Midpoint Representation of Fuzzy-Valued Functions and Applications

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***Abstract*—We present new results in the calculus for fuzzy-valued functions of a single real variable. We adopt extensively the midpoint-radius representation of intervals in the real half-plane and show its usefulness in fuzzy calculus. Concepts related to convergence and limits, continuity, level-wise gH-differentiability have interesting and useful midpoint expressions. Partial orders for fuzzy numbers and extremal points (min and max) for fuzzy functions associated to a partial order are discussed and analysed in detail. Graphical examples and pictures accompany the presentation.**

I. Fuzzy Intervals Notation

To describe and represent basic concepts and operations for real intervals, the well-known midpoint-radius representation (*midpoint* for short) is very useful (see [6]): for a given interval $A = [a^-, a^+]$, the midpoint $\widehat{a}$ and the radius $\widetilde{a}$ are, respectively,

$$\widehat{a} = \frac{a^+ + a^-}{2} \text{ and } \widetilde{a} = \frac{a^+ - a^-}{2},$$

so that $a^- = \widehat{a} - \widetilde{a}$ and $a^+ = \widehat{a} + \widetilde{a}$. We denote an interval by $A = [a^-, a^+]$ or, in midpoint notation, by $A = (\widehat{a}; \widetilde{a})$; so, the family of all compact intervals in $\mathbb{R}$ is denoted by

$$\mathcal{K}_C = \{(\widehat{a}; \widetilde{a}) \mid \widehat{a}, \widetilde{a} \in \mathbb{R} \text{ and } \widetilde{a} \geq 0\}.$$

Given $A = [a^-, a^+]$, $B = [b^-, b^+] \in \mathcal{K}_C$ and $\tau \in \mathbb{R}$, we have the following classical (Minkowski-type) addition, scalar multiplication and difference (see [1], [2], [8], [9], [10]):

- $A \oplus_M B = [a^- + b^-, a^+ + b^+]$,
- $\tau A = \{\tau a : a \in A\} = \begin{cases} [\tau a^-, \tau a^+], & \text{if } \tau \geq 0, \\ [\tau a^+, \tau a^-], & \text{if } \tau \leq 0 \end{cases}$,
- $-A = (-1)A = [-a^+, -a^-]$,
- $A \ominus_M B = A \oplus_M (-1)B = [a^- - b^+, a^+ - b^-]$.

Using midpoint notation, the previous operations, for $A = (\widehat{a}; \widetilde{a})$, $B = (\widehat{b}; \widetilde{b})$ and $\tau \in \mathbb{R}$ are:

- $A \oplus_M B = (\widehat{a} + \widehat{b}; \widetilde{a} + \widetilde{b})$,
- $\tau A = (\tau\widehat{a}; |\tau|\widetilde{a})$,
- $-A = (-\widehat{a}; \widetilde{a})$,
- $A \ominus_M B = (\widehat{a} - \widehat{b}; \widetilde{a} + \widetilde{b})$.

Generally, the subscript $(\cdot)_M$ in the notation of Minkowski-type operations will be removed, and classical addition and subtraction will be denoted by $\oplus$ and $\ominus$, respectively.

The gH-difference of two intervals always exists (see [7], [11], [12], [14]) and, in midpoint notation, is equal to

$$A \ominus_{gH} B = (\widehat{a} - \widehat{b}; |\widetilde{a} - \widetilde{b}|) \subseteq A \ominus_M B.$$

The Minkowski addition $\oplus$ is associative and commutative and with neutral element $\{0\}$; hereafter 0 will also denote the singleton $\{0\}$. In general, additive simplification is not valid, i.e., $(A \oplus B) \ominus B \neq A$. Instead, we always have $A \ominus_{gH} A = 0$ and $(A \oplus B) \ominus_{gH} B = A$, $\forall A, B \in \mathcal{K}_C$ (and other properties that will be given in the following, when needed).

If $A \in \mathcal{K}_C$, we will denote by $len(A) = a^+ - a^- = 2\widetilde{a}$ the *length* of interval A. Remark that $\alpha A \oplus \beta A = (\alpha + \beta)A$ only if $\alpha\beta \geq 0$ (except for trivial cases) and that $A \ominus_{gH} B = A \ominus B$ only if A and/or B are singletons.

The introduction of the addition $\oplus$ and two differences $\ominus$, $\ominus_{gH}$ for intervals is not motivated here as an attempt to define some "true" arithmetic in $\mathcal{K}_C$; these operations are each-other strongly related and their properties motivate their (appropriate) use in the context of interval and fuzzy analysis and calculus (see [3], [4], [17], [18]).

For two intervals $A, B \in \mathcal{K}_C$ the Pompeiu–Hausdorff distance $d_H : \mathcal{K}_C \times \mathcal{K}_C \to \mathbb{R}_+ \cup \{0\}$ is defined by

$$d_H(A, B) = \max\left\{\max_{a \in A} d(a, B), \max_{b \in B} d(b, A)\right\}$$

with $d(a, B) = \min_{b \in B} |a - b|$. The following properties are well known (see [3], [16]):

$$d_H(\tau A, \tau B) = |\tau| d_H(A, B), \forall \tau \in \mathbb{R},$$
$$d_H(A \oplus C, B \oplus C) = d_H(A, B),$$
$$d_H(A \oplus B, C \oplus D) \leq d_H(A, C) + d_H(B, D).$$

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It is known (see [12], [15], [16]) that $d_H(A,B) = \left\|A \ominus_{gH} B\right\|$ where for $C \in \mathcal{K}_C$, the quantity $\|C\| = \max\{|c|\,;c \in C\} = d_H(C,\{0\})$ is called the *magnitude* of C and an immediate property of the gH-difference for $A, B \in \mathcal{K}_C$ is

$$d_H(A,B) = 0 \iff A \ominus_{gH} B = 0 \iff A = B. \tag{1}$$

It is also well known that $(\mathcal{K}_C, d_H)$ is a complete metric space (see [3]). The concepts of a convergent sequence of intervals $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathcal{K}_C$ is considered in the metric space $\mathcal{K}_C$, endowed with the d_H distance (as in [2], [7], [10]):

Definition 1: We say that $\lim_{n\to\infty} A_n = A$ if and only if for any real $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}$ such that $d_H(A_n, A) < \varepsilon$ for all $n > n_\varepsilon$.

The following equivalence is always true, as it is a trivial application of (1):

$$\lim_{n\to\infty} A_n = A \text{ if and only if } \lim_{n\to\infty}\left(A_n \ominus_{gH} A\right) = 0. \tag{2}$$

A fuzzy set on $\mathbb{R}^n$ is a mapping ([3]) $u : \mathbb{R}^n \to [0,1]$; we denote its α-level set as $[u]_\alpha = \{x \in \mathbb{R}^n : \ u(x) \geq \alpha\}$ for any $\alpha \in [0,1]$. The *support* of u is $supp(u) = \{x \in \mathbb{R}^n | u(x) > 0\}$. 0-level of u is defined by $[u]_0 = cl(supp(u))$ where cl(M) means the closure of subset $M \subset \mathbb{R}^n$.

Definition 2: A fuzzy set u on $\mathbb{R}$ is a *fuzzy number* if:

(i) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(ii) u is a convex fuzzy set (i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}, \ \forall t \in [0,1], \ x, y \in \mathbb{R}$),
(iii) u is upper semi-continuous on $\mathbb{R}$,
(iv) $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the family of fuzzy numbers. For any $u \in \mathbb{R}_F$ we have $[u]_\alpha \in \mathcal{K}_C$ for all $\alpha \in [0,1]$ and thus the α-levels of a fuzzy number are given by $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$, $u_\alpha^-, u_\alpha^+ \in \mathbb{R}$ for all $\alpha \in [0,1]$; in midpoint notation, we will write $[u]_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$ so that $u_\alpha^- = \widehat{u}_\alpha - \widetilde{u}_\alpha$ and $u_\alpha^+ = \widetilde{u}_\alpha + \widehat{u}_\alpha$. If $[u]_1$ is a singleton then we say that u is a fuzzy number. Triangular fuzzy numbers are determined by three real numbers $a \leq b \leq c$ and denoted by $u =< a, b, c >$, with α-levels

$$[u]_\alpha = [a + (b-a)\alpha, c - (c-b)\alpha],$$

for all $\alpha \in [0,1]$.
Addition $u + v$ and scalar multiplication λu are defined level-wise, in terms of all the α-cuts of u, v: for every $\alpha \in [0,1]$ $[u+v]_\alpha = [u]_\alpha + [v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] = (\widehat{u}_\alpha + \widehat{v}_\alpha; \widetilde{u}_\alpha + \widetilde{v}_\alpha)$ and $[\lambda u]_\alpha = [\min\{\lambda u_\alpha^-, \lambda u_\alpha^+\}, \max\{\lambda u_\alpha^-, \lambda u_\alpha^+\}] = (\lambda\widehat{u}_\alpha; |\lambda|\widetilde{u}_\alpha)$.

A notion of fuzzy difference that is somewhat more general than the gH-difference, which we denotes as LgH-difference, is the following:

Definition 3: (see [4], [15]) For given two fuzzy numbers u, v, the level-wise generalized Hukuhara difference (LgH-difference, for short) is defined as the set of interval-valued gH-differences

$$u \ominus_{LgH} v = \left\{w_\alpha \quad | \quad w_\alpha = [u]_\alpha \ominus_{gH} [v]_\alpha, \quad \alpha \in [0,1]\right\},$$

that is, for each $\alpha \in [0,1]$, either $[u]_\alpha = [v]_\alpha + w_\alpha$ or $[v]_\alpha = [u]_\alpha - w_\alpha$.

In relation with the defined difference, we consider the concept of LgH-differentiability (see [4])

Definition 4: Let $x_0 \in]a,b[$ and h be such that $x_0 + h \in]a,b[$, then the level-wise gH-derivative (LgH-derivative for short) of a function $F :]a,b[\to \mathbb{R}_F$ at x_0 is defined as the set of interval-valued gH-derivatives, if they exist for all $\alpha \in [0,1]$,

$$F'_{LgH}(x_0)_\alpha = \lim_{h\to 0} \frac{1}{h}\left([F(x_0+h)]_\alpha \ominus_{gH} [F(x_0)]_\alpha\right); \tag{3}$$

more precisely, if $F'_{LgH}(x_0)_\alpha$ is a compact interval for all $\alpha \in [0,1]$, we say that F is level-wise generalized Hukuhara differentiable (LgH-differentiable for short) at x_0 and the family of intervals $\left\{F'_{LgH}(x_0)_\alpha \quad | \quad \alpha \in [0,1]\right\}$ is the LgH-derivative of F at x_0, denoted by $F'_{LgH}(x_0)$.

Also one-side derivatives can be considered; the right LgH-derivative of F at x_0 is $F'_{(r)LgH}(x_0) = \lim_{h\searrow 0} \frac{1}{h}\left([F(x_0+h)]_\alpha \ominus_{LgH} [F(x_0)]_\alpha\right)$ while, to the left, it is defined as $F'_{(l)LgH}(x_0) = \lim_{h\nearrow 0} \frac{1}{h}\left([F(x_0+h)]_\alpha \ominus_{gH} [F(x_0)]_\alpha\right)$.

The LgH-derivative exists at x_0 if and only if the left and right derivatives at x_0 exist and are the same interval.

In terms of midpoint representation $[F(x)]_\alpha = \left(\widehat{f}_\alpha(x); \widetilde{f}_\alpha(x)\right)$, for all $\alpha \in [0,1]$, we can write

$$\begin{aligned}
\frac{[F(x+h)]_\alpha \ominus_{gH} [F(x)]_\alpha}{h} &= \left(\widehat{\Delta}_{gH} F_\alpha(x,h); \widetilde{\Delta}_{gH} F_\alpha(x,h)\right) \\
\text{where } \widehat{\Delta}_{gH} F_\alpha(x,h) &= \frac{\widehat{f}_\alpha(x+h) - \widehat{f}_\alpha(x)}{h}, \\
\text{and } \widetilde{\Delta}_{gH} F_\alpha(x,h) &= \left|\frac{\widetilde{f}_\alpha(x+h) - \widetilde{f}_\alpha(x)}{h}\right|.
\end{aligned}$$

Taking the limit for $h \longrightarrow 0$, we obtain the LgH-derivative of F_α, if and only if the limits $\lim_{h\longrightarrow 0} \frac{\widehat{f}_\alpha(x+h) - \widehat{f}_\alpha(x)}{h}$, $\lim_{h\longrightarrow 0^+} \left|\frac{\widetilde{f}_\alpha(x+h) - \widetilde{f}_\alpha(x)}{h}\right|$ and $\lim_{h\longrightarrow 0^-} \left|\frac{\widetilde{f}_\alpha(x+h) - \widetilde{f}_\alpha(x)}{h}\right|$ exist in $\mathbb{R}$ and the last two have the same absolute value; remark that the midpoint function $\widehat{f}_\alpha$ is required to admit the ordinary derivative at x. With respect to the left and right limits above, the existence of the left and right derivatives $\widetilde{f}'_{l-\alpha}(x)$ and $\widetilde{f}'_{r-\alpha}(x)$ is required with $\left|\widetilde{f}'_{l-\alpha}(x)\right| = \left|\widetilde{f}'_{r-\alpha}(x)\right| = \widetilde{w}_\alpha(x) \geq 0$ (in particular $\widetilde{w}_\alpha(x) = \left|\widetilde{f}_\alpha{}'(x)\right|$ if $\widetilde{f}_\alpha{}'(x)$ exists) so that we have

$$F'_{LgH}(x)_\alpha = \left(\widehat{f}_\alpha{}'(x); \widetilde{w}_\alpha(x)\right) \tag{4}$$

or, in the standard interval notation,

$$F'_{LgH}(x)_\alpha = \left[\widehat{f}_\alpha{}'(x) - \widetilde{w}_\alpha(x), \widehat{f}_\alpha{}'(x) + \widetilde{w}_\alpha(x)\right]. \tag{5}$$

II. Orders for Fuzzy Numbers

The LU-fuzzy partial order is well known in the literature and frequently considered to be the standard order for fuzzy numbers (see, e.g., [5] and [13] for an account of its relations with LgH-difference and LgH-derivative).

Definition 5: [13] Given u,$v \in \mathbb{R}_\mathcal{F}$ and given $\alpha \in [0,1]$, we say that

(i) $u \precapprox_{\alpha-LU} v$ if and only if $u_\alpha \precapprox_{LU} v_\alpha$, that is, $u_\alpha^- \le v_\alpha^-$ and $u_\alpha^+ \le v_\alpha^+$,
(ii) $u \preceq_{\alpha-LU} v$ if and only if $u_\alpha \preceq_{LU} v_\alpha$,
(iii) $u \prec_{\alpha-LU} v$ if and only if $u_\alpha \prec_{LU} v_\alpha$.

Correspondingly, the analogous LU-fuzzy orders can be obtained by

(a) $u \precapprox_{LU} v$ if and only if $u \precapprox_{\alpha-LU} v$ for all $\alpha \in [0,1]$,
(b) $u \preceq_{LU} v$ if and only if $u \preceq_{\alpha-LU} v$ for all $\alpha \in [0,1]$,
(c) $u \prec_{LU} v$ if and only if $u \prec_{\alpha-LU} v$ for all $\alpha \in [0,1]$.

The reverse orders are, respectively, $u \succapprox_{LU} v \iff v \precapprox_{LU} u$, $u \succeq_{LU} v \iff v \preceq_{LU} u$ and $u \succ_{LU} v \iff v \prec_{LU} u$.

Using midpoint notation for α-levels, i.e., $u_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$, $v_\alpha = (\widehat{v}_\alpha; \widetilde{v}_\alpha)$ for all $\alpha \in [0,1]$, the partial orders (i) and (iii) above can be expressed for all $\alpha \in [0,1]$ as

$$(i) \begin{cases} \widehat{u}_\alpha \le \widehat{v}_\alpha \\ \widetilde{v}_\alpha \le \widetilde{u}_\alpha + (\widehat{v}_\alpha - \widehat{u}_\alpha) \\ \widetilde{v}_\alpha \ge \widetilde{u}_\alpha - (\widehat{v}_\alpha - \widehat{u}_\alpha) \end{cases} \quad \text{and}$$
$$(iii) \begin{cases} \widehat{u}_\alpha < \widehat{v}_\alpha \\ \widetilde{v}_\alpha < \widetilde{u}_\alpha + (\widehat{v}_\alpha - \widehat{u}_\alpha) \\ \widetilde{v}_\alpha > \widetilde{u}_\alpha - (\widehat{v}_\alpha - \widehat{u}_\alpha) \end{cases} ;$$

while (ii) can be expressed in terms of (i) with the additional requirement that at least one of the inequalities is strict.

In the sequel, the results are expressed without proof because they are similar to the ones in [17] and [18].

Proposition 6: Let $u, v \in \mathbb{R}_\mathcal{F}$ with $u_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$, $v_\alpha = (\widehat{v}_\alpha; \widetilde{v}_\alpha)$ for all $\alpha \in [0,1]$. We have

(i.a) $u \precapprox_{LU} v$ if and only if $\widehat{v}_\alpha - \widehat{u}_\alpha \ge |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$;
(ii,a) $u \preceq_{LU} v$ if and only if $\widehat{u}_\alpha < \widehat{v}_\alpha$ and $\widehat{v}_\alpha - \widehat{u}_\alpha \ge |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$;
(iii,a) $u \prec_{LU} v$ if and only if $\widehat{v}_\alpha - \widehat{u}_\alpha > |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$;
(i,b) $u \succapprox_{LU} v$ if and only if $\widehat{u}_\alpha - \widehat{v}_\alpha \ge |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$;
(ii,b) $u \succeq_{LU} v$ if and only if $\widehat{u}_\alpha > \widehat{v}_\alpha$ and $\widehat{u}_\alpha - \widehat{v}_\alpha \ge |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$;
(iii,b) $u \succ_{LU} v$ if and only if $\widehat{u}_\alpha - \widehat{v}_\alpha > |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for all $\alpha \in [0,1]$.

Proposition 7: Let $u, v \in \mathbb{R}_\mathcal{F}$ with $u_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$, $v_\alpha = (\widehat{v}_\alpha; \widetilde{v}_\alpha)$ for all $\alpha \in [0,1]$. We have

(i.a) $u \precapprox_{LU} v$ if and only if $u \ominus_{LgH} v \precapprox_{LU} 0$;
(ii.a) $u \preceq_{LU} v$ if and only if $u \ominus_{LgH} v \preceq_{LU} 0$;
(iii.a) $u \prec_{LU} v$ if and only if $u \ominus_{LgH} v \prec_{LU} 0$;
(i.b) $u \succapprox_{LU} v$ if and only if $u \ominus_{LgH} v \succapprox_{LU} 0$;
(ii.b) $u \succeq_{LU} v$ if and only if $u \ominus_{LgH} v \succeq_{LU} 0$;
(iii.b) $u \succ_{LU} v$ if and only if $u \ominus_{LgH} v \succ_{LU} 0$;
(iv) $u \prec_{LU} v \implies u \preceq_{LU} v \implies u \precapprox_{LU} v$;
(v) $u \succ_{LU} v \implies u \succeq_{LU} v \implies u \succapprox_{LU} v$.

Remark 8: Considering the distinction between type (i) and type (ii) of LgH-difference, several other implications can be established. For example in type (i), for all $\alpha \in [0,1]$, it is $\widetilde{u}_\alpha \ge \widetilde{v}_\alpha$ and we have

- $u \ominus_{LgH} v \precapprox_{LU} 0$ if and only if $(\widehat{u}_\alpha \le \widehat{v}_\alpha$ and $\widetilde{v}_\alpha \ge \widetilde{u}_\alpha + (\widehat{u}_\alpha - \widehat{v}_\alpha))$, for all $\alpha \in [0,1]$,
- $u \ominus_{LgH} v \succapprox_{LU} 0$ if and only if $(\widehat{u}_\alpha \ge \widehat{v}_\alpha$ and $\widetilde{v}_\alpha \ge \widetilde{u}_\alpha + (\widehat{v}_\alpha - \widehat{u}_\alpha))$, for all $\alpha \in [0,1]$.

Definition 9: Given $u, v \in \mathbb{R}_F$, we say that u and v are LU-incomparable if neither $u \precapprox_{LU} v$ nor $u \succapprox_{LU} v$ and u and v are α-LU-incomparable if neither $u \precapprox_{\alpha-LU} v$ nor $u \succapprox_{\alpha-LU} v$.

Proposition 10: Let $u, v \in \mathbb{R}_\mathcal{F}$ with $u_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$, $v_\alpha = (\widehat{v}_\alpha; \widetilde{v}_\alpha)$ for all $\alpha \in [0,1]$. The following are equivalent:

(i) u and v are α-LU-incomparable;
(ii) $u_\alpha \ominus_{gH} v_\alpha$ is not a singleton and $0 \in int(u_\alpha \ominus_{gH} v_\alpha)$;
(iii) $\left|\widehat{u}_\alpha - \widehat{v}_\alpha\right| < |\widetilde{v}_\alpha - \widetilde{u}_\alpha|$ for $\alpha \in [0,1]$;
(iv) $u_\alpha \subset int(v_\alpha)$ or $v_\alpha \subset int(u_\alpha)$.

It is possible to see that Proposition 10 is not always valid in the case of fuzzy LU-incomparability.

Proposition 11: If u, v, $w \in \mathbb{R}_\mathcal{F}$, then

(i) $u \precapprox_{LU} v$ if and only if $u + w \precapprox_{LU} v + w$;
(ii) if $u + v \precapprox_{LU} w$ then $u \precapprox_{LU} w \ominus_{LgH} v$;
(iii) if $u + v \succapprox_{LU} w$ then $u \succapprox_{LU} w \ominus_{LgH} v$;
(iv) $u \precapprox_{LU} v$ if and only if $(-v) \precapprox_{LU} (-u)$.

Definition 12: given $u \in \mathbb{R}_\mathcal{F}$ with $u_\alpha = (\widehat{u}_\alpha; \widetilde{u}_\alpha)$, for all $\alpha \in [0,1]$ we define the following sets of fuzzy numbers y which are

(a) $(\precapprox_{LU})$-dominated by u:
$$\mathbb{D}_<(u; LU) = \{y \in \mathbb{R}_\mathcal{F} | u \precapprox_{LU} y\}, \tag{6}$$
(b) $(\precapprox_{LU})$-dominating u:
$$\mathbb{D}_>(u; LU) = \{y \in \mathbb{R}_\mathcal{F} | y \precapprox_{LU} u\}, \tag{7}$$
(c) $(\precapprox_{LU})$-incomparable with u:
$$\mathbb{I}(u; LU) = \{y \in \mathbb{R}_\mathcal{F} | y \notin \mathbb{D}_<(u; LU), y \notin \mathbb{D}_>(u; LU)\}. \tag{8}$$

Proposition 13: For any fuzzy numbers $u, v \in \mathbb{R}_\mathcal{F}$, we have

a. $u \precapprox_{LU} v$ if and only if $\mathbb{D}_<(v; LU) \subseteq \mathbb{D}_<(u; LU)$;
b. $u = v$ if and only if $\mathbb{D}_<(u; LU) = \mathbb{D}_<(v; LU)$;
c. $\emptyset = \mathbb{D}_<(u; LU) \bigcap \mathbb{I}(u; LU) = \mathbb{D}_>(u; LU) \bigcap \mathbb{I}(u; LU)$;
d. $\{u\} = \mathbb{D}_<(u; LU) \bigcap \mathbb{D}_>(u; LU)$;
e. $\mathbb{R}_F = \mathbb{I}(u; LU) \bigcup \mathbb{D}_<(u; LU) \bigcup \mathbb{D}_>(u; LU)$.

III. Fuzzy-Valued Functions and midpoint representation

In midpoint representation, we write $[F(x)]_\alpha = \left(\widehat{f}_\alpha(x); \widetilde{f}_\alpha(x)\right)$ where $\widehat{f}_\alpha(x) \in \mathbb{R}$ is the midpoint value of interval $[F(x)]_\alpha$ and $\widetilde{f}_\alpha(x) \in \mathbb{R}^+ \cup \{0\}$ is the non-negative half-length of $F_\alpha(x)$:

$$\widehat{f}_\alpha(x) = \frac{f_\alpha^+(x) + f_\alpha^-(x)}{2} \quad \text{and}$$
$$\widetilde{f}_\alpha(x) = \frac{f_\alpha^+(x) - f_\alpha^-(x)}{2} \ge 0$$

so that

$$f_\alpha^-(x) = \widehat{f}_\alpha(x) - \widetilde{f}_\alpha(x) \text{ and } f_\alpha^+(x) = \widehat{f}_\alpha(x) + \widetilde{f}_\alpha(x).$$

Proposition 14: Let $F : T \longrightarrow \mathbb{R}_F$ be a fuzzy-valued function and $x_0 \in T \subseteq \mathbb{R}$ be an accumulation point of T. If $\lim_{x \to x_0} F(x) = L$ with $L_\alpha = [l_\alpha^-, l_\alpha^+]$, then $\lim_{x \to x_0} [F(x)]_\alpha = [l_\alpha^-, l_\alpha^+]$ for all α and, for all $\alpha \in [0,1]$, the limits and continuity can be expressed, respectively, as

$$\lim_{x \to x_0} [F(x)]_\alpha = L_\alpha \iff \begin{cases} \lim_{x \to x_0} \widehat{f}_\alpha(x) = \widehat{l}_\alpha \\ \lim_{x \to x_0} \widetilde{f}_\alpha(x) = \widetilde{l}_\alpha \end{cases} \tag{9}$$

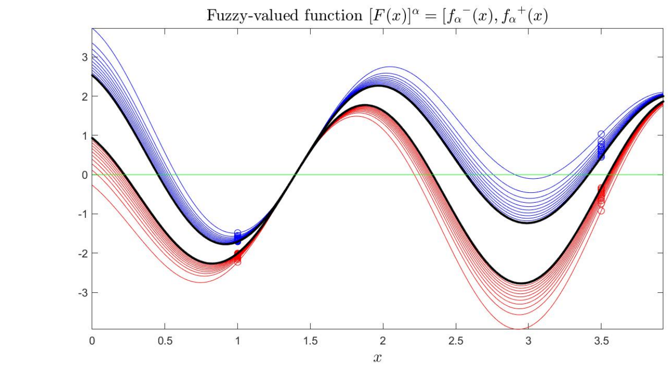

Fig. 1. Level-wise endpoint graphical representation of the fuzzy-valued function with α-cuts $[F(x)]_\alpha = \left(\widehat{f_\alpha}(x); \widetilde{f_\alpha}(x)\right)$ where $\widehat{f_\alpha}(x) = 2sin(-3x + \pi/3)$ and $\widetilde{f_\alpha}(x) = (1 + cos(9x/4)(1 - 0.6\sqrt{\alpha})$ for $x \in [0, 1.25\pi]$. The core, intercepted by the black-coloured curves, is the interval-valued function $x \longrightarrow [F(x)]_1 = [f_1^-(x), f_1^+(x)]$. The other α-cuts are represented by red-coloured curves for the left extreme functions $f_\alpha^-(x)$ and blue-coloured curves for the right extreme functions $f_\alpha^+(x)$.

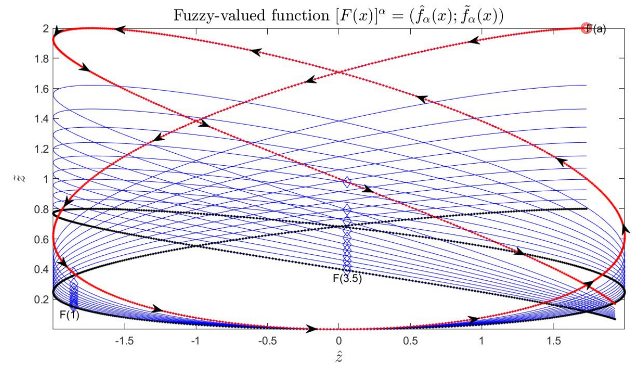

Fig. 2. Level-wise midpoint graphical representation in the half plane $(\widehat{z}; \widetilde{z})$ with $\widetilde{z} \geq 0$ as vertical axis, of the fuzzy-valued function $F(x)$, $x \in [0, 1.25\pi]$, described in Figure 1. In this representation, each curve corresponds to a single α-cut (only $n = 11$ curves are pictured with uniform $\alpha = 0, 0.1, ..., 1$; the core corresponds to the black-coloured curve, the support to the red-coloured one. The arrows give the *direction* of x from initial 0 to final 1.25π.

and

$$\lim_{x \to x_0} [F(x)]_\alpha = [F(x_0)]_\alpha \iff \begin{cases} \lim\limits_{x \to x_0} \widehat{f_\alpha}(x) = \widehat{f_\alpha}(x_0) \\ \lim\limits_{x \to x_0} \widetilde{f_\alpha}(x) = \widetilde{f_\alpha}(x_0). \end{cases} \quad (10)$$

The following proposition connects limits to the $\lessapprox_{LU}$ partial order of fuzzy numbers; analogous results can be obtained for the reverse partial order $\gtrapprox_{LU}$.

Proposition 15: Let $F, G, H : T \longrightarrow \mathbb{R}_F$ be fuzzy-valued functions and x_0 an accumulation point for T.

(i) If $F(x) \lessapprox_{LU} G(x)$ for all $x \in T$ in a neighbourhood of x_0 and $\lim_{x \to x_0} F(x) = L \in \mathbb{R}_F$, $\lim_{x \to x_0} G(x) = M \in \mathbb{R}_F$, then $L \lessapprox_{LU} M$;

(ii) If $F(x) \lessapprox_{LU} G(x) \lessapprox_{LU} H(x)$ for all $x \in T$ in a neighbourhood of x_0 and $\lim_{x \to x_0} F(x) = \lim_{x \to x_0} H(x) = L \in \mathbb{R}_F$, then $\lim_{x \to x_0} G(x) = L$.

Similar results as in Propositions 14 and 15 are valid for the left limit with $x \longrightarrow x_0$, $x < x_0$ ($x \nearrow x_0$ for short) and for the right limit $x \longrightarrow x_0$, $x > x_0$ ($x \searrow x_0$ for short); the condition that $\lim\limits_{x \to x_0} F(x) = L$ if and only if $\lim\limits_{x \nearrow x_0} F(x) = L = \lim\limits_{x \searrow x_0} F(x)$ is obvious.

Example Consider the fuzzy-valued function $F(x)$ having α-cuts $\widehat{f_\alpha}(x) = 2sin\left(-3x + \dfrac{\pi}{3}\right)$ and $\widetilde{f_\alpha}(x) = (1 + cos\left(\dfrac{9x}{4}\right)(1 - 0.6\sqrt{\alpha})$ for $x \in [0, 1.25\pi]$; remark that $\widehat{f_\alpha}(x)$ does not depend on α. Two alternative graphical representations of F are possible in terms either of the standard plot (see Figure 1), by picturing the level curves $y = f_\alpha^-(x)$ and $y = f_\alpha^+(x)$ in the plane (x, y), or, using midpoint representation in the half-plane $(\widehat{z}; \widetilde{z})$, by plotting the parametric curves $\widehat{z} = \widehat{f_\alpha}(x)$ and $\widetilde{z} = \widetilde{f_\alpha}(x)$ as in Figure 2; Figure 3 reproduces the membership functions (left pictures) and the midpoint α-cuts (right pictures) of the fuzzy values $F(1)$ (top) and $F(3.5)$ (bottom) of function F.

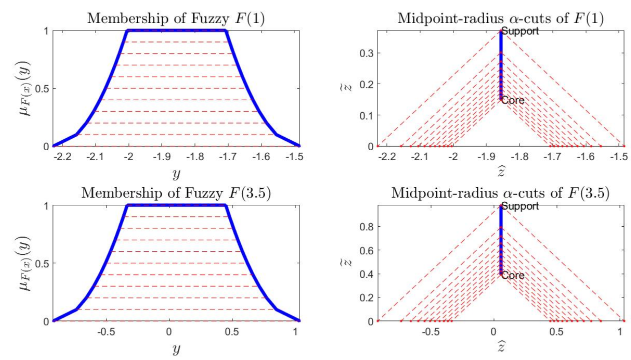

Fig. 3. Membership function and level-wise midpoint representations of two values $F(1)$ and $F(3.5)$ of the fuzzy-valued function $F(x)$ described in Figure 1. In the midpoint representation, a vertical curve corresponds to the displacement of the $n = 11$ computed α-cuts; the red lines on the right pictures reconstruct the α-cuts. Remark that y and $\widehat{z}$ represent the same domain and that a linear vertical segment in the midpoint representation corresponds to a symmetric membership function having the same value of $\widehat{f_\alpha}(x)$ for all α.

IV. Extrema of Fuzzy Valued Functions

The three types of partial orders defined above, $\lessapprox_{LU}$ (simple), $\preceq_{LU}$ (strict) and $\prec_{LU}$ (strong), lead to different concepts of extrema. We will adopt the following terminology:

Definition 16: If $F(x_0) \lessapprox_{LU} F(x)$, we say that $F(x_0)$ *dominates* $F(x)$ with respect to the *simple* partial order $\lessapprox_{LU}$ (for short, $F(x_0)$ ($\lessapprox_{LU}$)-dominated $F(x)$), or equivalently that $F(x)$ is ($\lessapprox_{LU}$)-dominated by $F(x_0)$. We say that $F(x)$ and $F(x_0)$ are *incomparable* with respect to $\lessapprox_{LU}$ if both $F(x_0) \lessapprox_{LU} F(x)$ and $F(x) \lessapprox_{LU} F(x_0)$ are not valid. Analogous domination rules are defined in terms of the *strict* and *strong* order relations $\preceq_{LU}$ and $\prec_{LU}$, respectively.

The corresponding important concepts of order-based minimum and maximum points for a fuzzy valued function are the following.

Definition 17: Let $F : [a, b] \to \mathbb{R}_F$ be a fuzzy-valued function and $x_0 \in [a, b]$. We say that, with respect to $\lessapprox_{LU}$,

(a) x_0 is a local lattice-minimum point of F (*min*-point for

short) if there exists $\delta > 0$ such that $F(x_0) \lessapprox_{LU} F(x)$ for all $x \in]x_0 - \delta, x_0 + \delta[\cap[a,b]$, i.e., if all $F(x)$ around x_0 are $(\lessapprox_{LU})$-dominated by $F(x_0)$;
(b) x_0 is a local lattice-maximum point of F (*max*-point for short) if there exists $\delta > 0$ such that $F(x) \lessapprox_{LU} F(x_0)$ for all $x \in]x_0 - \delta, x_0 + \delta[\cap[a,b]$, i.e., if all $F(x)$ around x_0 $(\lessapprox_{LU})$-dominate $F(x_0)$.

Conditions (a) or (b) in the definition above imply that if there exists $x' \in [a,b]$ such that $\widehat{f}_\alpha(x') = \widehat{f}_\alpha(x_0)$ and $\widetilde{f}_\alpha(x') \neq \widetilde{f}_\alpha(x_0)$ for all $\alpha \in [0,1]$, then it is impossible to have $F(x_0) \lessapprox_{LU} F(x')$ nor $F(x') \lessapprox_{LU} F(x_0)$, except for trivial cases, in particular, if $\widehat{f}_\alpha(x') = \widehat{f}_\alpha(x_0)$ for all $\alpha \in [0,1]$, then $F(x')$ and $F(x_0)$ are $(\lessapprox_{LU})$-incomparable or coincident.

It will be useful to explicitly write the conditions for $(\lessapprox_{LU})$-dominance of a general fuzzy function $F(x)$, with respect to fuzzy functions $F(x_m)$ and $F(x_M)$, that characterize the minimality of a point x_m and the maximality of a point x_M. Without explicit distinction between strict or strong dominance, we have, for all $\alpha \in [0,1]$:

$$F(x) \gtrapprox_{LU} F(x_m) \iff \begin{cases} \widehat{f}_\alpha(x) \geq \widehat{f}_\alpha(x_m) \\ \widetilde{f}_\alpha(x) \leq \widetilde{f}_\alpha(x_m) + \left(\widehat{f}_\alpha(x) - \widehat{f}_\alpha(x_m)\right) \\ \widetilde{f}_\alpha(x) \geq \widetilde{f}_\alpha(x_m) - \left(\widehat{f}_\alpha(x) - \widehat{f}_\alpha(x_m)\right), \end{cases} \tag{11}$$

and

$$F(x) \lessapprox_{LU} F(x_M) \iff \begin{cases} \widehat{f}_\alpha(x) \leq \widehat{f}_\alpha(x_M \\ \widetilde{f}_\alpha(x) \geq \widetilde{f}_\alpha(x_M) + \left(\widehat{f}_\alpha(x) - \widehat{f}_\alpha(x_M)\right) \\ \widetilde{f}(x) \leq \widetilde{f}_\alpha(x_M) - \left(\widehat{f}_\alpha(x) - \widehat{f}_\alpha(x_M)\right). \end{cases} \tag{12}$$

The following proposition shows that lattice-type minimality and maximality, with respect to the partial order $\lessapprox_{LU}$ can be recognized exactly in terms of functions f_α^- and f_α^+, for all $\alpha \in [0,1]$, as follows.

Proposition 18: Let $F : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function, where $[F(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$ for all $\alpha \in [0,1]$. Then
(a) $x_m \in [a,b]$ is a min-point of F if and only if it is a minimum of functions f_α^- and f_α^+ for all $\alpha \in [0,1]$;
(b) $x_M \in [a,b]$ is a max-point of F if and only if it is a maximum of functions f_α^- and f_α^+ for all $\alpha \in [0,1]$.

The discussion above highlights the restricting notion of a lattice-extreme point, as it is not frequent that simultaneous extrema occur for the two functions f_α^- and f_α^+. The following definition is more general, as it considers the possibility that fuzzy function values $F(x)$ for different x are locally incomparable with respect to the actual order relation.

Consider again the function $F(x)$ of the Example presented in section III; the midpoint function $\widehat{f}_\alpha(x)$ (independent on α) has two minimal points $x'_m = 0.87266461$, $x''_m = 2.96705971$ and a maximum at $x_M = 1.91986219$ (see Figure 4); x'_m and x''_m are not lattice-type minima and x_M is not a lattice-type maximum. These points are candidate to be best-type extrema of $F(x)$, according to the following definition.

Definition 19: Let $F : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function and $x_m, x_M \in [a,b]$. We say that, with respect to the

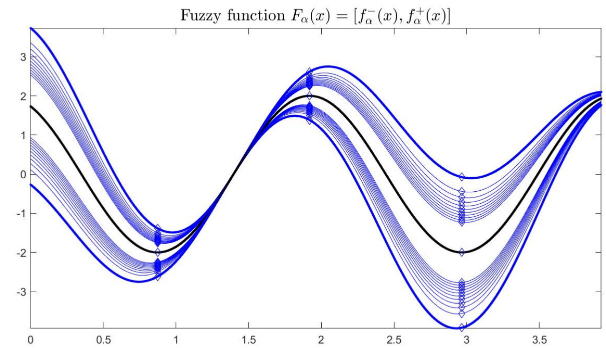

Fig. 4. Graphical representation of the fuzzy-valued function $F(x)$ in Example, evaluated at points $x_m^{(1)} = 0.87266$, $x_m^{(2)} = 2.96706$ and $x_M = 1.91986$. The black line corresponds to function $\widehat{f}(x)$, the core midpoint function, for which the three points are two minima and a maximum.

strict order $\preceq_{LU}$,
(c) x_m is a local best-minimum point of F (best-min for short) if:
(c.1) it is a local minimum for the midpoint function $\widehat{f}_\alpha$ for all $\alpha \in [0,1]$, and
(c.2) there exists $\delta > 0$ and no point $x \in]x_m - \delta, x_m + \delta[\cap[a,b]$ with $F(x) \neq F(x_m)$ such that $F(x) \lessapprox_{LU} F(x_m)$;
(d) x_M is a local best-maximum point of F (best-max for short) if:
(d.1) it is a local maximum for the midpoint function $\widehat{f}_\alpha$ for all $\alpha \in [0,1]$, and
(d.2) there exists $\delta > 0$ and no point $x \in]x_M - \delta, x_M + \delta[\cap[a,b]$ with $F(x) \neq F(x_M)$ such that $F(x_M) \lessapprox_{LU} F(x)$.

Remark 20: The definitions above are clearly valid also for points $x_0 \in [a,b]$ coincident with one of the end points a or b. It is also evident that a lattice-type extremum is also a best-type extremum.

Definitions of strict and strong (local) extremal points can be given by considering the strict $\preceq_{LU}$ or the strong $\prec_{LU}$ orders associated to the lattice order $\lessapprox_{LU}$.

Definition 21: Let $F : [a,b] \to \mathbb{R}_F$ be an fuzzy-valued function. With respect to an order $\lessapprox_{LU}$ and the associated strict order $\preceq_{LU}$ or strong order $\prec_{LU}$, we say that
- a best-min point x_m is a strict (respectively strong) best-minimum point if there exists $\delta > 0$ and no point $x \in]x_m - \delta, x_m + \delta[\cap[a,b]$ with $F(x) \preceq_{LU} F(x_m)$ (or $F(x) \prec_{LU} F(x_m)$, respectively);
- a best-max point x_M is a strict (respectively strong) best-maximum point if there exists $\delta > 0$ and no point $x \in]x_M - \delta, x_M + \delta[\cap[a,b]$ with $F(x_M) \preceq_{LU} F(x)$ (or $F(x_M) \prec_{LU} F(x)$, respectively).

Remark 22: It is clear that the definitions of lattice-type and best-type extremality do not require any assumptions on continuity of the fuzzy-valued function F on $[a,b]$; in the case of continuity (or left/right continuity) the existence of extreme points is also related to the local left and/or right monotonicity of F (with respect to the same partial order $\lessapprox_{LU}$).

If $x_m \in [a,b]$ is a lattice-minimum point, i.e., there exists a neighbourhood of x_m such that all $F(x)$ satisfy (11), then no such $F(x)$ is incomparable with $F(x_m)$; analogously, if $x_M \in [a,b]$ is a lattice-maximum point, i.e., there exists a neighbourhood of x_M such that all $F(x)$ satisfy (12), then no such $F(x)$ is incomparable with $F(x_M)$. We can express this fact by saying that the (local) min-efficient frontier for the *min*-point x_m is concentrated at the fuzzy value $F(x_m)$; analogously, the (local) max-efficient frontier for the max-point x_M is concentrated at the fuzzy value $F(x_M)$.

When x_m and x_M are best-type extrema and not lattice-type, then it is important to identify the fuzzy values $F(x)$, in particular with x in a neighbourhood of x_m or x_M, that are not min-dominated by $F(x_m)$ (or not max-dominated by $F(x_M)$); clearly, these $F(x)$ are necessarily ($\lessapprox_{LU}$)-incomparable with $F(x_m)$ (or with $F(x_M)$, respectively).

Corresponding to a minimum and to a maximum point of F, we are then interested to identify the locally (min/max)-efficient fuzzy values $F(x)$ and the local min or max efficient frontier for $F(x_m)$ and $F(x_M)$ around points x_m and x_M, respectively.

The first step in finding the efficient frontier for a strict minimum and a strict maximum is the following:

Proposition 23: Let $F : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function. Let $x_m, x_M \in [a,b]$ be local strict best-min and local strict best-max points of F. Then, there exist $x_m^L \leq x_m$, $x_m^R \geq x_m$, $x_M^L \leq x_M$ and $x_M^R \geq x_M$ (all belonging to $[a,b]$) such that, respectively,

1. $F(x)$ is incomparable with $F(x_m)$, for all $x \in [x_m^L, x_m^R]$, $x \neq x_m$;
2. $F(x)$ is incomparable with $F(x_M)$, for all $x \in [x_M^L, x_M^R]$, $x \neq x_M$.

A first consequence of Proposition 23 is a sufficient condition for a lattice type external point.

Proposition 24: Let $F : [a,b] \to \mathbb{R}_F$; if x_m (respectively, x_M) is a minimum point (a maximum point) of function $\widehat{f_\alpha}(x)$ for all $\alpha \in [0,1]$ and $x_m^L = x_m = x_m^R$ (or $x_M^L = x_M = x_M^R$) then x_m is a lattice min-point (respectively x_M is a lattice max-point) of $F(x)$ and vice versa.

A second consequence of the last proposition is that the efficient function $F(x)$, relative to the best-min point x_m or to the best-max point x_M, in the case where they are not lattice extrema, are to be searched among the points $x \in [x_m^L, x_m^R]$ and $x \in [x_M^L, x_M^R]$, respectively.

The next step is to characterize the points of $[x_m^L, x_m^R]$ and $[x_M^L, x_M^R]$ that contain, respectively, x_m, x_M and are such that all the corresponding $F(x)$ define the local efficient frontier of F around $F(x_m)$ and $F(x_M)$, respectively.

We start with a formal definition of the min/max efficient frontier:

Definition 25: Let $F : [a,b] \to \mathbb{R}_F$ be a fuzzy-valued function and let $x_m, x_M \in [a,b]$ be local strict best-min and local strict best-max points of F with respect to the partial order $\lessapprox_{LU}$.

(a) The (local) min-efficient frontier of function F associated to the best-min point x_m (or to the best-min fuzzy-value $F(x_m)$) is the set $E_{\min}(F, x_m)$ of fuzzy-values $F(x)$ such that:

(a.1) $F(x_m) \in E_{\min}(F, x_m)$,

(a.2) if $x', x'' \in [a,b]$ and $F(x'), F(x'') \in E_{\min}(F, x_m)$ then $F(x')$ and $F(x'')$ are ($\lessapprox_{LU}$)-incomparable,

(a.3) no other set E' containing $E_{\min}(F, x_m)$ has property (a.2).

(b) The (local) max-efficient frontier of function F associated to the best-max point x_M (or to the best-max fuzzy-value $F(x_M)$) is the set $E_{\max}(F, x_M)$ of fuzzy-values $F(x)$ such that:

(b.1) $F(x_M) \in E_{\max}(F, x_M)$,

(b.2) if $x', x'' \in [a,b]$ and $F(x'), F(x'') \in E_{\max}(F, x_M)$ then $F(x')$ and $F(x'')$ are ($\lessapprox_{LU}$)-incomparable,

(b.3) no other set E' containing $E_{\max}(F, x_m)$ has property (b.2).

The set of points $x \in [x_m^L, x_m^R]$ such that $F(x) \in E_{\min}(F, x_m)$ are the local min-efficient points corresponding to x_m and is denoted by $\mathit{Eff}_{\min}(F; x_m)$

The set of points $x \in [x_M^L, x_M^R]$ such that $F(x) \in E_{\max}(F, x_M)$ are the local max-efficient points corresponding to x_M and is denoted by $\mathit{Eff}_{\max}(F; x_M)$.

Clearly, the efficient frontiers $\mathit{Eff}_{\min}(F; x_m)$ or $\mathit{Eff}_{\max}(F; x_M)$ are subsets of the interval in Proposition 23; but their characterization is not easy, as we can imagine in cases where the function $F(x)$ has possible inflexion or angular points, or complex patterns.

For a fixed $\alpha \in [0,1]$, let C_{F_α} be the curve, in the half-plane $(\widehat{z}; \widetilde{z})$ with parametric equations $\widehat{z} = \widehat{f_\alpha}(x)$, $\widetilde{z} = \widetilde{f_\alpha}(x)$ and parameter $x \in [a,b]$ and assume that the curve is simple (no multiple points) and differentiable (i.e., both $\widehat{f_\alpha}(x)$ and $\widetilde{f_\alpha}(x)$ are differentiable at internal points); one says that the curve C_{F_α} has the convexity property if each of its points is such that the curve lies on one side of the tangent line to this point. In our setting, the convexity of C_{F_α} is required only locally, by considering the restriction of $F(x)$ to points around x_m (or x_M). More precisely, let's fix the notion of local convexity of C_{F_α} by distinguishing the case of a minimum to the case of a maximum point.

Assumption 1: For a minimum point x_m (not a lattice min) we will assume that there exist $\delta_m^{'}, \delta_m^{''} \geq 0$ (not both equal to zero) such that the curve corresponding to the restriction of $F(x)$ to the interval $\left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right]$ is simple and convex; this happens if the portion of plane on right of the curve, i.e., for each α, the set

$$P_{min}(x_m) = \bigcup_{x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right]} \left\{ \left(\widehat{z}; \widetilde{f_\alpha}(x)\right) | \widehat{z} \geq \widehat{f_\alpha}(x) \right\}, \quad (13)$$

is convex; in this case, the following portion of the half plane is convex and bounded

$$S_{min}(x_m) = P_{min}(x_m) \bigcap \left\{ (\widehat{z}; \widetilde{z}) | \widehat{z}_{min} \leq \widehat{z} \leq \widehat{z}_{max},\ \widetilde{z}_{min} \leq \widetilde{z} \leq \widetilde{z}_{max} \right\} \quad (14)$$

where $\widehat{z}_{min} = \min\left\{ \widehat{f_\alpha}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$,
$\widehat{z}_{max} = \max\left\{ \widehat{f_\alpha}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$,
$\widetilde{z}_{min} = \min\left\{ \widetilde{f_\alpha}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$ and
$\widetilde{z}_{max} = \max\left\{ \widetilde{f_\alpha}(x) | x \in \left[x_m - \delta_m^{'}, x_m + \delta_m^{''}\right] \right\}$.

Assumption 2: For a maximum point x_M (not a lattice max), assuming the existence of $\delta_M^{'}, \delta_M^{''} \geq 0$ such that the curve

$F(x)$ on interval $\left[x_M - \delta'_M, x_M + \delta''_M\right]$ is simple and convex, we obtain that the portion of plane on left of the curve, i.e., for all α,

$$P_{max}(x_M) = \bigcup_{x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]} \left\{ \left(\widehat{z}; \widetilde{f_\alpha}(x)\right) | \widetilde{z} \leq \widehat{f_\alpha}(x) \right\}, \quad (15)$$

is convex; in this case, the following set is convex and bounded

$$S_{max}(x_M) = P_{max}(x_M) \bigcap \{(\widehat{z}; \widetilde{z}) | \widehat{z}_{min} \leq \widehat{z} \leq \widehat{z}_{max}, \widetilde{z}_{min} \leq \widetilde{z} \leq \widetilde{z}_{max}\} \quad (16)$$

where, this time, $\widehat{z}_{min} = \min\left\{\widehat{f_\alpha}(x) | x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]\right\}$, $\widehat{z}_{max} = \max\left\{\widehat{f_\alpha}(x) | x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]\right\}$, and similarly for $\widetilde{z}_{min}$ and $\widetilde{z}_{max}$ in terms of $\widetilde{f_\alpha}(x)$.

Under Assumptions 1 or 2 (using the same notation) we can prove the following results:

Proposition 26: Let $\lessapprox_{LU}$ be a partial order on $\mathbb{R}_F$ and let $F : [a, b] \to \mathbb{R}_F$ be a fuzzy-valued function with $[F(x)]_\alpha = [f^-_\alpha(x), f^+_\alpha(x)]$ for all $\alpha \in [0, 1]$ such that $x_m \in]a, b[$ is a local min point of $\widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$ and Assumption (1) is satisfied. Then there exist two points $x'_m, x''_m \in [x^L_m, x^R_m]$ with $x'_m \leq x_m \leq x''_m$ and such that, for $x \in \left[x_m - \delta'_m, x_m + \delta''_m\right]$,
(1) either x'_m maximizes $\widetilde{f_\alpha}(x) - \widehat{f_\alpha}(x)$ or x''_m minimizes $\widetilde{f_\alpha}(x) + \widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$,
(2) or x'_m minimizes $\widetilde{f_\alpha}(x) + \widehat{f_\alpha}(x)$ and x''_m maximizes $\widetilde{f_\alpha}(x) - \widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$,
equivalently,
(i) either x'_m minimizes $f^-_\alpha(x)$ or x''_m minimizes $f^+_\alpha(x)$ for all $\alpha \in [0, 1]$,
(ii) or x'_m maximizes $f^+_\alpha(x)$ and x''_m minimizes $f^-_\alpha(x)$ for all $\alpha \in [0, 1]$.
Furthermore, interval $[x'_m, x''_m]$ is the local efficient frontier $Eff_{min}(F; x_m)$ of Definition 25.

In particular, if x'_m and x''_m are internal to the local convexity region and $\widehat{f_\alpha}(x)$, $\widetilde{f_\alpha}(x)$ are differentiable at x for all $\alpha \in [0, 1]$, then

$$\begin{cases} \widetilde{f_\alpha}'(x'_m) = \widehat{f_\alpha}'(x'_m) \\ \widetilde{f_\alpha}'(x''_m) = -\widehat{f_\alpha}'(x''_m) \end{cases} \text{or} \begin{cases} \widetilde{f_\alpha}'(x''_m) = \widehat{f_\alpha}'(x''_m) \\ \widetilde{f_\alpha}'(x'_m) = -\widehat{f_\alpha}'(x'_m) \end{cases}. \quad (17)$$

Proposition 27: Let $\lessapprox_{LU}$ be a partial order on $\mathbb{R}_F$ and let $F : [a, b] \to \mathbb{R}_F$ be fuzzy-valued function with $[F(x)]_\alpha = [f^-_\alpha(x), f^+_\alpha(x)]$ for all $\alpha \in [0, 1]$ such that $x_M \in]a, b[$ is a local max point of $\widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$ and Assumption (2) is satisfied. Then there exist two points $x'_M, x''_M \in [x^L_M, x^R_M]$ with $x'_M \leq x_M \leq x''_M$ and such that, for $x \in \left[x_M - \delta'_M, x_M + \delta''_M\right]$,
(1) either x'_M minimizes $\widetilde{f_\alpha}(x) - \widehat{f_\alpha}(x)$ or x''_M maximizes $\widetilde{f_\alpha}(x) + \widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$,
(2) or x'_M maximizes $\widetilde{f_\alpha}(x) + \widehat{f_\alpha}(x)$ and x''_M minimizes $\widetilde{f_\alpha}(x) - \widehat{f_\alpha}(x)$ for all $\alpha \in [0, 1]$.
equivalently,
(i) either x'_M maximizes $f^-_\alpha(x)$ or x''_M maximizes $f^+_\alpha(x)$, for all $\alpha \in [0, 1]$,
(ii) or x'_M maximizes $f^+_\alpha(x)$ and x''_M maximizes $f^+_\alpha(x)$ for all $\alpha \in [0, 1]$.

Furthermore, interval $[x'_M, x''_M]$ is the local efficient frontier $Eff_{max}(F; x_M)$ of Definition 25.

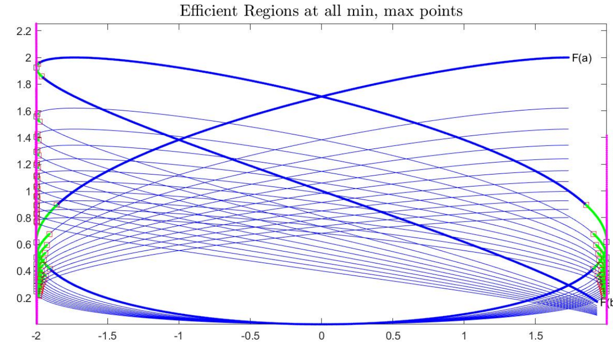

Fig. 5. Midpoint representation of the fuzzy-valued function $F(x)$ in Example, evaluated at the best-minimal points $x^{(1)}_m = 0.87266$, $x^{(2)}_m = 2.96706$ (on the left of picture) and the best-maximal point $x_M = 1.91986$ (on the right of the picture). The efficient regions are evidenced in green colour.

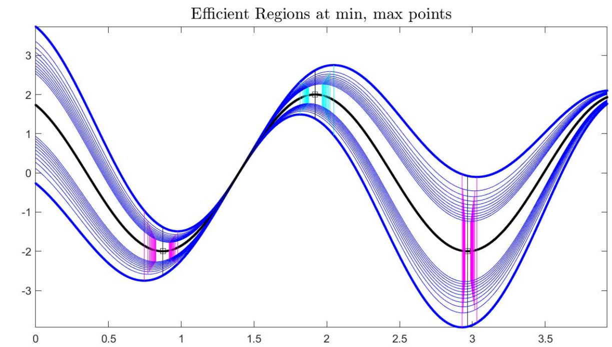

Fig. 6. Graphical representation of the fuzzy-valued function $F(x)$ in Example, evaluated at points $x^{(1)}_m = 0.87266$, $x^{(2)}_m = 2.96706$ and $x_M = 1.91986$. The black line corresponds to function $\widehat{f}(x)$. The efficient regions associated to the best-min and best-max points, in the x-domain, are delimited, for 11 α-cuts, by vertical lines around the three points.

In particular, if x'_M and x''_M are internal to the local convexity region and $\widehat{f_\alpha}(x)$, $\widetilde{f_\alpha}(x)$ are differentiable at x for all $\alpha \in [0, 1]$, then

$$\begin{cases} \widetilde{f_\alpha}'(x'_M) = \widehat{f}'(x'_M) \\ \widetilde{f_\alpha}'(x''_M) = -\widehat{f}'(x''_M) \end{cases} \text{or} \begin{cases} \widetilde{f_\alpha}'(x''_M) = \widehat{f_\alpha}'(x''_M) \\ \widetilde{f_\alpha}'(x'_M) = -\widehat{f_\alpha}'(x'_M) \end{cases}. \quad (18)$$

Returning to the the fuzzy-valued function $F(x)$ of Example, the efficient regions of the two best-minimal points $x^{(1)}_m = 0.87266$, $x^{(2)}_m = 2.96706$ and the best-maximum $x_M = 1.91986$ are obtained according to the propositions above; they are evidenced with green colour in Figure 5. The reconstruction of the efficient regions in the x-domain is easily obtained (from the midpoint representation) and pictured in Figure 6. The values of the first-order gH-derivatives $F'_{gH}(x^{(1)}_m)$, $F'_{gH}(x^{(2)}_m)$ and $F'_{gH}(x_M)$ are visualized in Figure 7; we see that in all cases, they contain the zero value internally to all α-cuts. Finally, the second-order gH-derivatives $F''_{gH}(x^{(1)}_m)$, $F''_{gH}(x^{(2)}_m)$ and $F''_{gH}(x_M)$ are visualized in Figure 8 and we see that $F''_{gH}(x^{(1)}_m) \succ_{LU} 0$, $F''_{gH}(x^{(2)}_m) \succ_{LU} 0$ and $F''_{gH}(x_M) \prec_{LU} 0$. Remark that these properties are analogous to the well-known sufficient

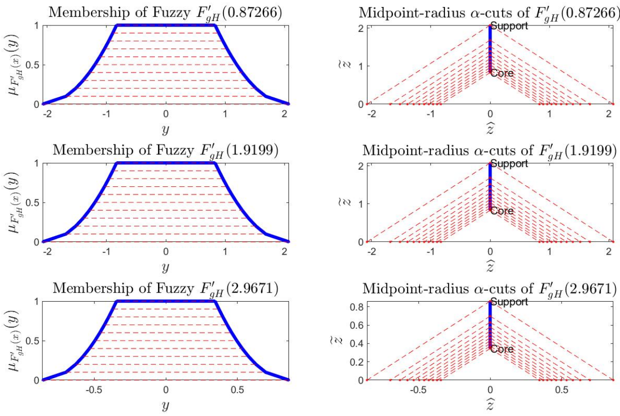

Fig. 7. Membership (left) and midpoint (right) values of the fuzzy-valued first-order gH-derivative $F'_{gH}(x)$ in Example at points $x_m^{(1)} = 0.87266$, $x_m^{(2)} = 2.96706$ and $x_M = 1.91986$.

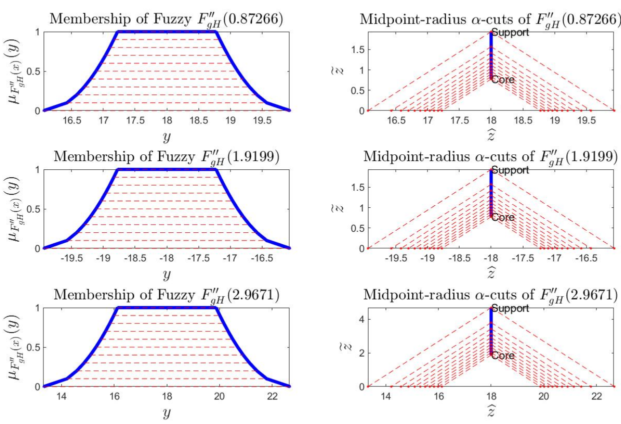

Fig. 8. Membership (left) and midpoint (right) values of the fuzzy-valued seconf-order gH-derivative $F''_{gH}(x)$ in Example at points $x_m^{(1)} = 0.87266$, $x_m^{(2)} = 2.96706$ and $x_M = 1.91986$.

conditions for minima and maxima for ordinary differentiable real-valued functions.

We conclude this section to see how local extremality of a point x_m (minimum) or x_M (maximum) is connected to the left and/or right LgH-derivatives or to the LgH-derivative $F'_{LgH}(x_0)$ if the two are equal. So, we have the following Fermat-like property:

Proposition 28: Let $F :]a,b[\rightarrow \mathbb{R}_F$ be LgH-differentiable at $x_0 \in]a,b[$ and $\precsim_{LU}$ be a partial order on $\mathbb{R}_F$. If x_0 is a lattice extremum for F (a lattice-min or a lattice-max point), then $F'_{LgH}(x_0) = 0$.

Proposition 29: Let $F : [a,b] \rightarrow \mathbb{R}_F$, $x_0 \in [a,b]$ and $\precsim_{LU}$ be a partial order on $\mathbb{R}_F$. Suppose that F has left and right LgH-derivatives at x_0 (if $x_0 = a$ or $x_0 = b$ we consider only the right or the left LgH-derivatives, respectively)

(1.a) If x_0 is a lattice minimum point for F, then $F'_{(l)LgH}(x_0) \precsim_{LU} 0$ and $F'_{(r)LgH}(x_0) \succsim_{LU} 0$;

(1.b) If x_0 is a lattice maximum point for F, then $F'_{(l)LgH}(x_0) \succsim_{LU} 0$ and $F'_{(r)LgH}(x_0) \precsim_{LU} 0$.

Proposition 30: Let $F :]a,b[\rightarrow \mathbb{R}_F$ be LgH-differentiable at $x_0 \in]a,b[$ and consider the partial order $\precsim_{LU}$ (or $\precsim_{\alpha-LU}$ level-wise) on $\mathbb{R}_F$.

(i) If x_0 is a best-minimum point for F, then $0 \in F'_{LgH}(x_0)$.

(ii) If x_0 is a best-maximum point for F, then $0 \in F'_{LgH}(x_0)$.

V. Conclusions and Further Work

We have developed new results to determine extremal points (local or global minima and maxima) of fuzzy-valued functions, in terms of the partial LU-order; the corresponding efficient regions are obtained (using standard dominance rules) from the newly introduced midpoint fuzzy representation. In further work, we will analyse monotonicity and convexity and we will extend our results to more general partial orders as suggested in [5], by the use of first-order and second-order LgH-derivatives.

References

[1] G. Alefeld, and J. Herzberger, *Introduction to Interval Computations*, Academic Press, New York, 1983.
[2] G. Alefeld, G. Mayer, Interval analysis: Theory and applications. *J. Comput. Appl. Math.*, 121 (2000) 421–464.
[3] B. Bede, *Mathematics of Fuzzy Sets and Fuzzy Logic*; Series: Studies in Fuzziness and Soft Computing n. 295; Springer: Berlin/Heidelberg, Germany, 2013.
[4] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions. *Fuzzy Sets Syst.* 230 (2013) 119–141.
[5] M.L. Guerra, L. Stefanini, A comparison index for intervals based on generalized Hukuhara difference, *Soft Computing*, 16 (2012) 1931-1943.
[6] Z. Kulpa, Diagrammatic representation for interval arithmetic, *Linear Algebra and its Applications*, 324 (2001) 55–80.
[7] S. Markov, Calculus for interval functions of real variable, *Computing*, 22 (1979) 325-337.
[8] R.E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1966.
[9] R.E. Moore, *Method and Applications of Interval Analysis*, SIAM, Philadelphia, 1979.
[10] R.E. Moore, R.B. Kearfott, J.M. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, PA, 2009.
[11] L. Stefanini, A generalization of Hukuhara difference. In D. Dubois et Al. (Eds.), *Soft Methods for Handling Variability and Imprecision*, Springer, 2008, pp. 205-210.
[12] L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. *Fuzzy Sets and Systems*, 161, (2010) 1564–1584.
[13] L. Stefanini, M. Arana-Jimenez, Karush-Kuhn-Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability. *Fuzzy Sets and Systems*, 362 (2019) 1–34.
[14] L. Stefanini, B. Bede, Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis*, 71 (2009) 1311–1328.
[15] L. Stefanini, B. Bede, Generalized fuzzy differentiability with LU-parametric representations. *Fuzzy Sets and Systems*, 257 (2014) 184–203.
[16] L. Stefanini, B. Bede, A new gH-difference for multi-dimensional convex sets and convex fuzzy sets. *Axioms* 2019, 8, 48, doi:10.3390/axioms8020048.
[17] L. Stefanini, M.L. Guerra, B. Amicizia, Interval analysis and calculus for interval-valued functions of a single variable. Part I: Partial orders, gH-derivative, monotonicity. *Axioms* 2019, 8, 113, doi:10.3390/axioms8040113
[18] L. Stefanini, L.Sorini, B. Amicizia, Interval analysis and calculus for interval-valued functions of a single variable. Part II: External points, convexity, periodicity. *Axioms*, 2019, 8, 114, doi:10.3390/axioms8040114