGn_S, given by $IGn_S(\vec{X}) = X_1 \dot{+} \ldots \dot{+} X_n$ is an n-dimensional iv-1-grouping function, but not an n-dimensional iv-grouping function.

Now, by combining the concepts of n-dimensional iv-0-grouping and iv-0-grouping functions, we present de definition of general interval-valued grouping function.

Definition 4.2: A general interval-valued (iv) grouping function is any mapping $\mathcal{IGG} : L([0,1])^n \to L([0,1])$ that satisfies following conditions, for all $\vec{X} \in L([0,1])^n$: ($\mathcal{IGG}1$) $\mathcal{IGG}$ is commutative; ($\mathcal{IGG}2$) If $X_1 = \ldots = X_n = [0,0]$ then $\mathcal{IGG}(\vec{X}) = [0,0]$; ($\mathcal{IGG}3$) If there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$ then $\mathcal{IGG}(\vec{X}) = [1,1]$; ($\mathcal{IGG}4$) $\mathcal{IGG}$ is $\leq_{Pr}$-increasing in the first component: $\mathcal{IGG}(X_1,\ldots,X_n) \leq_{Pr} \mathcal{IGG}(Y,X_2,\ldots,X_n)$ when $X_1 \leq_{Pr} Y$; ($\mathcal{IGG}5$) $\mathcal{IGG}$ is Moore continuous.

Example 4.2: The function defined as

$$\mathcal{IGG}_L(\vec{X}) = \begin{cases} [0,0] & \text{if } \overline{m} \leq 1/n, \\ [0, \min(1, n\cdot\overline{m})] & \text{if } \min(1,\underline{m}) \leq 1/n \text{ and} \\ & \min(1,\overline{m}) > 1/n, \\ n\cdot(X_1 \dot{+} \ldots \dot{+} X_n), & \text{otherwise,} \end{cases}$$

with $\underline{m} = \min\left(1, \sum_{i=1}^n \underline{X_i}\right)$ and $\overline{m} = \min\left(1, \sum_{i=1}^n \overline{X_i}\right)$, is a general iv-grouping function, which is neither an n-dimensional iv-0-grouping function, nor an n-dimensional iv-1-grouping function. Then, it is also not an n-dimensional iv-grouping function.

It is immediate that:

Proposition 4.1: If $F : L([0,1])^n \to L([0,1])$ is either an n-dimensional iv-grouping, iv-0-grouping or iv-1-grouping function, then F is also a general iv-grouping function.

Theorem 4.1: Let $\mathcal{GG}_1$ and $\mathcal{GG}_2$ be two general grouping functions such that $\mathcal{GG}_1 \leq \mathcal{GG}_2$. Then, the function $\widehat{\mathcal{GG}_1, \mathcal{GG}_2}$ is a general iv-grouping function.

Proof: Analogous to the proof of Theorem 3.1. ■

It is noteworthy that through Theorem 4.1 and Proposition 4.1, one can obtain a representable general iv-grouping function by constructing via n-dimensional grouping functions(and in this case it is called a g-representable general iv-grouping function) or any of its generalizations, such as 0-grouping, 1-grouping or general grouping functions. However, if a general iv-grouping function is representable, then its representatives must be general grouping functions.

Example 4.3: Consider the general grouping function $\mathcal{GG}_B$ defined by $\mathcal{GG}_B(\vec{x}) = \min\left(1, n - \sum_{i=1}^n (1-x_i)^2\right)$. Then, the representable general iv-grouping function $\mathcal{IGG}_B$ can be constructed by taking $\mathcal{GG}_B$ as both its representatives, given by $\mathcal{IGG}_B(\vec{X}) = \widehat{\mathcal{GG}_B}(\vec{X})$.

Proposition 4.2: Let $IA : L([0,1])^n \to L([0,1])$ be a commutative, Moore continuous interval-valued aggregation function. It holds that:

(i) If IA is conjunctive, then it is not a general iv-grouping function;

(ii) If IA is disjunctive, then it is a general iv-grouping function.

Proof: Let $IA\colon L([0,1])^n \to L([0,1])$ be a commutative Moore continuous interval-valued aggregation function. It is immediate that IA satisfies conditions ($\mathcal{IGG}1$), ($\mathcal{IGG}4$) and ($\mathcal{IGG}5$) from Definition 4.2. From Definition 2.9, it holds that $IA([0,0],\ldots,[0,0]) = [0,0]$, so IA also satisfies condition ($\mathcal{IGG}2$). Now, let us verify if it satisfies condition ($\mathcal{IGG}3$) when IA is conjunctive and, in the sequence, when IA is disjunctive.

(i) Suppose that IA is conjunctive. Then, one has that $IA([1,1],[0,0],\ldots,[0,0]) \leq_{Pr} \inf([0,0],[1,1],\ldots,[1,1]) = [0,0]$, which contradicts condition ($\mathcal{IGG}3$). Thus, IA cannot be a general iv-grouping function;

(ii) Suppose that IA is disjunctive. If $\vec{X} \in L([0,1])^n$ such that $\sup(\vec{X}) = [1,1]$, then there is at least one $X_i = [1,1]$. So, $IA(\vec{X}) \geq_{Pr} \sup(\vec{X}) = [1,1]$, meaning that IA also satisfies condition ($\mathcal{IGG}3$). Therefore, if IA is disjunctive, then it is a general iv-grouping function. ■

A. Characterization and Construction Methods of General IV-Grouping Functions

Here we present a characterization and some construction methods for general iv-grouping functions.

Theorem 4.2: The mapping $\mathcal{IGG} : L([0,1])^n \to L([0,1])$ is a general iv-grouping function if and only if

$$\mathcal{IGG}(\vec{X}) = F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X})),$$

for some $F, G : L([0,1])^n \to L([0,1])$ such that

(i) F and G are commutative;
(ii) If $X_1 = \ldots = X_n = [0,0]$ then $F(\vec{X}) = [0,0]$;
(iii) If there exists $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$ then $G(X_1,\ldots,X_n) = [0,0]$;
(iv) F is $\leq_{Pr}$-increasing in the first component and G is $\leq_{Pr}$-decreasing in the first component;
(v) F and G are Moore continuous;
(vi) $\underline{F(\vec{X}) \dot{+} G(\vec{X})} \neq 0$, for any $\vec{X} \in L([0,1])^n$.

Proof: $(\Rightarrow)$ Suppose that $\mathcal{IGG}$ is a general iv-grouping function, and consider $F(\vec{X}) = \mathcal{IGG}(\vec{X})$ and $G(\vec{X}) = [1 - \overline{\mathcal{IGG}(\vec{X})}, 1 - \underline{\mathcal{IGG}(\vec{X})}]$. By Definition 4.2, it is immediate that conditions (i) - (v) hold. Furthermore, $F(\vec{X}) \dot{+} G(\vec{X}) = \mathcal{IGG}(\vec{X}) \dot{+} [1 - \overline{\mathcal{IGG}(\vec{X})}, 1 - \underline{\mathcal{IGG}(\vec{X})}]$, which means that $\underline{F(\vec{X}) \dot{+} G(\vec{X})} = 1$, respecting condition (vi). Finally, it is clear that $\mathcal{IGG}(\vec{X}) = F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X}))$.

$(\Leftarrow)$ Consider that $F, G : L([0,1])^n \to L([0,1])$ satisfy the conditions (i)-(vi). Let's show that $\mathcal{IGG}(\vec{X}) = F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X}))$ is a general iv-grouping function, or in other words, that $\mathcal{IGG}$ is well defined and satisfies each condition from Definition 4.2. As $F(\vec{X}) \leq_{Pr} (F(\vec{X}) \dot{+} G(\vec{X}))$, by Remark 2.1 it is clear that $\mathcal{IGG}$ is well defined. Now, let's verify if it satisfies each condition from Definition 4.2:

($\mathcal{IGG}$1) It's trivial as F and G are both commutative;

($\mathcal{IGG}$2) Let $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$ such that $X_1 = \ldots = X_n = [0,0]$. From condition (ii) one has that $F(\vec{X}) = [0,0]$, and from condition (vi) it holds that $\underline{F(\vec{X}) \dot{+} G(\vec{X})} \neq 0$, so $F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X})) = [0,0] \div_H G(\vec{X})$.
Since $\underline{G(\vec{X})} \neq 0$, then $\mathcal{IGG}(\vec{X}) = [0,0]$;

($\mathcal{IGG}$3) Let $\vec{X} = (X_1,\ldots,X_n) \in L([0,1])^n$ such that $i \in \{1,\ldots,n\}$ with $X_i = [1,1]$. From condition (iii) one has that $G(\vec{X}) = [0,0]$, so $F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X})) = F(\vec{X}) \div_H (F(\vec{X}) \dot{+} [0,0])$. As, from condition (vi), $\underline{F(\vec{X}) \dot{+} G(\vec{X})} \neq 0$, then it holds that $\mathcal{IGG}(\vec{X}) = [1,1]$;

($\mathcal{IGG}$4) Let $\vec{X}, Y \in L([0,1])$ such that $X_1 \leq_{Pr} Y$. To simplify the notation, consider $F(\vec{X}) = A, F(Y, X_2, \ldots, X_n) = B, G(\vec{X}) = C$ and $G(Y, X_2, \ldots, X_n) = D$. From condition (iv) one has that $A \leq_{Pr} B$ and $D \leq_{Pr} C$. Then, by Lemma 4.2 in [18] it holds that

$$\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}} \leq \frac{\underline{B}}{\underline{B} \dot{+} \underline{D}} \text{ and } \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}} \leq \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}$$

Thus,

$$\min\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right) \leq \min\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right)$$

and

$$\max\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right) \leq \max\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right).$$

So,

$$\left[\min\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right), \max\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right)\right] \leq_{Pr}$$
$$\left[\min\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right), \max\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right)\right].$$

However,

$$A \div_H (A \dot{+} C) = \left[\min\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right), \max\left(\frac{\underline{A}}{\underline{A} \dot{+} \underline{C}}, \frac{\overline{A}}{\overline{A} \dot{+} \overline{C}}\right)\right]$$

and

$$B \div_H (B \dot{+} D) = \left[\min\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right), \max\left(\frac{\underline{B}}{\underline{B} \dot{+} \underline{D}}, \frac{\overline{B}}{\overline{B} \dot{+} \overline{D}}\right)\right]$$

meaning that

$$A \div_H (A \dot{+} C) \leq_{Pr} B \div_H (B \dot{+} D),$$

or in other words,

$$\mathcal{IGG}(X_1,\ldots,X_n) \leq_{Pr} \mathcal{IGG}(Y, X_2, \ldots, X_n),$$

proving that $\mathcal{IGG}$ is $\leq_{Pr}$-increasing in the first component;

($\mathcal{IGG}$5) Straightforward from Corollary 4.1 in [18] and the fact that F, G are Moore continuous.

■

Example 4.4: Let us apply the construction method presented in Theorem 4.2 to characterize some general iv-grouping functions through different pairs of functions $F, G : L([0,1])^n \to L([0,1])$ that satisfies conditions (i)-(vi).

1. For any F and G such that $G(\vec{X}) = [1 - \overline{F(\vec{X})}, 1 - \underline{F(\vec{X})}]$, we have that $\mathcal{IGG} = F$ is an general iv-grouping function;
2. For F and G defined, respectively, by $F(\vec{X}) = \begin{cases} [0,0] & \text{if } \max(m(X_1),\ldots,m(X_n)) \leq 0.5 \\ [2\mathfrak{m}, 2\mathfrak{m}] & \text{if } 0.5 \leq \max(m(X_1),\ldots,m(X_n)) \leq 1 \end{cases}$ and

$$G(\vec{X}) = [\min(1 - m(X_1),\ldots,1 - m(X_n)), \min(1 - m(X_1),\ldots,1 - m(X_n))],$$

with $m(X) = 0.5 \cdot (\underline{X} + \overline{X})$ and $\mathfrak{m} = \max(m(X_1),\ldots,m(X_n)) - 0.5$. Then, $\mathcal{IGG}(\vec{X}) = F(\vec{X}) \div_H (F(\vec{X}) \dot{+} G(\vec{X}))$ is a general iv-grouping function.

Proposition 4.3: Given a general iv-grouping function $\mathcal{IGG} : L([0,1])^n \to L([0,1])$ and a commutative, Moore continuous n-dimensional interval-valued aggregation function $IA : L([0,1])^n \to L([0,1])$, one has that

$$\mathcal{IGG}_{IA}(\vec{X}) = \sup(\mathcal{IGG}(\vec{X}), IA(\vec{X}))$$

is a general iv-grouping function.

Proof: It is immediate that $\mathcal{IGG}_{IA}$ is commutative ($\mathcal{IGG}1$), $\leq_{Pr}$-increasing in the first component ($\mathcal{IGG}4$) and Moore continuous ($\mathcal{IGG}5$), since $\mathcal{IGG}$, IA and the supremum share those same properties. Now, let us verify if $\mathcal{IGG}_{IA}$ satisfies conditions ($\mathcal{IGG}2$) and ($\mathcal{IGG}3$).

($\mathcal{IGG}2$) Consider $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$ such that $X_1 = \ldots = X_n = [0,0]$. Then, one has that $\mathcal{IGG}(\vec{X}) = [0,0]$ and $IA(\vec{X}) = [0,0]$ meaning that

$$\mathcal{IGG}_{IA}(\vec{X}) = \sup([0,0],[0,0]) = [0,0];$$

($\mathcal{IGG}3$) Take $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$ such that there exists $i \in \{1, \ldots, n\}$ with $X_i = [1,1]$. Then, one has that $\mathcal{IGG}(\vec{X}) = [1,1]$, and thus

$$\mathcal{IGG}_{IA}(\vec{X}) = \sup([1,1], IA(\vec{X})) = [1,1].$$

■

Corollary 4.1: Given a n-dimensional iv-grouping function $IGn : L([0,1])^n \to L([0,1])$ and a commutative, Moore continuous n-dimensional interval-valued aggregation function $IA : L([0,1])^n \to L([0,1])$, one has that

$$IGn_{IA}(\vec{X}) = \sup(IGn(\vec{X}), IA(\vec{X}))$$

is a general iv-grouping function.

Proof: It is immediate from Proposition 4.3.

■

Example 4.5: Considering the general iv-grouping function $\mathcal{IGG}_B$ as defined in Example 4.3 and the interval-valued aggregation function $IM : L([0,1])^n \to L([0,1])$ given by

$$IM(\vec{X}) = \frac{1}{n} \cdot \sum_{i=1}^{n} X_i,$$

the function $\mathcal{IGG}_{IMB} : L([0,1])^n \to L([0,1])$ defined by

$$\mathcal{IGG}_{IMB}(\vec{X}) = \sup(IM(\vec{X}), \mathcal{IGG}_B(\vec{X})),$$

is a general iv-grouping function, but not a n-dimensional iv-grouping function.

Theorem 4.3: Let $IA : L([0,1])^m \to L([0,1])$ be a Moore continuous m-dimensional interval-valued aggregation function and $\overrightarrow{\mathcal{IGG}} = (\mathcal{IGG}_1, \ldots, \mathcal{IGG}_m)$ a tuple of general iv-grouping functions. Then, the interval-valued function $IA_{\overrightarrow{\mathcal{IGG}}} : L([0,1])^n \to L([0,1])$, defined for all $\vec{X} \in L([0,1])^n$ by

$$IA_{\overrightarrow{\mathcal{IGG}}}(\vec{X}) = IA(\mathcal{IGG}_1(\vec{X}), \ldots, \mathcal{IGG}_m(\vec{X}))$$

is a general iv-grouping function.

Proof: It is immediate that $IA_{\overrightarrow{\mathcal{IGG}}}$ is commutative, since $\mathcal{IGG}_1, \ldots, \mathcal{IGG}_m$ are all commutative ($\mathcal{IGG}1$). Also, we have that $IA_{\overrightarrow{\mathcal{IGG}}}$ is $\leq_{Pr}$-increasing in the first component ($\mathcal{IGG}4$). As it is also Moore continuous ($\mathcal{IGG}5$), let us prove that $IA_{\overrightarrow{\mathcal{IGG}}}$ satisfies conditions ($\mathcal{IGG}2$) and ($\mathcal{IGG}3$).

($\mathcal{IGG}2$) Take $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$ such that $X_1 = \ldots = X_n = [0,0]$. Then, one has that $\mathcal{IGG}_j(\vec{X}) = [0,0]$ for all $j \in \{1, \ldots, m\}$, and, therefore,

$$IA(\mathcal{IGG}_1(\vec{X}), \ldots, \mathcal{IGG}_m(\vec{X})) = IA([0,0], \ldots, [0,0]) = [0,0];$$

($\mathcal{IGG}3$) Consider $\vec{X} = (X_1, \ldots, X_n) \in L([0,1])^n$ such that there exists $i \in \{1, \ldots, n\}$ with $X_i = [1,1]$. Then, one has that $\mathcal{IGG}_j(\vec{X}) = [1,1]$ for all $j \in \{1, \ldots, m\}$, and, therefore,

$$IA(\mathcal{IGG}_1(\vec{X}), \ldots, \mathcal{IGG}_m(\vec{X})) = IA([1,1], \ldots, [1,1]) = [1,1].$$

■

Then, it is immediate that:

Corollary 4.2: Consider the tuple $\overrightarrow{\mathcal{IGG}} = (\mathcal{IGG}_1, \ldots, \mathcal{IGG}_m)$ of general iv-grouping functions. Then, for $w_1, \ldots, w_m \in [0,1]$ such that $w_1 + w_2 + \ldots + w_m = 1$, the function given by

$$SUM_{\overrightarrow{\mathcal{IGG}}}(\vec{X}) = w_1 \cdot \mathcal{IGG}_1(\vec{X}) + \ldots + w_m \cdot \mathcal{IGG}_m(\vec{X})$$

is a general iv-grouping function.

Corollary 4.3: Given a Moore continuous n-dimensional interval-valued aggregation function $IA : L([0,1])^m \to L([0,1])$ and the tuple of n-dimensional iv-grouping functions $\overrightarrow{IGn} = (IGn_1, \ldots, IGn_m)$, the interval-valued function $IA_{\overrightarrow{IGn}} : L([0,1])^n \to L([0,1])$ defined for all $\vec{X} \in L([0,1])^n$ by

$$IA_{\overrightarrow{IGn}}(\vec{X}) = IA(IGn_1(\vec{X}), \ldots, IGn_m(\vec{X}))$$

is a general iv-grouping function.

Proof: Direct, from Proposition 4.1.

■

It is immediate that Corollary 4.3 could be rewritten by swapping the tuple of n-dimensional iv-grouping functions by either a tuple of n-dimensional iv-0-grouping functions or a tuple of n-dimensional iv-1-grouping functions.

V. Conclusion

In this paper, we first introduced the concept of n-dimensional iv-grouping functions and studied some of their properties and their representation, also introducing the concept of g-representability. Following that, we presented the definition, characterization and construction methods for general iv-grouping functions.

The theoretical developments presented here allow for a more flexible approach when dealing with decision making problems with multiple alternatives and interval-valued data, which is the kind of application we are going analyze in our future work. Also, we intend to further develop the presented concepts considering different interval orders.

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