A new family of Bonferroni mean-type pre-aggregation operators

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***Abstract*—The concept of pre-aggregation functions, which was oriented as an elementary attempt to outstretch the notion of monotonicity in aggregation functions, has enlarged the class of operators for information accumulation by considering directional monotonicity with respect to a vector. This consideration propels us to focus on the systematic investigation of the theoretical framework of different forms of pre-aggregation functions, particularly Bonferroni mean-type. In this regard, we propose the construction methodology of Bonferroni mean-type (BM-type) pre-aggregation functions by befitting suitable functions to provide a descriptive configuration, which is quite interpretable and understandable. Firstly, a construction method of BM-type pre-aggregation function has been propounded by utilizing a bivariate function M. Its properties are inspected in detail. To enrich its capacity, the proposed BM-type pre-aggregation function has been customized by utilizing two functions, namely M and M^*, respectively. Several illustrative examples have been presented in this regard.**

***Index Terms*—Aggregation operator, Directional monotonicity, Pre-aggregation operator, Bonferroni mean**

I. Introduction

Aggregation functions form an expeditiously emerging field of applied mathematics and information science, where a fusion of information from a considered ordered scale is deliberated to encapsulate the summarized representative of the available information. It plays a pivotal role in many processing problems, for instance, image processing [23], [22], pattern recognition [12], decision making [25], deep learning [30], data fusion [1], statistics [21]. A momentous property gratified by aggregation functions, while defining them is the boundary conditions, which corroborate that whenever complete evidence of the information is available, that happens when all input arguments equal 1, then the aggregated output also equals 1. Contrarily, if there is no evidence of information, i.e., all input arguments equal 0, then the aggregated value has zero evidence. Another pivotal property gratified by the aggregation function is the monotonically increasing property. However, this property is unexploited and ill-favored in many application problems, for instance, the mode function and the Lehmer mean function [27] are not monotonic.

An elementary attempt to outstretch the concept of monotonicity has been initiated by Wilkin and Beliakov [26] by introducing the concept of weak monotonicity that requires monotonicity along the direction of the first quadrant diagonal. Further augmentation in this line of research has been done by Bustince et al. [7], who revamped the notion of weak monotonicity by introducing the concept of directional monotonicity entitling monotonicity along a fixed ray. Lucca et al. [20] exploited this concept to define the notion of pre-aggregation functions. A different methodology has been proposed for the construction of pre-aggregation functions. Some special properties of it have been investigated by Dimuro et al. [15] and new construction methodology has been suggested. The notion of light pre-t-norms has been initiated by Dimuro, which is another pre-aggregation function mitigating the condition of associativity.

Choquet integral is a pre-eminent aggregation function in the existing literature of information fusion, that utilizes a fuzzy measure (non-additive) to aggregate the information in the data. Recently, this operator has been generalized to pre-aggregation function in many vivid ways, for instance, by utilizing t-norm T [20], by utilizing fusion function F satisfying several properties [19], by utilizing a copula C [16], [18], by utilizing pair of functions (F_1, F_2) [9] with some restrictive constraints on F_1 and F_2 [17], and generalized gC_{F_1,F_2}-integrals [9].

The structural interpretation of the BM operator was magnificently polished by Yager [28], where he analyzed the BM as a combination of averaging and an "anding" operator. The homogenization of attributes that BM operator deliberately model is realized through Yager's interpretation as the product of each input argument with the average of other interrelated input arguments. He also propounded a generalized version of the BM operator by subsequently substituting simple aver-

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aging operator by other mean-type aggregation operators like Choquet integral and OWA operator [28]. As inspired by the heuristic work of Yager, Beliakov et al. [2] further generalized BM by decomposing the operator into various subcomponents and subsequently substituting them with other averaging and conjunctive aggregation operators. The decomposable structure of generalized BM operator (GBM) is convenient for behavioral interpretation and facilitates modeling capability through the behavior of its subcomponents.

In this study, we initiated the notion of Bonferroni mean-type pre-aggregation functions, which amplifies the class of pre-aggregation functions. Firstly, a Bonferroni-mean type pre-aggregation function has been introduced by utilizing a bivariate function M. Some restrictions have been thrust on the operator M to make it satisfy certain properties like averaging, idempotency, etc. Then, a generalized form of Bonferroni mean-type pre-aggregation function has been proposed by bringing into limelight two functions, namely, M and M^*, respectively. Conditions have been imposed on these two operators to make it pre-aggregation function and to make it fulfill some beneficial properties. Section II summarizes some basic pre-requisites that are indispensable for understanding the related work. In section III, construction method of Bonferroni mean-type pre-aggregation function has been proposed that utilizes a bivariate function M. Section IV proposes the construction methodology of Bonferroni mean-type pre-aggregation functions by utilizing two operators, namely, M and M^*, respectively, with extensive study of its several preferable properties. In section V, some concluding remarks are provided along with the future scope of this work.

II. Basic Preliminaries

This section aims at providing the essential prerequisites that are useful in subsequent initiation of this work.

Definition 1: [19] Let $G : [0,1]^2 \mapsto [0,1]$ be a bivariate function. Then, G is said to satisfy left conjunctive property if $G(x,y) \leq x\ \forall x, y \in [0,1]$.

Definition 2: [14] An n-ary function $A : [0,1]^n \mapsto [0,1]$ is said to be an aggregation operator if it satisfies the following properties:

1) A is increasing in each of its arguments, i.e., if $y_i \leq x_i$ for every $i = 1, ..., n$, then $A(y_1, ..., y_n) \leq A(x_1, ..., x_n)$;
2) $A(0, ..., 0) = 0$ and $A(1, .., 1) = 1$, i.e., A satisfies boundary conditions.

Some of the special aggregation operators over $[0,1]^2$ are t-norms [14], overlap functions [19], and copula [19]. They can be suitably extended over $[0,1]^n$ [11], [10].

Definition 3: [20] Let $\vec{0} \neq \vec{r} = (r_1, ..., r_n)$ be an n-dimensional vector where all $r_i's$ are real numbers and let $G : [0,1]^n \mapsto [0,1]$ be an n-ary function. Then G is said to be $\vec{r}$-increasing if $\forall (y_1, ..., y_n) \in [0,1]^n$ and for any $c > 0$, such that $(y_1 + cr_1, ..., y_n + cr_n) \in [0,1]^n$, the inequality

$$G(y_1 + cr_1, ..., y_n + cr_n) \geq G(y_1, ..., y_n) \quad (1)$$

holds.

Similarly, one can define $\vec{r}$-decreasing functions.

Definition 4: [20] Let $G : [0,1]^n \mapsto [0,1]$ be n-ary function. Then G is said to be pre-aggregation function if the following conditions hold:

1) G is $\vec{r}$-increasing for some non-zero vector $\vec{r} = (r_1, .., r_n)$;
2) G satisfies boundary conditions, which are $G(0, ..., 0) = 0$ and $G(1, ..., 1) = 1$.

We call G as $\vec{r}$-pre-aggregation functions.

Definition 5: [20] An n-ary function $G : [0,1]^n \mapsto [0,1]$ is said to be averaging if it satisfies the inequalities:

$$min \leq G \leq max$$

.

Definition 6: [29] Let $F : [0,1]^n \mapsto [0,1]$ be an aggregation function and let $\sigma : \{1, ..., n\} \mapsto \{1, ..., n\}$ be a fixed permutation on the set $\{1, ..., n\}$. Then F is said to be σ-commutative (or σ-symmetric) if for any $(y_1, ..., y_n) \in [0,1]^n$, we have $F(y_1, ..., y_n) = F(y_{\sigma(1)}, ..., y_{\sigma(n)})$. Furthermore, F is said to be commutative (or symmetric) if it is σ-commutative (or σ-symmetric) for all permutations σ on $\{1, ..., n\}$.

Definition 7: [5], [28] With $p, q \geq 0$ and $p + q > 0$, the Bonferroni mean operator (BM) is a mapping BM:$[0,1]^n \rightarrow [0,1]$ which is defined as follows:

$$BM(y_1, y_2, ..., y_n) = \Big(\frac{1}{n}\sum_{i=1}^{n} y_i^p \Big(\frac{1}{n-1}\sum_{\substack{j \neq i \\ j=1}}^{n} y_j^q\Big)\Big)^{\frac{1}{p+q}}$$

III. Construction of Bonferroni Mean-Type Pre-Aggregation by Utilizing Bivariate Function M

This section proposes construction methodology of Bonferroni mean-type pre-aggregation function by utilizing a bivariate function $M : [0,1]^2 \mapsto [0,1]$. Throughout this section, it is assumed that diagonal of M, denoted by δ_M, be such that it is surjective as well as injective so that it is invertible. We denote the inverse diagonal of M by δ_M^{-1}. Clearly then, $\delta_M^{-1} : [0,1] \mapsto [0,1]$.

Definition 8: Let $M : [0,1]^2 \mapsto [0,1]$ be a bivariate function such that diagonal of M is surjective and injective so that it is invertible. Let $\delta_M^{-1} : [0,1] \mapsto [0,1]$ denote inverse diagonal of M. The Bonferroni mean-type operator based on the function M, denoted by B^M, is the function $B^M : [0,1]^n \mapsto [0,1]$ which is defined as follows:

$$B^M(y_1, ..., y_n) = \delta_M^{-1}\Big\{\frac{1}{n(n-1)}\sum_{\substack{i,j=1 \\ i \neq j}}^{n} M(y_i, y_j)\Big\} \quad (2)$$

where $(y_1, ..., y_n) \in [0,1]^n$.

Note that such an operator M always exists. For instance, one may take M as minimum operator given by $M(x,y) = \min\{x, y\}$. In that case, $\delta_M^{-1}(y) = y\ \forall y \in [0,1]$. Furthermore, it is to be noted that if M is idempotent and commutative (or symmetric) function with $n = 2$, then $B^M = M$.

Remark 3.1: For any eligible function M (means M is such that inverse diagonal $\delta_M^{-1} : [0,1] \mapsto [0,1]$), the following functions are also eligible: $1-M$, $M^d(x,y) = 1 - M(1-x, 1-y)$, and $1 - M^d(x,y) = M(1-x, 1-y)$. Then, we have $B^M = B^{1-M}$ (thus the same operator B^M can be generated by two different binary operators M and $1-M$) and $B^{M^d} = B^{(1-M^d)} = (B^M)^d$.

Example 3.1:

- Let $d : [0,1] \mapsto [0,1]$ be any injective and surjective function. Define the operator $M(x,y) = cd(x) + (1-c)d(y)$, $c \in [0,1]$. The corresponding B^M operator is given as follows:
$$B^M = d^{-1}\left\{\frac{d(y_1) + ... + d(y_n)}{n}\right\}$$
Thus, quasi-arithmetic mean is obtained provided d is monotone.
- On taking $M(x,y) = \min\{x,y\}$, the following form of B^M operator is obtained:
$$B^M(y_1, ..., y_n) = \left\{\frac{1}{n(n-1)}\{(2n-2)y_{\sigma(1)}+ (2n-4)y_{\sigma(2)} + ... + 2y_{\sigma(n-1)}\}\right\}$$
where $\sigma : \{1, ..., n\} \mapsto \{1, ..., n\}$ is a permutation such that $y_{\sigma(1)} \leq y_{\sigma(2)} \leq y_{\sigma(n-1)}$. Note that here B^M is an OWA operator [13].

Theorem 3.1: For the bivariate function M, such that it possesses inverse diagonal $\delta_M^{-1} : [0,1] \mapsto [0,1]$, the operator B^M is well defined and satisfies idempotency property; hence it satisfies the boundary conditions which are $B^M(0, ..., 0) = 0$ and $B^M(1, ..., 1) = 1$.

Proof 3.1: It is immediate to check that the operator B^M is well defined. To check that it is idempotent, let $y \in [0,1]$. Then,
$$B^M(y, ..., y) = \delta_M^{-1}\left\{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y,y)\right\}$$
$$= \delta_M^{-1}\{M(y,y)\} = \delta_M^{-1}(\delta_M(y)) = y.$$
Hence, the operator is idempotent, which further implies that boundary conditions are satisfied.

Theorem 3.2: For any function $M : [0,1]^2 \mapsto [0,1]$ such that M is (1,1)-increasing (M is weakly increasing), B^M is $\vec{1}$-increasing operator (i.e. B^M is weakly increasing operator), and hence a pre-aggregation function.

Proof 3.2: First of all, we show that if M is (1,1)-increasing, then δ_M is an increasing function. Let $c > 0$ be any real number. Consider,
$$\delta_M(y+c) = M(y+c, y+c) \geq M(y,y) = \delta_M(y) \quad \forall y \in [0,1].$$
Hence, δ_M is an increasing function.
To show that B^M is $\vec{1}$-increasing operator, let $(y_1, ..., y_n) \in [0,1]^n$ and consider,
$$B^M(y_1+c, ..., y_n+c) = \delta_M^{-1}\left\{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y_i+c, y_j+c)\right\}.$$
Since, M is (1,1)-increasing, we have $M(y_i + c, y_j + c) \geq M(y_i, y_j)\ \forall i \neq j$.
$$\Rightarrow \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y_i+c, y_j+c) \geq \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y_i, y_j).$$
Now, δ_M is increasing function, it follows that δ_M^{-1} is strictly increasing function. Hence, B^M is $\vec{1}$-increasing. By 3.1, B^M always satisfies boundary conditions, therefore a pre-aggregation function.

Example 3.2: Consider the Lehmer mean operator [27] given by
$$L(x,y) = \frac{x^2 + y^2}{x+y}$$
with convention that $\frac{0}{0} = 0$. Then, this operator is (1,1)-increasing function [3] (in fact, it is increasing only for $\vec{r} = (r,r)$, $r > 0$). The corresponding B^L operator is given by:
$$B^L(y_1, ..., y_n) = \left\{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \left(\frac{y_i^2 + y_j^2}{y_i + y_j}\right)\right\}$$
is an example of proper Bonferroni mean-type $\vec{1}$-pre-aggregation operator, which is not an aggregation operator.

Remark 3.2: B^M operator is σ-commutative (or σ-symmetric) for every permutation σ on $\{1, ..., n\}$, hence the operator is commutative (or symmetric).

It is to be noted that due to the commutativity of B^M operator, we have B^M is a pre-aggregation function if and only if it satisfies the boundary conditions and it is weakly increasing, i.e., it is $\vec{1}$-increasing. Therefore, our investigations concerning the pre-aggregation functions in the form of B^M can be reduced to the case of weak increasingness.

Theorem 3.3: Let a function $M : [0,1]^2 \mapsto [0,1]$ be such that it satisfies the following properties:

1) $\min(x,y) \leq M(x,y)\ \forall x,y \in [0,1]$;
2) M satisfies left conjunctive property.

Then, $\min \leq B^M \leq \max$, i.e., B^M is an averaging operator.

Proof 3.3: Since, M satisfies $\min(x,y) \leq M(x,y) \leq x$ $\forall x,y \in [0,1]$, so we have M is idempotent. Let $y_k = \min_{i=1}^n\{y_i\}$, and $\max_{i=1}^n\{y_i\} = y_p$. Then,
$$y_k = \min_{i=1}^{n}\{y_i\} \leq \min\{y_i, y_j\}$$
$$\leq M(y_i, y_j) \leq y_i \leq \max_{i=1}^{n}\{y_i\} = y_p$$
$$\Rightarrow y_k \leq \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y_i, y_j) \leq y_p.$$
Since M is idempotent, which implies that δ_M^{-1} is identity function hence strictly increasing, we have
$$\min_{i=1}^{n}\{y_i\} = y_k = \delta_M^{-1}\{\delta_M(y_k)\} = \delta_M^{-1}\{y_k\}$$

$$\leq \delta_M^{-1}\left\{\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} M(y_i,y_j)\right\}$$

$$\leq \delta_M^{-1}\{y_p\} = \delta_M^{-1}\{\delta_M(y_p)\} = y_p = \max_{i=1}^{n}\{y_i\}$$

Hence, the theorem follows.

IV. Construction of Bonferroni mean-type pre-aggregation operators by utilizing functions M and M^*

This section proposes construction methodology of Bonferroni mean-type pre-aggregation operators by utilizing two functions, namely M and M^* respectively. We assume that $M : [0,1]^2 \mapsto [0,1]$ be such that δ_M is surjective and injective function so that it is invertible. Denote inverse diagonal of M by $\delta_M^{-1} : [0,1] \mapsto [0,1]$. Further we assume that $M^* : [0,1]^{n(n-1)} \mapsto [0,1]$ is a commutative (or symmetric) function.

Definition 9: Let $M : [0,1]^2 \mapsto [0,1]$ be a binary function such that its diagonal $\delta_M : [0,1] \mapsto [0,1]$ is bijective and let $M^* : [0,1]^{n(n-1)} \mapsto [0,1]$ be a commutative (symmetric) function. The Bonferroni mean-type operator based on functions M and M^* respectively, denoted by $B_{M^*}^M$, is the function $B_{M^*}^M : [0,1]^n \mapsto [0,1]$ which is defined as follows:

$$B_{M^*}^M(y_1,...,y_n) = \delta_M^{-1}\left[M^*\{M(y_i,y_j) : i,j = 1,..,n; i\neq j\}\right] \tag{3}$$

where $(y_1,...,y_n) \in [0,1]^n$.
One wishes to find out conditions on M and M^* respectively, such that $B_{M^*}^M$ is a pre-aggregation operator. Unless otherwise stated, we are assuming conditions on δ_M^{-1} and M^* as given in Definition 9.

Remark 4.1: It is to be observed that if M^* is an arithmetic mean operator then the definitions 8 and 9 coincides, i.e., we have $B_{M^*}^M = B^M$.

Remark 4.2: It is to be observed that if M^* is a weighted arithmetic mean operator, and $w_{ij} \in [0,1]$ for every $i,j = 1,...,n,\ i\neq j$, satisfying $\sum_{\substack{i,j=1\\ i\neq j}}^{n} w_{ij} = 1$ then the definition 9 reduces to

$$B_{M^*}^M = (y_1,...,y_n) = \delta_M^{-1}\left[\sum_{\substack{i,j=1\\ i\neq j}}^{n} w_{ij}M(y_i,y_j)\right]$$

.

Theorem 4.1: The operator $B_{M^*}^M$ is well defined. Furthermore, the necessary and sufficient condition for $B_{M^*}^M$ to be idempotent is that M^* should be idempotent.

Proof 4.1: It is easy to show that $B_{M^*}^M$ is well defined. To check idempotency property, we first assume that M^* is idempotent. Consider,

$$B_{M^*}^M(y,..,y) = \delta_M^{-1}\left[M^*\{M(y,y) : i,j = 1,..,n; i\neq j\}\right]$$

$$= \delta_M^{-1}\left[M(y,y)\right] = \delta_M^{-1}\left[\delta_M(y)\right] = y$$

Now, assume that $B_{M^*}^M$ is idempotent. If possible let M^* is not idempotent. Consider,

$$B_{M^*}^M(y,..,y) = \delta_M^{-1}\left[M^*\{M(y,y) : i,j = 1,..,n; i\neq j\}\right]$$

If $M(y,y) = b$ (say) and $M^*(b,...,b) = c \neq b$, then

$$B_{M^*}^M(y,..,y) = \delta_M^{-1}\left[M^*(b,..,b)\right] = \delta_M^{-1}(c).$$

Now, it is given that $B_{M^*}^M$ is idempotent, then we have $c = \delta_M(y) = M(y,y) = b$, which is a contradiction. Hence, M^* has to be idempotent.

Remark 4.3: It is to be noted that on taking $M(x,y) = \min\{x\sqrt{y}, y\sqrt{x}\}$ and M^* as quadratic mean function, then the corresponding $B_{M^*}^M$ is given by the following form:

$$B_{M^*}^M(y_1,...,y_n) = \left[\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \min\{y_i^2y_j, y_j^2y_i\}\right]^{\frac{1}{3}}$$

The operator $B_{M^*}^M$ is idempotent in spite of taking M as non-idempotent operator.

Theorem 4.2: If the operator M is (1,1)-increasing and the operator M^* is an increasing function, then the operator $B_{M^*}^M$ is $\vec{1}$-increasing.

Proof 4.2: Since M is (1,1)-increasing, we have

$$M(y_i + c, y_j + c) \geq M(y_i,y_j) \quad \forall i\neq j, c > 0.$$

M^* is an increasing function, so we have

$$M^*\{M(y_i + c, y_j + c) : i,j = 1,..,n; i\neq j\}$$
$$\geq M^*\{M(y_i,y_j) : i,j = 1,..,n; i\neq j\}$$

Now, M is (1,1)-increasing, it follows that δ_M and hence δ_M^{-1} is strictly increasing. Thus,

$$B_{M^*}^M(y_1 + c,..,y_n + c)$$
$$= \delta_M^{-1}\left[M^*\{M(y_i + c, y_j + c) : i,j = 1,..,n; i\neq j\}\right]$$
$$\geq \delta_M^{-1}\left[M^*\{M(y_i,y_j) : i,j = 1,..,n; i\neq j\}\right] = B_{M^*}^M(y_1,..,y_n)$$

.

Theorem 4.3: If $M(0,0) = a$, $M(1,1) = b$, then the necessary and sufficient condition on $B_{M^*}^M$ to satisfy boundary conditions is that a and b are idempotent elements of M^*

Proof 4.3: To check the boundary condition, consider the following argument.
First, we assume that a and b are idempotent elements of M^*. Consider,

$$B_{M^*}^M(0,...,0) = \delta_M^{-1}\left[M^*\{M(0,0) : i,j = 1,..,n; i\neq j\}\right]$$
$$= \delta_M^{-1}\left[M^*(a,...,a)\right] = \delta_M^{-1}(a)$$
$$= \delta_M^{-1}(M(0,0)) = \delta_M^{-1}(\delta_M(0)) = 0$$

Now, let $B^M_{M^*}$ satisfies boundary condition at (0,...,0). Then,

$$B^M_{M^*}(0,...,0) = \delta_M^{-1}\Big[M^*\{M(0,0) : i,j = 1,..,n; i \neq j\}\Big] = 0$$

$$\Rightarrow M^*\{M(0,0) : i,j = 1,..,n; i \neq j\} = \delta_M(0) = M(0,0)$$

Thus, it implies that $M(0,0) = a$ is an idempotent element of M^*.
Similarly, we can prove it for $\vec{1}$.

Corollary 4.1: Let the operator M is (1,1)-increasing, and the operator M^* satisfies the following conditions:

1) $M(0,0) = a$ and $M(0,0) = b$ are idempotent elements of M^*.
2) M^* is an increasing function.

Then the operator $B^M_{M^*}$ is a pre-aggregation function.
Proof: Proof follows immediately by using theorems 4.2, 4.3.

Example 4.1: On taking M as Lehmer mean given by $L(x,y) = \frac{x^2+y^2}{x+y}$, and M^* as maximum operator, we get the following form of $B^L_{M^*}$:

$$B^L_{M^*}(y_1,..,y_n) = \max\left\{\frac{y_i^2+y_j^2}{y_i+y_j} : i,j = 1,...,n; \quad i \neq j\right\},$$

with the convention that $\frac{0}{0} = 0$ ([24], [4], [8]).
The operator $B^L_{M^*}$ is a proper pre-aggregation operator, which is not an aggregation operator.
On taking a particular case of $n = 2$, the corresponding $B^L_{M^*}$ operator is given as follows:

$$B^L_{M^*}(y_1,y_2) = \frac{y_1^2+y_2^2}{y_1+y_2},$$

which is Lehmer mean operator and is weakly monotone but not monotone, therefore a proper pre-aggregation operator is obtained.

Theorem 4.4: If the operator M satisfies the following conditions:

1) M satisfies left conjunctive property;
2) $M \geq \min$;

and the operator M^* is an idempotent and an increasing function, then we have $\min \leq B^M_{M^*} \leq \max$.

Proof 4.4: Since, M satisfies $\min\{x,y\} \leq M(x,y) \leq x$ $\forall x,y \in [0,1]$, so we have M as idempotent function. Now, consider

$$B^M_{M^*}(y_1,..,y_n) = \delta_M^{-1}\Big[M^*\{M(y_i,y_j) : i,j = 1,..,n; i \neq j\}\Big].$$

Let, $\min_{i=1}^n\{y_i\} = y_k$, and $\max_{i=1}^n\{y_i\} = y_p$. Then we have,

$$y_k = \min_{i=1}^n\{y_i\} \leq \min(y_i,y_j) \leq M(y_i,y_j)$$
$$\leq y_i \leq \max_{i=1}^n\{y_i\} = y_p \quad \forall i,j = 1,..,n; i \neq j.$$

Also, M^* is an increasing and idempotent function, we have

$$y_k = M^*\Big(y_k,...,y_k\Big) \leq M^*\Big(M(y_i,y_j) : i,j = 1,..,n; i \neq j\Big)$$
$$\leq M^*\Big(y_p,...,y_p\Big) = y_p$$

Since, δ_M^{-1} is an identity function, so we have

$$\min_{i=1}^n\{y_i\} = y_k = \delta_M^{-1}(y_k)$$
$$\leq \delta_M^{-1}\Big[M^*\Big(M(y_i,y_j) : i,j = 1,..,n; i \neq j\Big)\Big]$$
$$\leq \delta_M^{-1}(y_p) = y_p = \max_{i=1}^n\{y_i\}$$

Hence, the theorem follows.

Remark 4.4: The operator $B^M_{M^*}$ is σ-commutative (or σ-symmetric) for every permutation σ of the set $\{1,...,n\}$, hence the operator is commutative.

V. Conclusion

This paper develops a theoretical framework for Bonferroni mean-type pre-aggregation operators. Firstly, a family of operators is developed by utilizing a bivariate function M such that diagonal of M is surjective and invertible. By imposing several conditions on the function M, a family of Bonferroni mean-type $\vec{1}$ pre-aggregation functions, namely B^M, has been proposed. This family satisfies several advantageous properties, and it is illustrated by a set of examples. More generalized family of BM-type pre-aggregation functions has been constructed by suitably befitting two functions, namely M and M^* respectively.
Future scope in the direction of this work includes the proposition of partitioned Bonferroni mean-type pre-aggregation function and extended partitioned Bonferroni mean-type pre-aggregation function, along with its application in classification problem. Furthermore, the proposed operators can be generalized by constructing ordered directionally monotone Bonferroni-mean type functions [6], [9].

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