# The ordering methods of interval-valued fuzzy cardinal numbers with application in an uncertain decision making


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***Abstract*—In this contribution we propose new methodology to compare interval-valued fuzzy cardinal numbers (IVFCN). The new methods are based on interval subsethood measures which take into account widths of the intervals. An application of introduced methodology is presented on an example of decision algorithm for medical diagnosis support.**

***Index Terms*—interval-valued cardinal number, interval-valued aggregation function, selection of interval-valued cardinal numbers**


## I. Introduction

Many new methods and theories take into account imprecision and uncertainty since Zadeh introduced fuzzy sets [1] in 1965. Especially, in the literature, many authors have proposed different approaches to the definition of different types of distance measures, similarity measures and subsethood, inclusion or equivalence measures between fuzzy sets (e.g. [2], [3], [4], [5], [6]). We focus on subsethood measures, many applications of them have been proposed and they have been adapted and applied in different settings [7], [8]. As extensions of classical fuzzy set theory, intuitionistic fuzzy sets [9] and interval-valued fuzzy sets [10], [11] they are very useful in dealing with imprecision and uncertainty (see [12] for more details). In this setting, different proposals for subsethood measures between interval-valued fuzzy sets have been proposed [5], [13].

The motivation of the present paper is to propose a more natural tool for estimating the degree of subsethood between interval-valued fuzzy sets taking into account the widths of the intervals and we assume that the precise membership degree of an element in a given set is a number included in the membership interval. For such interpretation, the width of the membership interval of an element reflects the lack of precise membership degree of that element. Hence, the fact that two elements have the same membership intervals does not necessarily mean that their corresponding membership values are the same. This is why we have taken into account the importance of the notion of width of intervals while defining new types of subsethood measures. This approach reflects the meaning of the interval values and is better adapted to real applications. That allowed to construct an effective method for comparing and ordering interval valued fuzzy cardinal numbers. Such numbers are of great importance in solving decision problems in which uncertainty occurs (see [14], [15], [16], [17], [18], [19]).

The paper is organized as follows. In Section 2, basic information of interval-valued fuzzy setting are recalled. Next, in Section 3, an interval subsethood measure for interval-valued fuzzy values by using partial or linear orders is presented. Especially, some construction methods are proposed. Finally, we propose the methodology to compare of interval-valued fuzzy cardinal numbers and its application in decision model of medical diagnosis support.

## II. Preliminaries

### *A. Interval-valued fuzzy set theory. Orders in the interval-valued fuzzy settings*

We use the following notation for the set of intervals

$$L^I = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in [0, 1] \ \ and \ \ \underline{x} \leq \overline{x}\},$$

which are the basis of interval-valued fuzzy sets introduced by L. A. Zadeh [10] and R. Sambuc [11].

**Definition 1** (cf. [11], [10])**.** An interval-valued fuzzy set IVFS $\widetilde{A}$ in $X$ is a mapping $\widetilde{A} : X \to L^I$ such that $\widetilde{A}(x) = [\underline{A}(x), \overline{A}(x)] \in L^I$ for $x \in X$, where

$$\widetilde{A} \cap \widetilde{B} = \{\langle x, [\min\{\underline{A}(x), \underline{B}(x)\}, \min\{\overline{A}(x), \overline{B}(x)\}]\rangle : x \in X\},$$

$$\widetilde{A} \cup \widetilde{B} = \{\langle x, [\max\{\underline{A}(x), \underline{B}(x)\}, \max\{\overline{A}(x), \overline{B}(x)\}]\rangle : x \in X\}$$

The well-known classical monotonicity (partial order) for intervals is of the form

$$[\underline{x}, \overline{x}] \leq_{L^I} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \leq \underline{y} \ \text{ and } \ \overline{x} \leq \overline{y}, \tag{1}$$

where $[\underline{x}, \overline{x}] <_{L^I} [\underline{y}, \overline{y}] \Leftrightarrow$
$[\underline{x}, \overline{x}] \leq_{L^I} [\underline{y}, \overline{y}]$ *and* $(\underline{x} < \underline{y}$ *or* $\overline{x} < \overline{y})$.

In $L^I$ the operations joint and meet are defined respectively

$$[\underline{x}, \overline{x}] \vee [\underline{y}, \overline{y}] = [\max(\underline{x}, \underline{y}), \max(\overline{x}, \overline{y})], \tag{2}$$

$$[\underline{x}, \overline{x}] \wedge [\underline{y}, \overline{y}] = [\min(\underline{x}, \underline{y}), \min(\overline{x}, \overline{y})]. \tag{3}$$

Note that the structure $(L^I, \vee, \wedge)$ is a complete lattice, with the partial order $\leq_{L^I}$, where

$$\mathbf{1} = [1, 1] \ \text{ and } \ \mathbf{0} = [0, 0]$$

are the greatest and the smallest element of $(L^I, \leq_{L^I})$, respectively.

We are interested in extending the partial order $\leq_{L^I}$ to a linear order, solving the problem of existence of incomparable elements. We recall the notion of an admissible order, which was introduced in [20] and studied, for example, in [21] and [22]. The linearity of the order is needed in many applications of real problems, in order to be able to compare any two interval data [23].

**Definition 2** ([20])**.** An order $\leq_{Adm}$ in $L^I$ is called admissible if it is linear and satisfies that for all $x, y \in L^I$, such that $x \leq_{L^I} y$, then $x \leq_{Adm} y$.

**Proposition 1** ([20])**.** *Let $B_1, B_2 : [0, 1]^2 \to [0, 1]$ be two continuous aggregation functions, such that, for all $x = [\underline{x}, \overline{x}], y = [\underline{y}, \overline{y}] \in L^I$, the equalities $B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y})$ and $B_2(\underline{x}, \overline{x}) = B_2(\underline{y}, \overline{y})$ hold if and only if $x = y$. If the order $\leq_{B_{1,2}}$ on $L^I$ is defined by $x \leq_{B_{1,2}} y$ if and only if*

$$B_1(\underline{x}, \overline{x}) < B_1(\underline{y}, \overline{y}) \quad \text{or}$$

$$(B_1(\underline{x}, \overline{x}) = B_1(\underline{y}, \overline{y}) \ \text{ and } B_2(\underline{x}, \overline{x}) \leq B_2(\underline{y}, \overline{y})),$$

*then $\leq_{B_{1,2}}$ is an admissible order on $L^I$.*

**Example 1** ([20])**.** The following are special cases of admissible linear orders on $L^I$:

- The Xu and Yager order:

$$\begin{aligned} [\underline{x}, \overline{x}] \leq_{XY} [\underline{y}, \overline{y}] \Leftrightarrow & \ \underline{x} + \overline{x} < \underline{y} + \overline{y} \ \text{or} \\ & (\overline{x} + \underline{x} = \overline{y} + \underline{y} \ \text{ and } \ \overline{x} - \underline{x} \leq \overline{y} - \underline{y}). \end{aligned} \tag{4}$$

- The first lexicographical order (with respect to the first variable), $\leq_{Lex1}$ defined as:

$$\begin{aligned} [\underline{x}, \overline{x}] \leq_{Lex1} [\underline{y}, \overline{y}] \Leftrightarrow & \ \underline{x} < \underline{y} \ \text{or} \\ & (\underline{x} = \underline{y} \ \text{ and } \ \overline{x} \leq \overline{y}). \end{aligned} \tag{5}$$

- The second lexicographical order (with respect to the second variable), $\leq_{Lex2}$ defined as:

$$\begin{aligned} [\underline{x}, \overline{x}] \leq_{Lex2} [\underline{y}, \overline{y}] \Leftrightarrow & \ \overline{x} < \overline{y} \ \text{or} \\ & (\overline{x} = \overline{y} \ \text{ and } \ \underline{x} \leq \underline{y}). \end{aligned} \tag{6}$$

- The $\alpha\beta$ order, $\leq_{\alpha\beta}$ defined as:

$$\begin{aligned} [\underline{x}, \overline{x}] \leq_{\alpha\beta} [\underline{y}, \overline{y}] \Leftrightarrow & K_\alpha(\underline{x}, \overline{x}) < K_\alpha(\underline{y}, \overline{y}) \ \text{or} \\ & (K_\alpha(\underline{x}, \overline{x}) = K_\alpha(\underline{y}, \overline{y}) \ \text{and} \\ & K_\beta(\underline{x}, \overline{x}) \leq K_\beta(\underline{y}, \overline{y})) \end{aligned} \tag{7}$$

for some $\alpha \neq \beta \in [0, 1]$ and $x, y \in L^I$, where $K_\alpha : [0, 1]^2 \to [0, 1]$ is defined as $K_\alpha(x, y) = \alpha x + (1 - \alpha)y$.

The orders $\leq_{XY}$, $\leq_{Lex1}$ and $\leq_{Lex2}$ are special cases of the order $\leq_{\alpha\beta}$ with $\leq_{0.5\beta}$ (for $\beta > 0.5$), $\leq_{1,0}$, $\leq_{0,1}$, respectively. The orders $\leq_{XY}$, $\leq_{Lex1}$, $\leq_{Lex2}$, and $\leq_{\alpha\beta}$ are admissible linear orders $\leq_{B_{1,2}}$ defined by pairs of aggregation functions, namely weighted means. In the case of the orders $\leq_{Lex1}$ and $\leq_{Lex2}$, the aggregations that are used are the projections $P_1$, $P_2$ and $P_2$, $P_1$, respectively.

**Remark 1.** In the later part we will use the notation $\leq$ both for the partial or admissible linear order, with **0** and **1** as minimal and maximal element of $L^I$, respectively. Notation $\leq_{L^I}$ will be used while the results for the admissible linear orders will be used with the notation $\leq_{Adm}$.

### *B. Interval-valued aggregation functions*

Let us now recall the concept of an interval-valued aggregation function, or an aggregation function on $L^I$ , which is an important notion for many applications. We consider interval-valued aggregation functions both with respect to $\leq_{L^I}$ and $\leq_{Adm}$.

**Definition 3** ([22], [24])**.** An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an interval-valued aggregation function if it is increasing with respect to the order $\leq$ (partial or total) and

$$\mathcal{A}(\underbrace{\mathbf{0}, ..., \mathbf{0}}_{n\times}) = \mathbf{0}, \quad \mathcal{A}(\underbrace{\mathbf{1}, ..., \mathbf{1}}_{n\times}) = \mathbf{1}.$$

A special class of interval-valued aggregation functions is the one formed by the so-called representable interval-valued aggregation functions.

**Definition 4** ([25], [26])**.** An interval-valued aggregation function $\mathcal{A} : (L^I)^n \to L^I$ is said to be representable if there exist aggregation functions $A_1, A_2 : [0, 1]^n \to [0, 1]$ such that

$$\mathcal{A}(x_1, \ldots, x_n) = [A_1(\underline{x}_1, \ldots \underline{x}_n), A_2(\overline{x}_1, \ldots, \overline{x}_n)]$$

for all $x_1, \ldots, x_n \in L^I$, provided that

$$A_1(\underline{x}_1, \ldots \underline{x}_n) \leq A_2(\overline{x}_1, \ldots, \overline{x}_n).$$

**Example 2.** Lattice operations $\wedge$ and $\vee$ on $L^I$ are examples of representable aggregation functions on $L^I$ with respect to the partial order $\leq_{L^I}$, with $A_1 = A_2 = \min$ in the first case and $A_1 = A_2 = \max$ in the second one. However, $\wedge$ and $\vee$ are not interval-valued aggregation functions with respect to $\leq_{Lex1}$, $\leq_{Lex2}$ or $\leq_{XY}$. The following are other examples of representable interval-valued aggregation functions with respect to $\leq_{L^I}$.

- The projections:

$$\mathcal{A}_L([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\underline{x},\overline{x}],\ \mathcal{A}_R([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\underline{y},\overline{y}]. \quad (8)$$

- The representable product:

$$\mathcal{A}_p([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\underline{x}\underline{y},\overline{x}\overline{y}]. \quad (9)$$

- The representable arithmetic mean:

$$\mathcal{A}_{mean}([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\frac{\underline{x}+\underline{y}}{2},\frac{\overline{x}+\overline{y}}{2}]. \quad (10)$$

- The representable geometric mean:

$$\mathcal{A}_{gmean}([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\sqrt{\underline{x}\underline{y}},\sqrt{\overline{x}\overline{y}}]. \quad (11)$$

- The representable power mean:

$$\mathcal{A}_{power}([\underline{x},\overline{x}],[\underline{y},\overline{y}])=[\sqrt{\frac{\underline{x}^2+\underline{y}^2}{2}},\sqrt{\frac{\overline{x}^2+\overline{y}^2}{2}}]. \quad (12)$$

Representability is not the only possible way to build interval-valued aggregation functions with respect to $\leq_{L^I}$ or $\leq_{Adm}$.

**Example 3.** Let $A:[0,1]^2\to[0,1]$ be an aggregation function.

- The function $\mathcal{A}_1:(L^I)^2\to L^I$, where

$$\mathcal{A}_1(x,y)=\begin{cases}[1,1], & if\ (x,y)=([1,1],[1,1]),\\ [0,A(\underline{x},\overline{y})], & otherwise,\end{cases}$$

is a non-representable interval-valued aggregation function with respect to $\leq_{L^I}$.
- The function $\mathcal{A}_2:(L^I)^2\to L^I$ ([27]), where

$$\mathcal{A}_2(x,y)=\begin{cases}[1,1], & if\ (x,y)=([1,1],[1,1])\\ [0,A(\underline{x},\underline{y})], & otherwise\end{cases}$$

is non-representable interval-valued aggregation function with respect to $\leq_{Lex1}$.
- The function $\mathcal{A}_3:(L^I)^2\to L^I$ ([27]), where

$$\mathcal{A}_3(x,y)=\begin{cases}[0,0], & if\ (x,y)=([0,0],[0,0])\\ [A(\overline{x},\overline{y}),1], & otherwise,\end{cases}$$

is non-representable interval-valued aggregation function with respect to $\leq_{Lex2}$.

- $\mathcal{A}_{mean}$ is an aggregation function with respect to $\leq_{\alpha\beta}$ (cf. [21]).
- The following function

$$\mathcal{A}_\alpha(x,y)=[\alpha\underline{x}+(1-\alpha)\underline{y},\alpha\overline{x}+(1-\alpha)\overline{y}] \quad (13)$$

is an interval-valued aggregation function on $L^I$ with respect to $\leq_{Lex1}$, $\leq_{Lex2}$ and $\leq_{XY}$ for $x,y\in L^I$ and $\alpha\in[0,1]$ (cf. [22]).

## III. Subsethood measure

Subsethood, or inclusion measures have been studied mainly from constructive and axiomatic approaches and have been introduced successfully into the theory of fuzzy sets and their extensions. Many researchers have tried to relax the rigidity of Zadeh's definition of subsethood to get a soft approach which is more compatible with the spirit of fuzzy logic. For instance, Zhang and Leung (1996) defended that quantitative methods were the main approaches in uncertainty inference, a key problem in artificial intelligence, so they presented a generalized definition for subsethood measures, called *including degrees*. There exist several works regarding subsethood measures in the interval-valued fuzzy setting [28], [4], [22], [5], [6], [29], however the condition regarding the width of the intervals, with which we deal in this paper, has not been so far considered.

### A. Precedence indicator

We use the notion of an interval subsethood measure for a pair of intervals with the partial and admissible orders and the width of intervals introduced and examined in [30].

**Definition 5.** A function $\mathrm{Prec}:(L^I)^2\to L^I$ is said to be a **precedence indicator** if it satisfies the following conditions for any $a,b,c\in L^I$

- P1 if $a=1_{L^I}$ and $b=0_{L^I}$, then $\mathrm{Prec}(a,b)=0_{L^I}$
- P2 if $a<b$, then $\mathrm{Prec}(a,b)=1_{L^I}$ for any $a,b\in L^I$
- P3 $\mathrm{Prec}(a,a)=[1-w(a),1]$ for any $a\in L^I$
- P4 if $a\leq b\leq c$ and $w(a)=w(b)=w(c)$, then $\mathrm{Prec}(c,a)\leq\mathrm{Prec}(b,a)$ and $\mathrm{Prec}(c,a)\leq\mathrm{Prec}(c,b)$, for any $a,b,c\in L^I$,

where $w(a)=\overline{a}-\underline{a}$.

Let us present two construction methods for such an interval subsethood measure. The first one is given in the following result.

**Proposition 2** ([30])**.** *For $a,b\in L^I$ the operation $\mathrm{Prec}_z:(L^I)^2\to L^I$ is the precedence indicator*

$$\mathrm{Prec}_z(a,b)=\begin{cases}[1-w(a),1], & a=b,\\ 1_{L^I}, & a<b,\\ 0_{L^I}, & otherwise.\end{cases} \quad (14)$$

The second construction method is based on the aggregation and negation functions which play important rule in many applications (e.g. [25], [26], [31], [32]) and is presented in the next theorem. Recall that an interval-valued fuzzy negation $N_{IV}$ is an antytonic operation that satisfies $N_{IV}(0_{L^I})=1_{L^I}$ and $N_{IV}(1_{L^I})=0_{L^I}$ ([33], [34])

**Proposition 3** ([30])**.** *For $a,b\in L^I$ the operation $\mathrm{Prec}_{\mathcal{A}}:(L^I)^2\to L^I$ is the precedence indicator*

$$\mathrm{Prec}_{\mathcal{A}}(a,b)=\begin{cases}[1-w(a),1], & a=b,\\ 1_{L^I}, & a<b,\\ \mathcal{A}(N_{IV}(a),b), & otherwise\end{cases} \quad (15)$$

*for $a, b \in L^I$ and the interval-valued fuzzy negation $N_{IV}$, such that*

$$N_{IV}(a) = [N(\overline{a}), N(\underline{a})] \leq [1-\overline{a}, 1-\underline{a}],$$

*where $N$ is a fuzzy negation and $\mathcal{A}$ is a representable interval-valued aggregation such that $\mathcal{A} \leq \vee$.*

Using the construction methods from Proposition 3 we obtain the following examples.

**Example 4.** The following function is an interval subsethood measure with respect to $\leq_{L^I}$:

$$\mathrm{Prec}_{\mathcal{A}_{meanL^I}}(x,y) = \begin{cases} [1-w(x), 1], & x = y, \\ \mathbf{1}, & x <_{L^I} y, \\ [\frac{1-\overline{x}+\underline{y}}{2}, \frac{1-\underline{x}+\overline{y}}{2}], & otherwise, \end{cases} \tag{16}$$

where $N_{IV}(x) = [1-\overline{x}, 1-\underline{x}]$.
Moreover, the following function is a subsethood measure with respect to $\leq_{Lex2}$:

$$\mathrm{Prec}_{\mathcal{A}_{meanLex2}}(x,y) = \begin{cases} [1-w(x), 1], & x = y, \\ \mathbf{1}, & x <_{Lex2} y, \\ [\frac{\underline{y}}{2}, \frac{1-\overline{x}+\overline{y}}{2}], & otherwise. \end{cases} \tag{17}$$

Using the interval-valued aggregation function $\mathcal{A}_\alpha$ for $\alpha \in [0,1]$, we get the subsethood measure

$$\mathrm{Prec}_{\mathcal{A}_\alpha Lex2}(x,y) = \begin{cases} [1-w(x), 1], & x = y, \\ \mathbf{1}, & x <_{Lex2} y, \\ [(1-\alpha)\underline{y}, \\ \quad \alpha(1-\overline{x}) + (1-\alpha)\overline{y}], & otherwise, \end{cases} \tag{18}$$

where

$$N_{IV}(x) = \begin{cases} \mathbf{1}, & x = \mathbf{0}, \\ [0, 1-\overline{x}], & otherwise, \end{cases}$$

is an interval-valued fuzzy negation with respect to $\leq_{Lex2}$.

**Remark 2** ([22])**.** The aggregation $\mathcal{A}_\alpha$ preserves the width of the intervals of the same width.

Another construction method, which is inspired by the construction presented for generalization of the subsethood measure at paper [30], presents the following proposition.

**Proposition 4.** *The operation*

$$\mathrm{Prec}_w(a,b) = \begin{cases} 1_{L^I}, & a < b, \\ [1-\max(w(a), r(a,b)), 1-r(a,b)], & else \end{cases} \tag{19}$$

*is the precedence indicator with respect to $\leq$, where $r(a,b) = \max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\}$ for $a, b \in L^I$.*

We would like to point out the connection between interval-valued implication functions [22] and the examined interval subsethood measures.

**Remark 3** ([30])**.** Let $a, b, c \in L^I$ and $w(a) = w(b) = w(x)$. If Prec is precedence indicator, then is an interval-valued implication function.

Moreover, we observe that $w(a) < w(b)$ (respectively, $w(a) = w(b)$) if and only if $\mathrm{Prec}(b,b) < \mathrm{Prec}(a,a)$ (respectively, $\mathrm{Prec}(b,b) = \mathrm{Prec}(a,a)$) for $a, b \in L^I$.

## IV. Interval-valued fuzzy cardinal numbers

In this section we briefly introduce main ideas about cardinalities of IVFSs. More details can be found in the monographs [35], [15].

In further part we will use the following notations:

- For given fuzzy set $A$ a symbol $[A]_i$ is defined as:

$$[A]_i := \bigvee\{t \in (0,1] : |A_t| \geq i\} \text{ for } i \in \mathbb{N}.$$

- Function $f : [0,1] \to [0,1]$ is called cardinality pattern if it meets the following conditions:
  1) is nondecreasing i.e. $\forall_{a,b\in[0,1]} f(a) \leq f(b)$ if $a \leq b$,
  2) and meets limit conditions $f(0) = 0$ i $f(1) = 1$.
- Symbol $\cap_T$ means the triangular norm and $N$ the fuzzy negation.

### A. Generalized fuzzy cardinal numbers

1) **Generalized cardinal number FGCount** is interpreted as a degree to which fuzzy set $A$ has at least $k$ elements

$$FG_f(k) := f([A]_1) \cap_T f([A]_2) \cap_T \ldots \cap_T f([A]_k)$$

for $k \in \mathbf{N}$.

2) **Generalized cardinal number FLCount** is interpreter as a degree to which $A$ includes at most $k$ elements

$$FL_f(k) :=$$

$$N(f([A]_{k+1})) \cap_T N(f([A]_{k+2})) \cap_T \cdot \cap_T \; N(f([A]_n))$$

for $k \in \mathbf{N}$.

3) **Generalized cardinal number FECount** expresses the degree to which $A$ has exactly $k$ elements where

$$FE_f(k) := f([A]_1) \cap_T f([A]_2) \cap_T \ldots \cap_T f([A]_k) \cap_T$$

$$N(f([A]_{k+1})) \cap_T N(f([A]_{k+2})) \cap_T \ldots \cap_T N(f([A]_n))$$

for $k \in \mathbf{N}$.

$FE_f$ is the intersection of $FG_f$ and $FL_f$. It may be perceived as the **'actual' generalized cardinal number** of a fuzzy set $A$

### B. Fuzzy Cardinality of IVFS

Cardinalities of interval-valued fuzzy sets are defined in a natural manner using cardinalities of fuzzy sets described in in previous section.

For a finite interval-valued fuzzy set $\widetilde{A} = (\underline{A}, \overline{A})$ fuzzy type cardinalities are defined as interval-valued fuzzy sets in $\mathbb{N}$ (see [35]).

**Definition 6.** 1) *f-FGCount* of IVFS $\widetilde{A}$ for a given cardinality pattern $f$ is defined as:

$$\widetilde{FG}_f(\widetilde{A}) = [FG_f(\underline{A}), FG_f(\overline{A})], \tag{20}$$

i.e. for $k \in \mathbb{N}$:

$$\begin{aligned}\widetilde{FG}_f(\widetilde{A})(k) =& [FG_f(\underline{A})(k), FG_f(\overline{A})(k)] = \\ =& [f([\underline{A}]_k), f([\overline{A}]_k)],\end{aligned} \tag{21}$$

where $FG_f(\underline{A})$ and $FG_f(\overline{A})$ are the fuzzy cardinalites defined in previous section.

2) *f-FLCount* of IVFS $\widetilde{A}$ for a given cardinality pattern $f$ is defined as:

$$\widetilde{FL}_f(\widetilde{A}) = [FL_f(\overline{A}), FL_f(\underline{A})], \tag{22}$$

i.e. for $k \in \mathbb{N}$:

$$\begin{aligned}\widetilde{FL}_f(\widetilde{A})(k) =& [FL_f(\underline{A})(k), FL_f(\overline{A}(k)] = \\ =& [1 - f([\overline{A}]_{k+1}), 1 - f([\underline{A}]_k)],\end{aligned} \tag{23}$$

where $FL_f(\underline{A})$ and $FL_f(\overline{A})$ are the fuzzy cardinalites defined in previous section.

3) *f-FECount* of IVFS $\widetilde{A}$ for a given cardinality pattern $f$ is defined as:

$$\widetilde{FE}_f(\widetilde{A}) = \widetilde{FG}_f(\widetilde{A}) \cap \widetilde{FL}_f(\widetilde{A}), \tag{24}$$

i.e. for $k \in \mathbb{N}$:

$$\begin{aligned}\widetilde{FE}_f(\widetilde{A})(k) = [&f([\underline{A}]_k) \wedge (1 - f([\overline{A}]_{k+1})), \\ &f([\overline{A}]_k) \wedge (1 - f([\underline{A}]_{k+1})].\end{aligned} \tag{25}$$

To simplify the notations, f-FECount of an IVFS will be denoted by $\widetilde{\sigma}$ and we will call it Interval-Valued Fuzzy Cardinal Number (in short IVFCN).

### C. An idea of comparability algorithm

In many decision-making applications, an important problem to solve is comparing the cardinalities of IVFSs. The example on Figure 1 shows two IVFCN $\widetilde{\sigma}_1$, $\widetilde{\sigma}_2$ and illustrates how non-trivial task is to effectively compare such numbers, if we accept the epistemic interpretation of meanings of the intervals, where the real values are somewhere in the given ranges.

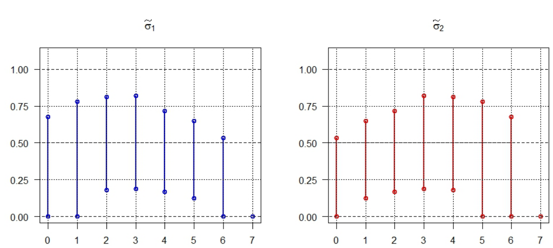


Fig. 1. Two $IVFCNs$ $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$

An idea of our proposed algorithm is as follows. In the first step divide a sum of supports of both IVFCNs $S = supp(\widetilde{6\sigma_1}) \cup supp(\widetilde{\sigma}_2)$ into two equal parts $S_1$ and $S_2$. In the second step compare precedence indicators from both parts separately. The idea is that: if $\widetilde{\sigma}_1$ to be greater than $\widetilde{\sigma}_2$ the appropriate measure of precedence should be smaller on $S_1$, and greater on $S_2$ and vice versa if $\widetilde{\sigma}_1$ to be smaller than $\widetilde{\sigma}_2$ the appropriate measure of precedence should be greater on $S_1$, and smaller on $S_2$. Figure 2 presents example of two $IVFCNs$ with appropriate values of precedence indicators.

To define algorithm more formally, we need to introduce some basics notations.

**Definition 7.** Representative $Rep(x) \in \mathbb{R}$ of an interval $x \in L^I$ for $a \in [0, 1]$ is defined as:

$$Rep(x) = \underline{x} + a * w(x).$$

In a special case value of representative of an interval can lead to middle or bounds of interval:

1) if $a = 0$ then $Rep(x) = \underline{x}$ - lower bound of the interval;
2) if $a = 1$ then $Rep(x) = \overline{x}$ - upper bound of the interval;
3) if $a = 0.5$ then $Rep(x) = (\underline{x} + \overline{x})/2$ - middle of the interval.

**Definition 8.** Set of precedence indicators for given interval-valued fuzzy sets $X, Y$ is defined as follows:

$$Prec_I(X, Y) := \{Prec(x_i, y_i) : \text{ for all } x_i \in X, y_i \in Y\},$$

where $Prec \in \{\mathrm{Prec}_z, \mathrm{Prec}_{\mathcal{A}}, \mathrm{Prec}_w\}$.

Next, we define the Immersion measure using the previously introduced concept of the set representative.

**Definition 9.** Immersion measure for given interval-valued fuzzy sets $X, Y$ and support set $S$ is defined as follows:

$$Conn(X, Y, S) = \sum_{i \in S} Comp(x_i, y_i),$$

where

$$Comp(x, y) = \begin{cases} 1, & Rep(x) <= Rep(y), \\ 0, & otherwise. \end{cases}$$

### D. The comparability algorithm

Algorithm to rank two interval-valued fuzzy cardinal numbers (IVFCNs) $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ more formally we can define in a following steps:

Step 1 Divide support $S$ of sum of both interval-valued fuzzy cardinal numbers $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$, i.e.,

$$S = supp(\widetilde{\sigma}_1 \cup \widetilde{\sigma}_2)$$

into two parts:
$S_1 = [min(S), (max(S) - min(S)/2)]$,
$S_2 = [(max(S) - min(S))/2, max(S)]$.

Step 2 Construct precedence sets on both parts of support, i.e.,
$I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1} = Prec_I(\widetilde{\sigma}_1, \widetilde{\sigma}_2, S_1)$,
$I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1} = Prec_I(\widetilde{\sigma}_2, \widetilde{\sigma}_1, S_1)$,
$I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_2} = Prec_I(\widetilde{\sigma}_1, \widetilde{\sigma}_2, S_2)$,
$I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2} = Prec_I(\widetilde{\sigma}_2, \widetilde{\sigma}_1, S_2)$.

Step 3 Calculate immersion measures.
$C_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1} = Conn(I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1}, I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1})$,

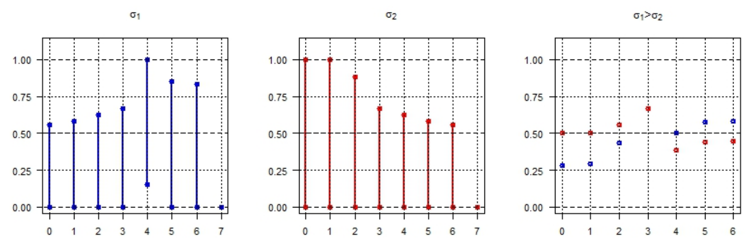


Fig. 2. Two $IVFCNs$ $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ with set of precedence indicators ($\widetilde{\sigma}_1 > \widetilde{\sigma}_2$)

$$C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1} = Conn(I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1}, I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1}),$$
$$C_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_2} = Conn(I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_2}, I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2}),$$
$$C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2} = Conn(I_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2}, I_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_2}).$$

Step 4 Compare Immersion measure on the both parts of the support.

**if** $C_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1} > C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1}$
*and* $C_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_2} < C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2}$ **then**
  $\widetilde{\sigma}_1 > \widetilde{\sigma}_2$
**else if** $C_{\widetilde{\sigma}_1,\widetilde{\sigma}_2,S_1} < C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_1}$
*and* $C_{\sigma_1,\sigma_2,S_2} > C_{\widetilde{\sigma}_2,\widetilde{\sigma}_1,S_2}$ **then**
  $\widetilde{\sigma}_1 < \widetilde{\sigma}_2$
**else**
  $\widetilde{\sigma}_1 = \widetilde{\sigma}_2$

Figure 3 presents result of algorithm for given IVFCN, with corresponding precedence sets.

## V. Application in Decision Making

The presented methodology for comparing IVFCNs can be applied in the decision model used in the OvaExpert system. OvaExpert is an intelligent decision support system for the diagnosis of ovarian tumors. The system was developed as a result of joint research of two Polish research centers: the Division of Gynecologic Surgery of the Poznan University of Medical Sciences and the Department of Imprecise Information Processing Methods, Faculty of Mathematics. More detailed information about the system can be found in [15].

Figure 4 presents diagram showing OvaExpert counting approach for making decisions. This method of decision making utilised in the system is based on voting strategy with counting. On input system gets incomplete information about patient. In step 1 many diagnostic models are computed and it results as IVFS of decisions. Then in step 2 two IVFSNs are computed (which represents positive and negative diagnosis). And finally in step 3 comparison of this two IVFSNs resulting decision.

### *A. Decision Making Algorithm based on bipolar voting strategy*

The idea behind decision algorithm is to use bipolar perspective on IVFS. Because such an IVFS contains information on uncertainty level, it carries both information supporting and rejecting the decision. This property of IVFS is used in decision algorithm. The basic idea behind this algorithm consists of a couple of steps:

- As an input we have two IVFS's $P$ and $C$ (representing number of decision's $D_{pro}$ and $D_{contra}$ supporting given decision):
  - $P = \sigma(D_{pro})$ - representing the number of decisions 'for';
  - $C = \sigma(D_{contra})$ - representing the number of decisions 'against';
- To make decisions, we must choose a set that is more numerous e.g. decide if (or vice versa):

$$P < C.$$

### *B. Performance of the model*

The presented algorithm have been tested on real medical data. These data described 388 cases of patients diagnosed and treated in the Division of Gynecological Surgery, Poznan University of Medical Sciences, between 2005 and 2015. Out of them 61% have been diagnosed as suffering from benign tumours and 39% as suffering from malign tumours. Moreover, 56% of patients had full diagnostic (no test required by diagnostic scales was missing), 40% had significant amounts of missing data varying from (0%, 50%], and for the remaining ones 50% of data was missing. Detailed description of data used for evaluation can be found in [36]. More information on the data format used and technical details can be found in [16].

The goal of evaluation was to select a decision algorithm that would best classify malignity cases with the top possible decisiveness. There are many measures of classification quality. In the medical problem under consideration high sensitivity and specificity is crucial. Additionally, selection of a right, unified quality measure constitutes a difficult task (see [37]). Thus, we decided to use a cost matrix for main model assessment.

Specific value of costs matrix have been selected in cooperation with experts in ovarian cancer diagnosis. The Table I presents costs (penalties) attributed to classifiers for incorrect decisions. Correct decisions (TP and TN) do not receive a penalty. A classifier receives top penalty in the case of committing type II error (FN), i.e., if a patient with malign tumour is classified as a benign case. Penalty for FP type errors was half of it, as unjustified operation is still dangerous for a patient but death risk is much lower. Additionally, there are

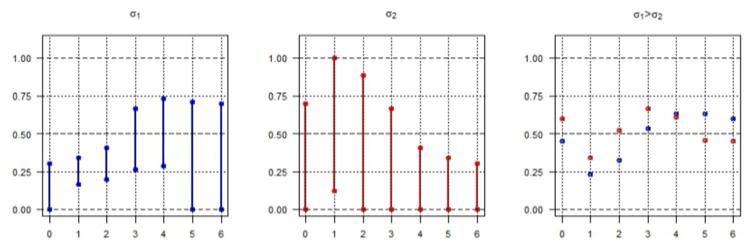


Fig. 3. Two $IVFCNs$ $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ with set of precedence indicators and comparison result ($\widetilde{\sigma}_1 > \widetilde{\sigma}_2$)

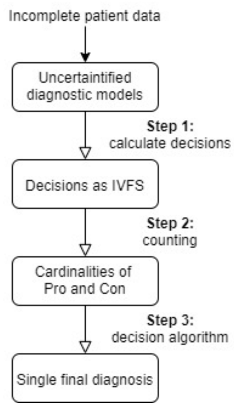


Fig. 4. OvaExpert counting approach for making medical decision

also penalties for the classifier for failure to make a decision (NA). The penalty is lower, as in such a case the patient needs additional diagnostics and will probably be directed to a more experienced specialist who would make a correct diagnosis. However, we differentiated penalties for lack of decision in positive (malign) is twice as high as in the negative (benign) case. The Figure 5 presents performance result for the best

TABLE I
COST MATRIX USED FOR MODELS ASSESSMENT.

| | | **predicted** | | |
|---|---|---|---|---|
| | | benign | malignant | NA |
| **acual** | benign | 0 | 2.5 | 1 |
| | malignant | 5 | 0 | 2 |

four algorithms based on counting (with different cardinality functions and with the precedence indicator $\mathrm{Prec}_w$, which gave the optimal results) with comparison to OWA baseline model (which is current decision method in OvaExpert system). As can be seen the best version reach better result in total cost with very similar values of other performance measures (like sensitivity and specificity). Which allows us to initially positively evaluate the new method, but it requires further evaluation on larger data sets. We are currently working on a more in-depth analysis of results based on other data sets.

## VI. CONCLUSIONS AND FUTURE PLANS

In this presentation, we discuss possible axiomatically definitions of inclusion for interval-valued fuzzy setting, where the notion with widths of intervals involved. Moreover, the inclusion measure and new algorithm of comparing and ranking cardinalities of interval-valued fuzzy sets were applied in decision making algorithm. The considered model used a intelligent decision support system (OvaExpert) for the diagnosis of ovarian tumors. We are currently working on several other types of ranking methods that uses inclusion measures and aggregation methods.

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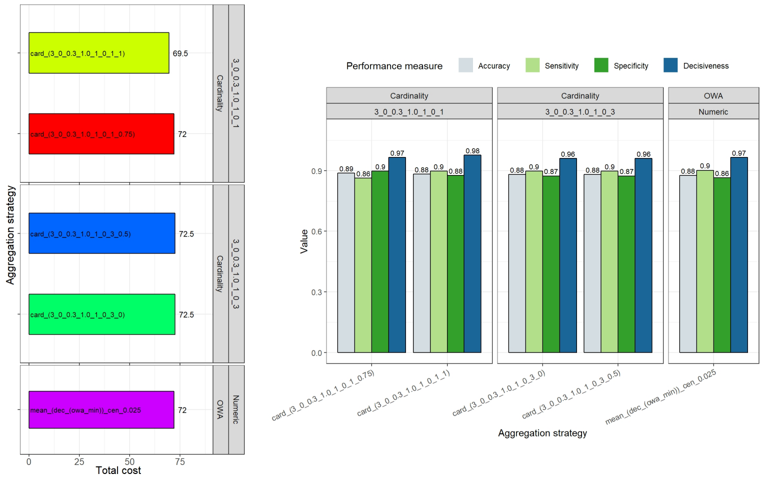


Fig. 5. Performance of counting methods and comparison to baseline method


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