Conditioned Monotonicity for Generalized Pre-Aggregations and Aggregations

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***Abstract*—The concept of pre-aggregation function defined in $[0,1]^n$ has been recently extended to that of generalized pre-aggregation function in the framework of a totally ordered set $\mathcal{T}$ with maximum and minimum value. To do so, the concept of monotonicity is transformed in that of conditioned monotonicity based on the chains in $\mathcal{T}^n$, generalizing the idea of directional monotonicity. In the present paper we explore the concept of conditioned monotonicity considering some specific conditioning structures (covers, partitions and projections). On this basis we consider some situations where conditioned monotonicity ensures monotonicity. Finally we use these definitions and properties to define some pre-aggregation and aggregation functions that are applied to image preprocessing problems.**

***Index Terms*—Aggregation functions, pre-aggregation functions, directional monotonicity, conditioned monotonicity.**

I. INTRODUCTION

Aggregation functions [1] play a central role in information fusion and decision making. This question has been mostly focused on by means of variables taking values on $[0,1]$ interval, aggregated through monotonic functions. Only in recent years aggregation processes have been opened to new approaches where unit interval and monotonic functions are replaced by other options.

The $[0,1]$ interval can be extended to any complete lattice [2] (defining generalized aggregation functions), or even to a bounded poset as in [3] (working with type-2 fuzzy sets). A different option is that of relaxing monotonicity through weak monotonicity [4], directional monotonicity [5], or conditioned monotonicity [6]. Directional monotonicity and conditioned monotonicity are the basis for the definition of pre-aggregation functions [7] and generalized pre-aggregation functions [6].

The present paper will continue with the analysis of conditioned monotonicity and generalized pre-aggregation functions started in [6]. We will work with a totally ordered set $\mathcal{T}$ with maximum and minimum value. This set can even be discrete and we need to reinterpret the concept of monotonicity in terms of chains or linear orders in $\mathcal{T}^n$ (conditioned monotonicity). In this new framework (totally ordered sets and conditioned monotonicity), we will concentrate on some specific cases. It could be also interesting to transfer the existing relation between directional monotonicity and monotonicity, to the case of conditioned monotonicity.

The rest of the paper is structured as follows. Section II presents some preliminary concepts related to Aggregation operators, as well as the recently introduced concepts of conditioned monotonicity and generalized pre-aggregation. Then, in Section III, the concepts of conditioned monotonicity and generalized pre-aggregation are considered in the framework of partitions related to a prototype, and particularly for those partitions linked to projections. This idea leads us to analyze the monotonicity of a function in terms of conditioned monotonicity over its projections in Section IV. Then, its application to define generalized pre-aggregation and aggregation functions useful in image processing is shown in Section V. Finally, some conclusions and open questions are considered.

II. PRELIMINARIES

A. Aggregation operators

The usual definition of an aggregation operator is that of a function

$$Ag_n : [0,1]^n \to [0,1]$$

which holds the following properties:

1) Ag_n is monotone, non decreasing.
2) $Ag_n(0,\dots,0) = 0$.
3) $Ag_n(1,\dots,1) = 1$.

The previous concept can be extended to a more general case by replacing the lattice $[0,1]$ with a complete lattice (with a maximum and minimum), leading to what is usually known as generalized aggregation function (see for example [2], [8]).

Definition 1: Let $(\mathcal{T},\leq)$ be a complete lattice (with maximum and minimum elements, $1_\mathcal{T}$ and $0_\mathcal{T}$ respectively) and let $(\mathcal{T}^n,\leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$ (i.e. $\mathcal{T}^n = \underbrace{\mathcal{T} \times \ \dots \times \mathcal{T}}_{n\ times}$).

A **generalized aggregation function** is a mapping $Ag : \mathcal{T}^n \to \mathcal{T}$ such that it satisfies:

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1) $Ag(\underbrace{0_{\mathcal{T}}, 0_{\mathcal{T}}..., 0_{\mathcal{T}}}_{n\ times}) = 0_{\mathcal{T}}$ and $Ag(\underbrace{1_{\mathcal{T}}, 1_{\mathcal{T}}, .., 1_{\mathcal{T}}}_{n\ times}) = 1_{\mathcal{T}}$.
2) Ag is monotonic with respect to the lattice's order ($\leq^n$).

But this is not the only extension that can be defined. As previously said weak monotonicity [4] extends classical monotonicity property by requiring monotonicity only along the direction of the first quadrant diagonal. This idea is further extended with the concept of directional monotonicity [5] based on that of r-increasing functions.

Definition 2: [5] Let f be a function from $[0, 1]^n$ to $[0, 1]$. And let r be an element of $\mathbb{R}^n$. Then we will say that f is **r-increasing** if for all $x \in [0, 1]^n$ and for all $\lambda \geq 0$, the following holds:

$$\text{If} \quad x + \lambda r \in [0, 1]^n \quad \text{then} \quad f(x + \lambda r) \geq f(x).$$

Then, the concept of pre-aggregation function relies on that of r-increasing functions.

Definition 3: [7] A mapping $Ag_n : [0, 1]^n \to [0, 1]$ is an n-dimensional **pre-aggregation** function if it satisfies:

1) There exists a real vector $r \in [0, 1]^n$ with $r \neq 0$ such that Ag_n is r-increasing.
2) $Ag_n(0, \ldots, 0) = 0$ and $Ag_n(1, \ldots, 1) = 1$.

It is also possible to merge the two previously mentioned extensions. Considering that pre-aggregation functions have already produced good results (see for example [9]) in aggregation processes working over the complete lattice $[0, 1]$, it could be interesting to extend pre-aggregations to problems defined on $\mathcal{T}$ (a complete lattice with a maximum and minimum element).

B. Conditioned Monotonicity

The idea now is to extend the concept of pre-aggregation from $[0, 1]^n$ to $\mathcal{T}^n$. To do so, the idea of directional monotonicity is adapted to a universe where *directions* may not exist. Consequently, it is transformed in the more general concept of *conditioned monotonicity*.

Definition 4: [6] Let $(\mathcal{T}, \leq)$ be a totally ordered set with maximum and minimum elements ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, let $\mathcal{P}_{\leq}(\mathcal{T}^n)$ be the set of all totally ordered subsets (chains) of $\mathcal{T}^n$, and let C be a subset of $\mathcal{P}_{\leq}(\mathcal{T}^n)$. Consequently $C = \{C_i : i \in I\}$ where C_i are chains of $\mathcal{T}^n$.

A mapping $F : \mathcal{T}^n \to \mathcal{T}$ is said to be a C**-conditioned increasing (decreasing) function** if it satisfies that $\forall a, b \in \mathcal{T}^n$, if $a \leq^n b$ and $\exists C_i \in C$ such that $a, b \in C_i$, then $F(a) \leq F(b)$ ($F(a) \geq F(b)$).

In other words, the function F is said to be a C-conditioned increasing (decreasing) function if $F_{|C_i}$ is an increasing (decreasing) function, for every chain C_i in C. For this idea it is important to notice that, first, any two elements in C_i are comparable since chains are linearly ordered sets, and second, any two images are comparable being the range of F a subset of a totally ordered set ($\mathcal{T}$).

Definition 5: [6] Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, let $\mathcal{P}_{\leq}(\mathcal{T}^n)$ be the set of all totally ordered subsets (chains) of $\mathcal{T}^n$, and let C be a subset of $\mathcal{P}_{\leq}(\mathcal{T}^n)$.

A mapping $F : \mathcal{T}^n \to \mathcal{T}$ is said to be a C**-conditioned monotonic function** if it is either a C-conditioned increasing function or a C-conditioned decreasing function.

Theorem 1: [6] Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, let $\mathcal{P}_{\leq}(\mathcal{T}^n)$ be the set of all totally ordered subsets (chains) of $\mathcal{T}^n$.

If $F : \mathcal{T}^n \to \mathcal{T}$ is a $\mathcal{P}$-conditioned monotonic function, i.e., it is a C-conditioned monotonic function with $C = \mathcal{P}_{\leq}(\mathcal{T}^n)$, then F is a monotonic function.

C. Generalized pre-aggregation functions

Definition 6: [6] Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, let $\mathcal{P}_{\leq}(\mathcal{T}^n)$ be the set of all totally ordered subsets (chains) of $\mathcal{T}^n$, and let C be a subset of $\mathcal{P}_{\leq}(\mathcal{T}^n)$.

A **generalized pre-aggregation function over** C is a mapping $Ag : \mathcal{T}^n \to \mathcal{T}$ such that it satisfies:

1) $Ag(\underbrace{0_{\mathcal{T}}, 0_{\mathcal{T}}..., 0_{\mathcal{T}}}_{n\ times}) = 0_{\mathcal{T}}$ and $Ag(\underbrace{1_{\mathcal{T}}, 1_{\mathcal{T}}, .., 1_{\mathcal{T}}}_{n\ times}) = 1_{\mathcal{T}}$.
2) Ag is a C-conditioned increasing function.

Proposition 1: [6] A generalized pre-aggregation function over $\mathcal{P}_{\leq}(\mathcal{T}^n)$ is a generalized aggregation function.

Definition 7: [6] A mapping $Ag : \mathcal{T}^n \to \mathcal{T}$ is said to be a **partitioning generalized pre-aggregation function** over C when it is a generalized pre-aggregation function over C, and C is a partition of $\mathcal{T}^n$.

Proposition 2: [6] Any pre-aggregation function in the sense of Definition 3 from [7], is a partitioning generalized pre-aggregation function.

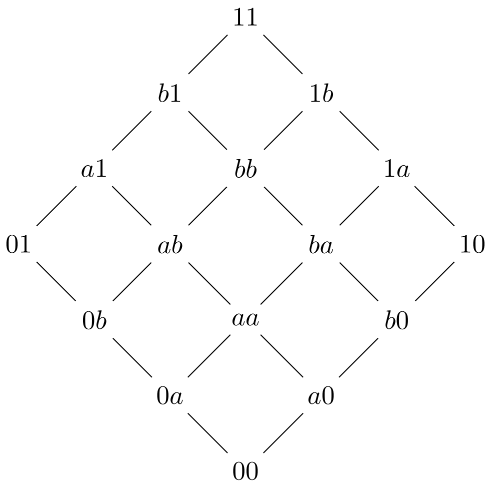
Definition 8: [6] A mapping $Ag : \mathcal{T}^n \to \mathcal{T}$ is said to be a **covering generalized pre-aggregation function** over C when it is a generalized pre-aggregation function over C, and C covers $\mathcal{T}^n$.

We say that C covers $\mathcal{T}^n$, or C is a cover of $\mathcal{T}^n$, when C is a family of nonempty (and non-duplicated) subsets of $\mathcal{T}^n$ whose union is equal to $\mathcal{T}^n$.

Proposition 3: [6] Any partitioning generalized pre-aggregation function over C is a covering generalized pre-aggregation function over C.

In summary, conditioned monotonicity offers many different options for the analysis according to the relation between the conditioning set C, and the overall set of chains in $\mathcal{T}^n$ ($\mathcal{P}_{\leq}(\mathcal{T}^n)$). These options go from monotonicity when $C = \mathcal{P}_{\leq}(\mathcal{T}^n)$ (see Proposition 1) to fusion functions (where monotonicity is eliminated) when $C = \emptyset$.

Example 2.1: Let us consider the totally ordered set $(\mathcal{T}, \leq)$ with $\mathcal{T} = \{0, a, b, 1\}$ and the order $0 < a < b < 1$. In $(\mathcal{T}^2, \leq^2)$, corresponding to the following Hasse diagram



we define the function $F : \mathcal{T}^2 \to \mathcal{T}$ as:
$F(0,0) = F(0,a) = F(0,b) = F(a,0) = F(b,0) = F(b,a) = 0$,
$F(0,1) = F(a,a) = F(1,0) = F(b,b) = a$,
$F(a,b) = F(b,1) = F(1,a) = b$,
$F(a,1) = F(1,b) = F(1,1) = 1$.

If we consider the following chains in $\mathcal{T}^2$: $C_1 = \{00, 0a, 0b, 01, a1\}, C_2 = \{00, a0, aa, ab, a1\}, C_3 = \{0a, aa, ab, bb, b1, 11\}, C_4 = \{a0, b0, ba, bb, b1\}$ and $C_5 = \{b0, 10, 1a, 1b, 11\}$; the function F is monotonic when restricted to C_1, to C_2, to C_4 and to C_5, but is not monotonic when restricted to C_3. Consequently, if we consider $C = \{C_1, C_2, C_4, C_5\}$, F is a C-conditioned monotonic function. In addition, as $F(0,0) = 0$ and $F(1,1) = 1$, F is a generalized pre-aggregation function over C. Finally, considering that C covers $\mathcal{T}^2$, F is a covering generalized pre-aggregation function over C. On the other hand, being not increasing over C_3, F is not a Generalized aggregation function.

III. Conditioned monotonicity on prototype-based partitions

In previous section we have considered the concept of conditioned monotonicity having its upper bound in monotonicity when $C = \mathcal{P}_{\leq}(\mathcal{T}^n)$. On the other hand, applying the concept of conditioned monotonicity, we have established different levels of generalized pre-aggregations including covering and partitioning pre-aggregation functions. Having in mind the relation between conditioned monotonicity and pre-aggregation, we can replicate these same intermediate levels in conditioned monotonicity.

In a *covering generalized pre-aggregation function*, the conditioning set C covers $\mathcal{T}^n$. It makes sense to say that the conditioned monotonicity appearing in this case is a *covering conditioned monotonicity*. With a similar reasoning it is possible to establish the idea of *partitioning conditioned monotonicity*.

Covering conditioned monotonicity seems to be an interesting option since being C a cover ensures that the union of all elements in C is $\mathcal{T}^n$. Consequently, every element x in $\mathcal{T}^n$ will be in, at least, one chain (C_i) in C. This situation ensures that for any $x \in \mathcal{T}^n$ there is at least a linear order $C_x \in C$, with $x \in C_x$, that is, every element in $\mathcal{T}^n$ could be a matter of analysis in what concerns monotonicity. But, it could also happen the linear order not being unique.

To cope with that potential problem could be interesting to force the subsets in C being pairwise disjoint. Under this restriction the cover becomes a partition of $\mathcal{T}^n$, and the covering conditioned monotonicity is transformed into a partitioning conditioned monotonicity. The function will be C-conditioned monotonic, being C a partition of $\mathcal{T}^n$. In this case we can assure that for any $x \in \mathcal{T}^n$ there exist a unique linear order $C_x \in C$, with $x \in C_x$. It is interesting to notice that the $r-$directional monotonicity defined in [5] is a particular case of partitioning conditioned monotonicity where $\mathcal{T} = [0,1]$, and the chains C_x are defined by the straight line with direction r that contains the element x.

A direction r is a natural way to define a partition in $[0,1]^n$. The question is how to define a partition in $\mathcal{T}^n$ in a natural way similar to that of the straight line used in $r-$directional monotonicity. The main idea is to generate a partition on the basis of a certain property or condition. We will refer to this kind of partition a prototype-based partition, and to the building property or condition as the prototype of the partition.

A. Conditioned monotonicity on projective partitions

A particular and interesting case could be that of partitions where the prototype generates chains which elements are constant in every component except for one, i.e, $c \in C$ implies $c = \{(a_1, \ldots, x_j, \ldots, a_n) : x_j \in \mathcal{T}$, and a_i is constant $(i \neq j)\}$. To explore this concept we will relate it to the idea of projection.

Definition 9: Let T be a set, and let x be an element of T^n, i.e., $x = (x_1, x_2, \ldots, x_n)$, we define the i-th projection map in T^n as a mapping $\text{proj}_i : T^n \to T$ such that $\text{proj}_i(x) = x_i$.

We can now generalize this idea by selecting more than one component of x, so producing a mapping onto T^m with $m \leq n$.

Definition 10: Let T be a set, let x be an element of T^n, let $J \subset I = \{1, 2, , \ldots, n\}$, with $|J| = m$. We define the J-th projection map in T^n as a mapping $\text{proj}_J : T^n \to T^m$ such that $\text{proj}_J(x) = (x_{J_1}, x_{J_2}, \ldots, x_{J_m})$, where $J_j, J_k \in J$, and $J_j < J_k$, when $j < k$.

On the basis of these definitions of projection we can generate equivalence relations by simply grouping in an equivalence class all those elements having the same projection.

Definition 11: Let T be a set, let x be an element of T^n, let $J \subset I = \{1, 2, , \ldots, n\}$, let $a, b \in T^n$, and let proj_J be the J-th projection map in T^n. We define the equivalence relation Π_J as $(a,b) \in \Pi_J$, if and only if $\text{proj}_J(a) = \text{proj}_J(b)$.

If we denote now the set $\bar{i}$ as $\bar{i} = I - \{i\}$, i.e., the complementary of $\{i\}$ in I, the relation $\Pi_{\bar{i}}$ will produce equivalence classes where all members will be identical except for the value of their i-th component, i.e, $(a,b) \in \Pi_{\bar{i}}$, if and only if $\text{proj}_{\bar{i}}(a) = \text{proj}_{\bar{i}}(b)$, and consequently $a_j = b_j, \forall j \neq i$. We will represent this equivalence class as $[(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)]$. This equivalence class is made up of all elements having as common projection $(a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

The set of all possible equivalence classes of T^n by $\Pi_{\bar{i}}$, denoted $T^n/\Pi_{\bar{i}}$, is the quotient set of T^n by $\Pi_{\bar{i}}$. And any quotient set (in this case $T^n/\Pi_{\bar{i}}$) is a partition of T^n. We can then define a partition by means of a projection.

Let us replace now the generic set T used in previous definitions with a totally ordered set $(\mathcal{T}, \leq)$. We will first consider the structure of its equivalence classes (being chains), and apply then the concept of conditioned monotonicity by conditioning through a projection (that generates a partition as previously shown).

Proposition 4: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, and let $\text{proj}_{\bar{i}} : \mathcal{T}^n \to \mathcal{T}$ be the projection map over $\bar{i}$. Then, each equivalence class defined by $\Pi_{\bar{i}}$ is a chain in $(\mathcal{T}^n, \leq^n)$.

Proof: Each equivalence class defined by $\Pi_{\bar{i}}$ can be represented by $[(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)]$. As x_i takes values in $\mathcal{T}$, if $(a_1, \ldots, a_i, \ldots, a_n), (a_1, \ldots, b_i, \ldots, a_n) \in [(a_1, a_2, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)]$, and $(a_1, \ldots, a_i, \ldots, a_n) \neq (a_1, \ldots, b_i, \ldots, a_n)$, it should be either $a_i < b_i$ or $b_i < a_i$ (a_i and b_i are comparable). In addition, $\forall a \in \mathcal{T}^n$ and $b_i \in \mathcal{T}$, $a_i < b_i \iff (a_1, \ldots, a_i, \ldots, a_n) <^n (a_1, \ldots, b_i, \ldots, a_n)$, and $b_i < a_i \iff (a_1, \ldots, b_i, \ldots, a_n) <^n (a_1, \ldots, a_i, \ldots, a_n)$ (being $(a_1, \ldots, a_i, \ldots, a_n)$ and $(a_1, \ldots, b_i, \ldots, a_n)$ also comparables). Consequently, each pair of elements in the equivalence class defined by $\Pi_{\bar{i}}$ are comparable, and the equivalence class is a linear order (a chain). ∎

If we consider again the set, and the function defined in Example 2.1, it is possible to apply the concept of projection to generate a partition and analyze conditioned monotonicity.

Example *3.1:* Let us consider $\mathcal{T} = \{0, a, b, 1\}$, with the order $0 < a < b < 1$, and $(\mathcal{T}^2, \leq^2)$. We define the function $F : \mathcal{T}^2 \to \mathcal{T}$ as:
$F(0,0) = F(0,a) = F(0,b) = F(a,0) = F(b,0) = F(b,a) = 0$,
$F(0,1) = F(a,a) = F(1,0) = F(b,b) = a$,
$F(a,b) = F(b,1) = F(1,a) = b$,
$F(a,1) = F(1,b) = F(1,1) = 1$.

The function F is a partitioning generalized pre-aggregation function over $\mathcal{T}^2/\Pi_{\bar{2}}$. The function is increasing over any chain of the form $\{x0, xa, xb, x1\}$, with $x \in \mathcal{T}$, that is, over any equivalence classes of $\mathcal{T}^2$.

On the other hand, F is not a partitioning generalized pre-aggregation function over $\mathcal{T}^2/\Pi_{\bar{1}}$ considering that F is not increasing over the chains $\{0a, aa, ba, 1a\}$, $\{0b, ab, bb, 1b\}$, and $\{01, a1, b1, 11\}$.

The projections we have considered to this point produce the dimensional reduction by means of removing part of the components of x. If $(\mathcal{T}^n, \leq^n)$ has additional properties, it is possible to consider other projections when generating our partitions. We can even define a different interaction in between partitions and projections.

Definition 12: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_{\mathcal{T}}$ and $0_{\mathcal{T}}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, and let $x = (x_1, \ldots, x_n)$ be an element of $\mathcal{T}^n$. We define the lower surface of $(\mathcal{T}^n, \leq^n)$, denoted by $S_l(\mathcal{T}^n)$ as:

$$S_l(\mathcal{T}^n) = \{x \in \mathcal{T}^n : \exists i \in \{1, \ldots, n\} \text{ with } x_i = 0_{\mathcal{T}}\}.$$

In a similar way we can define the upper surface of $(\mathcal{T}^n, \leq^n)$, denoted by $S_u(\mathcal{T}^n)$ as:

$$S_u(\mathcal{T}^n) = \{x \in \mathcal{T}^n : \exists i \in \{1, \ldots, n\} \text{ with } x_i = 1_{\mathcal{T}}\}.$$

If we define now the set $C \subset \mathcal{P}_{\leq}(\mathcal{T}^n)$ in such a way that it was a partition, and that every chain $C_i \in C$ has one and only one element of $S_l(\mathcal{T}^n)$ (noted as C_{il}), and has one and only one element of $S_u(\mathcal{T}^n)$ (noted as C_{iu}), the functions $\Pi_{Cl} : \mathcal{T}^n \to S_l(\mathcal{T}^n)$ and $\Pi_{Cu} : \mathcal{T}^n \to S_u(\mathcal{T}^n)$ defined as $\Pi_{Cl}(x) = C_{il}\ \forall x \in C_i$ and $\Pi_{Cu}(x) = C_{iu}\ \forall x \in C_i$ respectively, are projections of $\mathcal{T}^n$.

This kind of partition related to a prototype/projection being not the canonical projection could also be of interest.

In particular we have said that $r-$directional monotonicity defined in [5] is a particular case of partitioning conditioned monotonicity where $\mathcal{T} = [0,1]$, and the chains C_x are defined by the straight line with direction r that contains the element x. Is it possible to define such a partition as a prototype based partition? Let us consider now this question.

In this case the definition of the partition C and the corresponding projection is quite simple.

Definition 13: Let us consider $\mathcal{T} = [0,1]$, let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, given $v = (v_1, \ldots, v_n) \in \mathcal{T}^n$, $v \neq (0, \ldots, 0)$, we define the projection in the direction of v as a mapping $\text{proj}_v : \mathcal{T}^n \to S_l(\mathcal{T}^n)$, such that for any $x = (x_1, \ldots, x_n) \in \mathcal{T}^n$, $\text{proj}_v(x) = x - \lambda_x v$, where $\lambda_x = \min_{i, v_i \neq 0} \left\{ \dfrac{x_i}{v_i} \right\}$.

But we don't need to consider such a complicated structure if we want to create partitions in spaces as $[0,1]^n$. We can simply create a prototype based partition on $[0,1]^n$ by considering a prototype being a straight line with a certain slope (that at the end is also a chain). This situation corresponds to the so called directional monotonicity, being the direction that of the line (prototype). In fact, the previous idea of prototypes where only one component changes are related to this concept by taking the directions of the coordinate axes.

This idea can be easily extended to other kind of patterns or conditions (prototype chain) as a parabola or any other curve or variety, with the only restriction that the prototype should generate chains (linearly ordered sets).

IV. From Conditioned Monotonicity to Monotonicity

When the conditioned monotonicity of a function appears over different partitions, it is in some cases possible to generalize it to other chains not directly considered in the partitions.

In the case of monotonicity related to different projections we can scale up the property by merging the projections. When the conditioned monotonicity appears on every projection, the function is monotonic, as will be shown below.

Remark 1: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_\mathcal{T}$ and $0_\mathcal{T}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, let x be an element of $\mathcal{T}^n$, and let $\text{proj}_{\bar{i}} : \mathcal{T}^n \to \mathcal{T}$ be the projection map over $\bar{i}$. If $F : \mathcal{T}^n \to \mathcal{T}$ is a $\mathcal{T}^n/\Pi_{\bar{i}}$-conditioned increasing function then $\forall a \in \mathcal{T}^n$ and $b_i \in \mathcal{T}$, $(a_1, \ldots, a_i, \ldots, a_n) \leq^n (a_1, \ldots, b_i, \ldots, a_n)$ implies that $F(a_1, \ldots, a_i, \ldots, a_n) \leq F(a_1, \ldots, b_i, \ldots, a_n)$.

Theorem 2: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_\mathcal{T}$ and $0_\mathcal{T}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, and let $\text{proj}_{\bar{i}} : \mathcal{T}^n \to \mathcal{T}$ be the projection map over $\bar{i}$. If $F : \mathcal{T}^n \to \mathcal{T}$ is a $\mathcal{T}^n/\Pi_{\bar{i}}$-conditioned increasing function $\forall i \in \{1, 2, \ldots, n\}$, then F is an increasing function in $\mathcal{T}^n$.

Proof: Let us consider $a, b \in \mathcal{T}^n$ such that $a \leq^n b$, consequently $a_i \leq b_i \forall i \in \{1, \ldots, n\}$. As F is $\mathcal{T}^n/\Pi_{\bar{i}}$-conditioned increasing $\forall i \in \{1, 2, \ldots, n\}$, it holds that

$$F(a_1, a_2, \ldots, a_n) \leq F(b_1, a_2, \ldots, a_n) \leq$$
$$F(b_1, b_2, \ldots, a_n) \leq \ldots \leq F(b_1, b_2, \ldots, b_n)$$

where the first inequality follows from F being $\mathcal{T}^n/\Pi_{\bar{1}}$-conditioned increasing, the second from F being $\mathcal{T}^n/\Pi_{\bar{2}}$-conditioned increasing, and so on until the last inequality following from F being $\mathcal{T}^n/\Pi_{\bar{n}}$-conditioned increasing. Then, as

$$a \leq^n b \text{ implies } F(a_1, a_2, \ldots, a_n) \leq F(b_1, b_2, \ldots, b_n),$$

F is an increasing function in $\mathcal{T}^n$.

■

Lemma 1: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_\mathcal{T}$ and $0_\mathcal{T}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$, and let $\text{proj}_{\bar{i}} : \mathcal{T}^n \to \mathcal{T}$ be the projection map over $\bar{i}$. If $F(0_\mathcal{T}, 0_\mathcal{T}, \ldots, 0_\mathcal{T}, 0_\mathcal{T}) = 0_\mathcal{T}$, $F(1_\mathcal{T}, 1_\mathcal{T}, \ldots, 1_\mathcal{T}, 1_\mathcal{T}) = 1_\mathcal{T}$, and $F : \mathcal{T}^n \to \mathcal{T}$ is a $\mathcal{T}^n/\Pi_{\bar{i}}$-conditioned increasing function $\forall i \in \{1, 2, \ldots, n\}$, then F is a generalized aggregation function in $\mathcal{T}^n$.

Remark 2: It has been previously stated ([5], [10]) that for a function $F : [0,1]^n \to [0,1]$, directional monotonicity in the direction of every vector in the canonical basis, implies monotonicity. Theorem 2 could be considered as a way to adapt that result to the more general framework of the functions $\mathcal{T}^n \to \mathcal{T}$.

A. Searching for generalized aggregations

The previous results can help us to define generalized aggregation functions in the framework of applications that work on lattices with some specific characteristics. This is the case, as an example, of several image processing problems requiring to merge the information of a tuple in $\{0, \ldots, 255\}^n$ in a single value in $\{0, \ldots, 255\}$. In this scenario, it is possible to find several functions that fulfill the conditions of Lemma 1 and consequently are generalized aggregation functions in $\{0, \ldots, 255\}^n$.

Let us consider as an example, the Median of n values in $\mathcal{T}$.

Lemma 2: Let $(\mathcal{T}, \leq)$ be a totally ordered set with a maximum and a minimum element ($1_\mathcal{T}$ and $0_\mathcal{T}$ respectively), let $(\mathcal{T}^n, \leq^n)$ be the natural lattice of n elements of type $\mathcal{T}$ with n an odd number, and let $\phi_{median} : \mathcal{T}^n \longrightarrow \mathcal{T}$ be the median function, $\phi_{median}(x_1, \ldots, x_n) = Median\{x_1, \ldots, x_n\}$. Then, ϕ_{median} is an increasing function in $\mathcal{T}^n$.

Proof: By Theorem 2, if $\phi_{median} : \mathcal{T}^n \to \mathcal{T}$ is a $\mathcal{T}^n/\Pi_{\bar{i}}$-conditioned increasing function $\forall i \in \{1, 2, \ldots, n\}$, then ϕ_{median} is an increasing function in $\mathcal{T}^n$.

To find the median of a set of data, the data have to be first arranged in order from least to greatest, and the median will be the middle number in the ordered set, so $\phi_{median}(x_1, \ldots, x_n) = \phi_{median}(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, being $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ any permutation. Then,

$$\phi_{median}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) =$$
$$\phi_{median}(x, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

Thus, we can take $\bar{i} = \bar{1}$ and show that ϕ_{median} is a $\mathcal{T}^n/\Pi_{\bar{1}}$-conditioned increasing function.

Given $(x, a_2, \ldots, a_n) \in \mathcal{T}^n$ with $a_2 \leq \cdots \leq a_n$ and n an odd natural number, it holds that:

- if $x \leq a_{\frac{n+1}{2}} \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = a_{\frac{n+1}{2}}$,
- if $a_{\frac{n+1}{2}} < x \leq a_{\frac{n+1}{2}+1} \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = x$, and
- if $x > a_{\frac{n+1}{2}+1} \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = a_{\frac{n+1}{2}+1}$.

Let us take, $(x, a_2, \ldots, a_n) \leq^n (y, a_2, \ldots, , a_n)$ and, we can consider $\{a_2, \ldots, a_n\}$ an ordered set. As $x \leq y$, we have the following cases:

- if $y \leq a_{\frac{n+1}{2}} \Rightarrow x \leq y \leq a_{\frac{n+1}{2}}$ and $\phi_{median}(x, a_2, \ldots, a_n) = a_{\frac{n+1}{2}} = \phi_{median}(y, a_2, \ldots, a_n)$,
- if $x \leq a_{\frac{n+1}{2}} < y \leq a_{\frac{n+1}{2}+1} \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = a_{\frac{n+1}{2}}$, $\phi_{median}(y, a_2, \ldots, a_n) = y$ and $a_{\frac{n+1}{2}} < y$.
- if $x \leq a_{\frac{n+1}{2}} \leq a_{\frac{n+1}{2}+1} < y \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = a_{\frac{n+1}{2}}$, $\phi_{median}(y, a_2, \ldots, a_n) = a_{\frac{n+1}{2}+1}$, and $a_{\frac{n+1}{2}} \leq a_{\frac{n+1}{2}+1}$ since $\{a_2, \ldots, a_n\}$ is an ordered set.
- if $a_{\frac{n+1}{2}} < x \leq y \leq a_{\frac{n+1}{2}+1} \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = x$, $\phi_{median}(y, a_2, \ldots, a_n) = y$, and $x \leq y$.
- if $a_{\frac{n+1}{2}} < x \leq a_{\frac{n+1}{2}+1} < y \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = x$, $\phi_{median}(y, a_2, \ldots, a_n) = a_{\frac{n+1}{2}+1}$, and $x \leq a_{\frac{n+1}{2}+1}$.
- if $a_{\frac{n+1}{2}+1} < x \leq y \Rightarrow \phi_{median}(x, a_2, \ldots, a_n) = \phi_{median}(y, a_2, \ldots, a_n) = a_{\frac{n+1}{2}+1}$.

Therefore, ϕ_{median} is a $\mathcal{T}^n/\Pi_{\bar{1}}$-conditioned increasing function. ■

Remark 3: The demonstration can be extended to the case of n being an even number by defining the median as either: $\phi_{median}(x_1, \ldots, x_n) = x_{\sigma(\lceil \frac{n+1}{2} \rceil)}$ or $\phi_{median}(x_1, \ldots, x_n) = x_{\sigma(\lfloor \frac{n+1}{2} \rfloor)}$, being σ an ordering permutation such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$.

Proposition 5: The function ϕ_{median} is a generalized aggregation function for $\mathcal{T} = \{0, \ldots, 255\}$.

Proof: Following Lemma 2, ϕ_{median} is a monotonic non-decreasing function. In addition, $\phi_{median}(0, 0, \ldots, 0) = 0$ and $\phi_{median}(255, 255, \ldots, 255) = 255$. Consequently $\phi_{median} : \{0, \ldots, 255\}^n \to \{0, \ldots, 255\}$ is a generalized aggregation function. ∎

It is important to emphasize that in the same way that the concept of directional monotonicity for Riesz spaces defined in [5] allows to extend the set of aggregation operators that can be used for soft computing, the directional monotonicity concept here introduced extends in a more general way the set of pre-aggregation functions allowing now to deal with any space T in which a partial order is defined.

An example of how this concept of directional monotony allows us to include functions as aggregation operators is in the field of image processing. The information that is extracted from a pixel belongs to the set $\mathcal{T} = \{0, \ldots, 255\}$. Therefore, the functions that operate with this class of information and type of data is not a vector or Riesz space. In the following two examples, two functions frequently considered in image processing applications are analyzed, demonstrating that can be characterized as a generalized aggregation and a generalized pre-aggregation respectively.

V. An application to edge detection problems

Image processing and aggregation problems are very well related topics during recent years. In image processing recent literature [11]–[13], we can find the use of aggregation techniques in the developing of more efficient algorithms.

From a mathematical point of view, let us denote by I a digital image, and by (i, j) the pixel coordinates of the *spatial domain*. The coordinates are integers, where each point (i, j) represents a pixel with $i = 0, \ldots, r$ and $j = 0, \ldots, s$. Therefore, the size of an image, $r \times s$, is the number of its horizontal pixels multiplied by its number of verticals. As we are dealing with color images, then a $k = 1, \ldots, \tilde{k}$ index is needed for expressing the number of channels in the image. Thus, let us denote by I=$P^k_{i,j}$ the spectral information associated with each pixel (i,j) at channel k. As well, $I = P^k_{i,j}$ is equivalent to $\tilde{k}$ images of one single channel (gray-scale images) $I = \{I^1, \ldots, I^{\tilde{k}}\}$. The values range of this information depends on the type of image considered.

- *Binary map*: $I_{bin} = P_{i,j} \in \{0, 255\}$ (as well it is usually expressed as $\{0, 1\}$).
- *Gray-scale*: $I_{gray} = P_{i,j} \in \{0, 1, \ldots, 255\}$.
- *RGB*: $I_{RGB} = P_{i,j} \in \{0, 1, \ldots, 255\}^3$. (R=*Red*; G=*Green* and B=*Blue*).
- *Fuzzy image*: $I_{fuzzy} = P_{i,j} \in [0, 1]$.

As it can be deduced from the previous formulation, we can conclude that when working with gray-scale images or color images, the associated domain for each pixel with coordinates (i, j) for a channel k (that is, $P^k_{i,j}$) should be the discrete set $\mathcal{T} = \{0, \ldots, 255\}$. Taking into account this fact, it would be natural to think that any aggregation operator that deals with this class of spectral information should have the same set.

Nevertheless, most of aggregation operators used in this topic are defined on the real interval $[0, 255]$ or its normalized version $[0, 1]$. Let us analyze the following two very well-known examples in which aggregation operators should be defined over a discrete domain.

***Example** 5.1:* The first example that we present in this section is a classical smoothed aggregation of a grey-scale image. Let us assume that we have a gray-scale image $I_{gray} = P_{i,j} \in \{0, 1, \ldots, 255\}$.

Now let us define the neighborhood $N(i, j)$ of each pixel with coordinates (i, j). The idea of the smoothing process is to define a new image that is the result of aggregate to each pixel the spectral information of its neighborhood $N(i, j)$. Frequently, with the intention of softening the original image I_{gray}, another $I^s_{gray} = \{I^s(i, j)\ \forall i = 1, \ldots, s; j = 1, \ldots, r\}$ where

$$I^s(i, j) = \phi\left(I(u, v), (u, v) \in N(i, j)\right)$$

with

$$\phi : [0, 255]^n \longrightarrow [0, 255]$$

being an aggregation function in the classical sense.

Given a pixel $P_{i,j}$, let us suppose that $N(i, j) = \{(i-1, j-1), (i-1, j), (i-1, j+1), (i, j-1), (i, j), (i, j+1), (i+1, j-1), (i+1, j), (i+1, j+1)\}$. If we denote by $P_{N(i,j)}$ the 9 dimensional vector

$$\big(P_{(i-1,j-1)}, P_{(i-1,j)}, P_{(i-1,j+1)}, P_{(i,j-1)},\\ P_{(i,j)}, P_{(i,j+1)}, P_{(i+1,j-1)}, P_{(i+1,j)}, P_{(i+1,j+1)})\big),$$

a smoothed aggregation operator could be defined in a natural way as:

$$\phi_{smoo} : \underset{P_{N(i,j)}}{\{0, \ldots, 255\}^9} \longrightarrow \underset{Median\{P_{(u,v)}\ ,(u,v)\in N(i,j)\}}{\{0, \ldots, 255\}}$$

With this definition, as shown in previous section, ϕ_{smoo} is a Generalized aggregation function.

The smoothing methods of an image are especially useful in edge detection problems. Just to test the effectiveness of the smoothing aggregation here presented, the edge detection algorithm known as Sobel has been applied after smoothing the image using the ϕ_{smoo} aggregator and compared with the output provided directly by the Sobel algorithm. The following table shows the results obtained for the first 50 known images of the Berkeley [14] database with the maximum human pairing method.

As it can be seen, the results obtained after the application of ϕ_{smoo} significantly improve the overall performance, since they decrease the average recall but increasing the precision and the F measured.

***Example** 5.2:* Let us now present an aggregation operator that can be viewed as a Generalized Pre-Aggregation but is neither a pre-aggregation in the sense of [5] nor a generalized aggregation operator. In edge detection problems it is frequent to identify changes in luminosity in different directions. To do so it is common to transform the original image I_{gray} into

TABLE I
PRECISION AND RECALL IN THE SOBEL ALGORITHM WITH AND WITHOUT SMOOTHING.

	Sobel Smooth			Sobel		
Imag	F	Prec	Rec	F	Prec	Rec
100075.jpg'	0,4951	0,5060	0,5848	0,4447	0,4501	0,5344
100080.jpg'	0,1219	0,3217	0,0853	0,2991	0,4720	0,3959
100098.jpg'	0,7183	0,6532	0,7977	0,6285	0,4792	0,9147
103041.jpg'	0,3884	0,7431	0,2629	0,4714	0,5894	0,3927
104022.jpg'	0,4429	0,6692	0,3428	0,5621	0,7434	0,5563
105019.jpg'	0,5562	0,7707	0,4402	0,6117	0,7272	0,5279
105053.jpg'	0,3672	0,5571	0,3129	0,3552	0,4942	0,3185
106020.jpg'	0,2264	0,1857	0,2899	0,2113	0,1418	0,4147
106025.jpg'	0,6143	0,9612	0,4525	0,6337	0,9320	0,4855
108041.jpg'	0,2473	0,2158	0,4498	0,2346	0,1791	0,5135
108073.jpg'	0,3232	0,2724	0,3973	0,3603	0,2519	0,6321
109034.jpg'	0,2895	0,2239	0,5851	0,2144	0,1319	0,7154
112082.jpg'	0,4162	0,4018	0,4367	0,3335	0,2219	0,6754
113009.jpg'	0,6387	0,8460	0,5619	0,6800	0,8305	0,6280
113016.jpg'	0,7377	0,7935	0,7506	0,4015	0,2646	0,8691
113044.jpg'	0,6786	0,7816	0,6378	0,4001	0,2700	0,8509
117054.jpg'	0,5983	0,6158	0,6732	0,4540	0,3385	0,6891
118020.jpg'	0,4692	0,5085	0,4655	0,4443	0,3629	0,6376
118035.jpg'	0,7419	0,7689	0,9541	0,7715	0,7805	0,9684
12003.jpg'	0,6073	0,5986	0,7439	0,5536	0,4778	0,7928
12074.jpg'	0,5986	0,4782	0,8261	0,5279	0,3827	0,8509
122048.jpg'	0,5009	0,5039	0,6062	0,4711	0,4104	0,6436
124084.jpg'	0,7006	0,7910	0,7858	0,6587	0,6534	0,8586
126039.jpg'	0,5481	0,7132	0,5945	0,5698	0,7283	0,6270
130034.jpg'	0,3274	0,3003	0,3600	0,2172	0,1312	0,7707
134008.jpg'	0,2910	0,2030	0,5134	0,2056	0,1241	0,6216
134052.jpg'	0,4327	0,3070	0,7328	0,2943	0,1814	0,9130
135037.jpg'	0,4730	0,8256	0,3721	0,4501	0,8320	0,4179
135069.jpg'	0,9247	0,8855	0,9883	0,9367	0,8891	0,9927
138032.jpg'	0,3833	0,4839	0,3685	0,3103	0,2387	0,4800
138078.jpg'	0,7029	0,7823	0,7062	0,6734	0,6947	0,8536
140055.jpg'	0,2123	0,4340	0,1628	0,2785	0,2287	0,3621
140075.jpg'	0,6065	0,7306	0,7527	0,5436	0,5053	0,7737
144067.jpg'	0,3672	0,3143	0,4421	0,2593	0,1640	0,6662
145014.jpg'	0,5557	0,5292	0,5910	0,5206	0,4094	0,7262
145053.jpg'	0,5650	0,5835	0,6928	0,4131	0,2965	0,7947
147021.jpg'	0,4073	0,5916	0,3248	0,3998	0,3490	0,5151
147062.jpg'	0,4547	0,3760	0,6548	0,3643	0,2285	0,9154
15004.jpg'	0,4303	0,4189	0,4422	0,3837	0,2798	0,6165
15088.jpg'	0,2908	0,1897	0,6743	0,3384	0,2103	0,8791
151087.jpg'	0,6741	0,7831	0,6341	0,6220	0,5669	0,7489
153077.jpg'	0,5276	0,6976	0,4925	0,4953	0,6074	0,4963
153093.jpg'	0,5531	0,7281	0,5582	0,5311	0,5941	0,6663
155060.jpg'	0,6154	0,7256	0,5774	0,5984	0,5139	0,7804
156079.jpg'	0,5214	0,6561	0,5420	0,4937	0,5452	0,5782
157036.jpg'	0,8226	0,8796	0,8921	0,6585	0,5416	0,9395
159029.jpg'	0,3707	0,3553	0,3875	0,2920	0,1978	0,5836
159045.jpg'	0,5281	0,4864	0,5929	0,5334	0,4565	0,6480
159091.jpg'	0,6006	0,5307	0,7751	0,4771	0,3937	0,8797
16052.jpg'	0,4746	0,4039	0,5754	0,3421	0,3963	0,3010
	0,56	0,56	0,56	0,53	0,44	0,67
	Average			Average		

another that gathers the differences in some specific directions (see for example [15]–[17]).

Given a pixel $P_{i,j}$, with intensity $I_{gray}(i,j)$, let us denote by $I_{gray}(i,j)^{\phi} \in \{0,\ldots,255\}$ the gray scale value obtained after applying the function ϕ to the intensity of the pixel $P_{i,j}$. And let us denote by I^{ϕ}_{gray} the associated image. In [17], two functions ϕ_v and ϕ_h were defined to detect directional changes in luminosity. Formally, given a pixel with coordinates (i,j) we have:

- $\phi_v : \{0,\ldots,255\}^9 \longrightarrow \{0,\ldots,255\}$ defined as
$$\phi_v\left(I_{gray}(i-1,j-1),\ldots,,I_{gray}(i+1,j+1)\right)$$
$$= Max\{0, I_{gray}(i+1,j) - I_{gray}(i-1,j)\}.$$

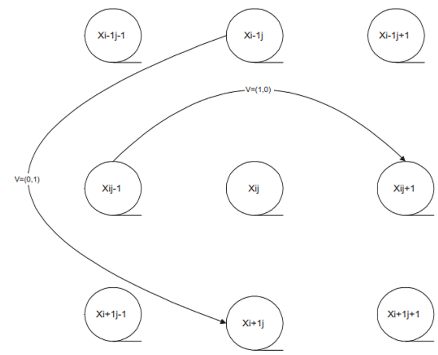


Fig. 1. Directional aggregation operator in discrete space

- $\phi_h : \{0,\ldots,255\}^9 \longrightarrow \{0,\ldots,255\}$
$$\phi_h\left(I_{gray}(i-1,j-1),\ldots,,I_{gray}(i+1,j+1)\right)$$
$$= Max\{0, I_{gray}(i,j+1) - I_{gray}(i,j-1)\}.$$

Figure 1 illustrates these two operators detecting directional changes. ϕ_h corresponds to the changes in the *direction* described as $V(1,0)$ while ϕ_v corresponds to $V(0,1)$.

It is very easy to check that ϕ_v and ϕ_h are not classical generalized aggregation operators since the monotonicity condition is not satisfied.

For example if we take the inputs $x = (0,0,0,70,0,120,0,0,0), y = (0,0,0,80,0,120,0,0,0) \in \{0,\ldots,255\}^9$. It its clear that $x \leq y$ in the partial order of the set $\{0,\ldots,255\}^9$, but $\phi_h(x) = 120 - 70 > \phi_h(y) = 120 - 80 = 40$, and $\phi_h(x)$ is not a monotonic increasing function. Also it is easy to check that these functions are not pre-agregation in the sense of [5] since we have a discrete space.

Nevertheless, we can see that there exist directions (the sixth coordinate i.e. $r = (0,0,0,0,0,1,0,0,0)$) in which the ϕ_h is monotone.

So it is not very difficult to see that the functions ϕ_v and ϕ_h are Generalized Pre-Aggregations.

The Generalized Pre-Aggregation function ϕ_h is growing in the sixth coordinate since we are searching the changes from right to left in the image (see Figure 2 and Figure 3) and we want that the function increases in the position (i,j+1) that corresponds with the sixth coordinate.

VI. CONCLUSIONS

The present paper explores the concept of conditioned monotonicity considering some specific conditioning structures (covers, partitions and projections). On this basis we study some situations where conditioned monotonicity ensures monotonicity. Finally we use these definitions and properties to consider some generalized pre-aggregation and aggregation functions that correspond to operators commonly applied to image preprocessing problems. In this way we include under the umbrella of generalized pre-aggregation functions those operators that were previously not considered as aggregations.

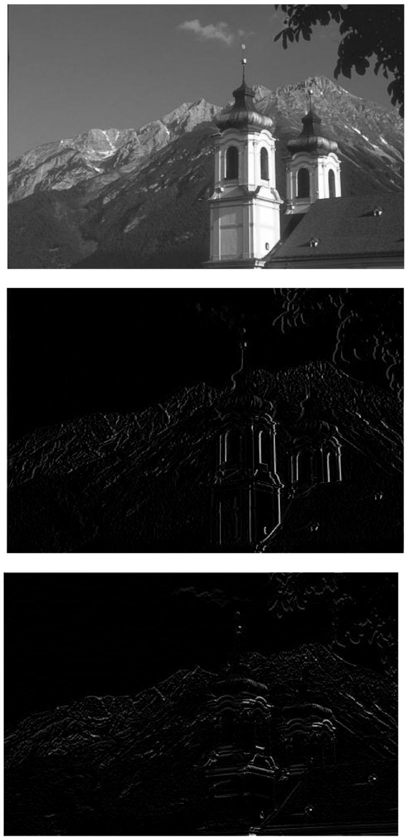


Fig. 2. Original, vertical and horizontal images after applying the Generalized Pre-Agregation ϕ_v and ϕ_h ($I^{\phi_v}_{126007}$ image from Berkeley dataset).

Fig. 3. Original, vertical and horizontal images after applying the Generalized Pre-Agregation ϕ_v and ϕ_h ($I^{\phi_v}_{119082}$ image from Berkeley dataset).

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