On Fréchet and Gateaux derivatives for interval and fuzzy-valued functions in the setting of gH-differentiability

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***Abstract*—We present new results in the calculus for interval functions of multiple real variables in the setting of gH-differentiability and, based on a new concept of interval linear functions, we define and analyse Gateaux and Fréchet differentiability and their properties. Finally, the results obtained for interval-valued functions are easily extended (level-wise) to define linear fuzzy-valued functions and corresponding Gateaux-Fréchet fuzzy differentiability.**

I. Intervals and gH-linear functions

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e. (see, e.g., [1], [8], [9], [10], [11]),

$$\begin{aligned}\mathcal{K}_C &= \left\{\left[\underline{a},\overline{a}\right] \mid \underline{a},\overline{a}\in\mathbb{R} \text{ and } \underline{a}\leq\overline{a}\right\}\\ &= \left\{(\widehat{a};\widetilde{a}) \mid \widehat{a},\widetilde{a}\in\mathbb{R} \text{ and } \widetilde{a}\geq 0\right\}\end{aligned}$$

where $\widehat{a}=\frac{\underline{a}+\overline{a}}{2}$, $\widetilde{a}=\frac{\overline{a}-\underline{a}}{2}$, $\underline{a}=\widehat{a}-\widetilde{a}$ and $\overline{a}=\widehat{a}+\widetilde{a}$.

The pair $(\widehat{a};\widetilde{a})$ (with a semicolon ; separator) is called midpoint notation for intervals. Minkowski operations on intervals are the following, given $A=[\underline{a},\overline{a}]=(\widehat{a};\widetilde{a})$, $B=[\underline{b},\overline{b}]=(\widehat{b};\widetilde{b})$ and $\tau\in\mathbb{R}$:

- $A\oplus_M B=(\widehat{a}+\widehat{b};\widetilde{a}+\widetilde{b})$,
- $\tau A=(\tau\widehat{a};|\tau|\widetilde{a})$,
- $-A=(-\widehat{a};\widetilde{a})$,
- $A\ominus_M B=A\oplus_M(-B)=(\widehat{a}-\widehat{b};\widetilde{a}+\widetilde{b})$.

The gH-difference of two intervals A and B is

$$A\ominus_{gH} B=C \Longleftrightarrow \begin{cases} & (a)\ A=B+C \\ \text{or} & (b)\ B=A+(-1)C \end{cases};$$

it always exists (see [4], [9], [13]) with

$$A\ominus_{gH} B=(\widehat{a}-\widehat{b};|\widetilde{a}-\widetilde{b}|)\subseteq A\ominus_M B. \quad (1)$$

Given $A,B\in\mathcal{K}_C$, the Pompeiu-Hausdorff distance is defined by (see, e.g., [7], [9], [13])

$$H(A,B)=\max\left[\max_{a\in A} d(a,B),\max_{b\in B} d(b,A)\right] \quad (2)$$

where $d(a,B)=\min_{b\in B}\|a-b\|$. We have $H(A,B)=\left\|A\ominus_{gH}B\right\|$ where, for $C\in\mathcal{K}_C$, $\|C\|_H=\max\{|c|\,;c\in C\}$; remark that, in particular, if $C=(0;\widetilde{c})$ (i.e., the midpoint of C is zero), then $\|C\|_H=\widetilde{c}$.

Given $A,B\in\mathcal{K}_C$, we define also the gH-addition (it always exists) as

$$A\oplus_{gH} B=A\ominus_{gH}(-B)=(\widehat{a}+\widehat{b};|\widetilde{a}-\widetilde{b}|)\subseteq A\oplus_M B. \quad (3)$$

An interval-valued function $f:\mathbb{X}\subseteq\mathbb{R}^n\to\mathcal{K}_C$ is of the form

$$f(x)=\left[\underline{f}(x),\overline{f}(x)\right]=\left(\widehat{f}(x);\widetilde{f}(x)\right)$$

with $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$.

The space $\mathcal{K}_C$ of intervals can be ordered by several partial orders (see [2], [3], [7], [16], [17]); we will use the so called LU (lower-upper) partial order, defined by $A\preceq B \iff \left(\underline{a}\leq\underline{b} \text{ and } \overline{a}\leq\overline{b}\right)$; it is well known that $A\preceq B \iff A\ominus_{gH}B\preceq 0$; remark that two intervals with the same midpoint values are either equal or LU-incomparable.

In terms of the LU partial order, the LU-convexity of an interval-valued function $f(x)=\left[\underline{f}(x),\overline{f}(x)\right]$ on a convex set $\mathbb{X}\subseteq\mathbb{R}^n$ is defined by requiring $f(\alpha x+\beta y)\preceq\alpha f(x)\oplus_M\beta f(x)$ for all $x,y\in\mathbb{X}$ and all $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$.

Our definition of linearity for interval-valued functions is the following; it will be called gH-linearity.

Definition 1: Given an interval-valued function $L:\mathbb{R}^n\to\mathcal{K}_C$, we say that L is gH-linear if there exist two linear functions $\widehat{L},W:\mathbb{R}^n\to\mathbb{R}$ such that $L(x)=\left(\widehat{L}(x);|W(x)|\right)$ for all $x\in\mathbb{R}^n$, i.e., $L(x)=\left[\widehat{L}(x)-|W(x)|,\widehat{L}(x)+|W(x)|\right]$.

The following property is immediate to prove:

Proposition 2: A necessary condition for a function $F:\mathbb{R}^n\to\mathcal{K}_C$ to be gH-linear is that
(1) it is homogeneous, i.e., $F(\lambda x)=\lambda F(x)$ for all $\lambda\in\mathbb{R}$ and for all $x\in\mathbb{R}^n$, and
(2) the following inclusions are satisfied for all $x,y\in\mathbb{R}^n$: $F(x)\oplus_{gH}F(y)\subseteq F(x+y)\subseteq F(x)\oplus_M F(y)$.

Example 1: *Given n intervals $A_j=\left(\widehat{a}_j;\widetilde{a}_j\right)$ with $\widehat{a}_j,\widetilde{a}_j\in\mathbb{R}$, $\widetilde{a}_j\geq 0$ for $j=1,2,...,n$, the following gH-combination of the intervals with coefficients $x_1,x_2,...,x_n$, defined by $f(x)=\left(\sum_{j=1}^n x_j\widehat{a}_j;\left|\sum_{j=1}^n x_j\widetilde{a}_j\right|\right)$ is a gH-linear function.*

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Notation: In the rest of the paper, we denote a gH-combination of n intervals $A_j = \left(\widehat{a}_j; \widetilde{a}_j\right)$ with coefficients $x_1, x_2, ..., x_n$ as

$$\bigoplus_{j=1}^{n} x_j A_j = \left(\sum_{j=1}^{n} x_j \widehat{a}_j; \left|\sum_{j=1}^{n} x_j \widetilde{a}_j\right|\right) \tag{4}$$

while the Minkowski combination (M-combination for short) is denoted as

$$\sum_{j=1}^{n} x_j A_j = \left(\sum_{j=1}^{n} x_j \widehat{a}_j; \sum_{j=1}^{n} \left|x_j\right| \widetilde{a}_j\right);$$

remark that, for any set of intervals $A_j = \left(\widehat{a}_j; \widetilde{a}_j\right)$, $j = 1, 2, ..., n$ and for any set of real coefficients $x_1, x_2, ..., x_n$ we have

$$\bigoplus_{j=1}^{n} x_j A_j \subseteq \sum_{j=1}^{n} x_j A_j$$

and the inclusion reduces to an equality if and only if the nonzero coefficients in $\{x_1, x_2, ..., x_n\}$ are all of the same sign.

Example 2: *Let us consider the interval-valued function* $f : \mathbb{R}^2 \to \mathcal{K}_C$*, where* $f(x) = f(x_1, x_2) = x_1[2, 5] \oplus_M x_2[-1, 3]$*, i.e., f is the Minkowski combination of two intervals; it is easy to see that f satisfies the conditions of proposition 2 but it is not gH-linear; indeed, we have* $f(x) = \left(\frac{7}{2}x_1 + x_2; \frac{3}{2}|x_1| + 2|x_2|\right)$ *and the radius function* $\widetilde{f}(x_1, x_2) = \frac{3}{2}|x_1| + 2|x_2|$ *cannot be expressed as the absolute value of a linear function* $W(x_1, x_2)$*.*

More generally, given n intervals $A_j = \left(\widehat{a}_j; \widetilde{a}_j\right)$ with $\widehat{a}_j, \widetilde{a}_j \in \mathbb{R}$, $\widetilde{a}_j \geq 0$ for $j = 1, 2, ..., n$, the M-combination $\sum_{j=1}^{n} x_j A_j$ of the intervals with coefficients $x_1, x_2, ..., x_n$ is not a gH-linear function, unless $n = 1$.

We can easily prove that an interval-valued function $f : \mathbb{R}^n \to \mathcal{K}_C$ is gH-linear if and only if there exist two real vectors $\widehat{C}, C \in \mathbb{R}^n$, say $\widehat{C} = (\widehat{c_1}, \dots, \widehat{c_n})$ and $C = (c_1, \dots, c_n)$, with no restrictions in the sign of the components $\widehat{c_i}, c_i \in \mathbb{R}, i = 1, ..., n$, such that

$$f(x) = \left(\widehat{f}(x); \widetilde{f}(x)\right) = \left(\sum_{i=1}^{n} x_i \widehat{c_i}; \left|\sum_{i=1}^{n} x_i c_i\right|\right). \tag{5}$$

Note that any gH-linear interval-valued function f with interval representation $f(x) = \left(\sum_{i=1}^n x_i \widehat{c_i}; \left|\sum_{i=1}^n x_i c_i\right|\right)$, with vectors $\widehat{C}, C \in \mathbb{R}^n$, admits (obviously) also a second representation with vectors $\widehat{C}, -C \in \mathbb{R}^n$, given by

$$f(x) = \left(\sum_{i=1}^{n} x_i \widehat{c_i}; \left|\sum_{i=1}^{n} x_i (-c_i)\right|\right).$$

The property above is indeed coherent with the definition of interval gH-differentiability proposed in [16].

There exist several definitions of convexity for interval-valued functions; we will adopt the one introduced in [17], based on the LU partial order. It is easy to see that, as desirable, a gH-linear function is convex.

Proposition 3: Any gH-linear function $L : \mathbb{R}^n \to \mathcal{K}_C$, $L(x) = \left(\widehat{L}(x); |W(x)|\right)$, is LU-convex on $\mathbb{R}^n$. Furthermore, for all $\alpha, \beta \in \mathbb{R}$ and all $x, y \in \mathbb{R}^n$ it is

$$\alpha L(x) \oplus_{gH} \beta L(y) \preceq L(\alpha x + \beta y) \preceq \alpha L(x) \oplus_M \beta L(y).$$

Proof 1: From the linearity of $\widehat{L}(x)$ and $W(x)$ and from the triangular inequality, we have

$$\begin{aligned} L(\alpha x + \beta y) &= \left(\alpha \widehat{L}(x) + \beta \widehat{L}(y); |\alpha W(x) + \beta W(y)|\right) \\ &\preceq \left(\alpha \widehat{L}(x) + \beta \widehat{L}(y); |\alpha W(x)| + |\beta W(y)|\right) \\ &= \alpha L(x) \oplus_M \beta L(y); \end{aligned}$$

by choosing $\alpha, \beta \in [0, 1]$ with $\alpha + \beta = 1$ we obtain the convexity of L. The left inequality is deduced analogously.

II. Interval Fréchet and Gateaux gH-differentiability

We have now all elements to define Fréchet and Gateaux type differentiability for an interval-valued function $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$, $F(x) = (\widehat{F}(x); \widetilde{F}(x))$.

Definition 4: Let $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$, $F(x) = (\widehat{F}(x); \widetilde{F}(x)) = [\widehat{F}(x) - \widetilde{F}(x), \widehat{F}(x) + \widetilde{F}(x)]$ and let $x^{(0)} \in K$ such that $x^{(0)} + h \in K$, for all $h \in \mathbb{R}^n$ with $||h|| < \delta$ for a given $\delta > 0$. We say that F is Fréchet gH-differentiable at $x^{(0)}$ if and only if there exist a (continuous) gH-linear function $L_{x^{(0)}} : \mathbb{R}^n \to \mathcal{K}_C$ such that

$$\lim_{||h|| \to 0} \frac{|| \left(F(x^{(0)} + h) \ominus_{gH} F(x^{(0)}\right) \ominus_{gH} L_{x^{(0)}}(h))||_H}{||h||} = 0 \tag{6}$$

The linear interval-valued function $L_{x^{(0)}} : h \longrightarrow L_{x^{(0)}}(h)$, $h \in \mathbb{R}^n$, is called the Fréchet gH-derivative (or Fréchet differential function) of F at $x^{(0)}$, denoted as $D_{gH}F(x^{(0)})$, i.e.,$D_{gH}F(x^{(0)})(h) = L_{x^{(0)}}(h)$ for all $h \in \mathbb{R}^n$.

The following are fundamental properties of Fréchet differentiability (the proof of the first Theorem is easy and we omit it).

Theorem 5: Let $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$, $F(x) = (\widehat{F}(x); \widetilde{F}(x))$. If F is Fréchet differentiable at $x^{(0)} \in K$, then it is continuous at $x^{(0)}$.

Theorem 6: Let $F : \mathbb{R}^n \to \mathcal{K}_C$, $F(x) = (\widehat{F}(x); \widetilde{F}(x))$ be gH-linear. Then, at any points $x \in \mathbb{R}^n$, F is Fréchet differentiable and its differential is exactly the function $D_{gH}F(x)(h) = F(h)$ for all $h \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n$.

Proof 2: Let $F(x) = (\widehat{F}(x); \widetilde{F}(x))$ with $\widehat{F}(x) = \sum_{i=1}^n x_i \widehat{c_i}$ and $\widetilde{F}(x) = \left|\sum_{i=1}^n x_i c_i\right|$ with given (fixed) $\widehat{c_i}, c_i \in \mathbb{R}^n$. We then have the following equalities for all $x, h \in \mathbb{R}^n$ with $h \neq 0$:

$$F(x) = \left(\sum_{i=1}^{n} x_i \widehat{c_i}; \left|\sum_{i=1}^{n} x_i c_i\right|\right)$$

$$F(x + h) = \left(\sum_{i=1}^{n} (x_i + h_i) \widehat{c_i}; \left|\sum_{i=1}^{n} (x_i + h_i) c_i\right|\right)$$

$$F(h) = \left(\sum_{i=1}^{n} h_i \widehat{c_i}; \left|\sum_{i=1}^{n} h_i c_i\right|\right)$$

so that, the second component being the absolute value of the difference of two absolute values,

$$F(x+h)\ominus_{gH}F(x)=\left(\sum_{i=1}^{n}h_i\widehat{c_i};\left\|\left|\sum_{i=1}^{n}(x_i+h_i)c_i\right|-\left|\sum_{i=1}^{n}x_ic_i\right|\right\|\right);$$

denoting the interval $A_{x,h}=\left(F(x+h)\ominus_{gH}F(x)\right)\ominus_{gH}F(h)$ by $A_{x,h}=\left(\widehat{A}_{x,h};\widetilde{A}_{x,h}\right)$ in midpoint notation, it follows that

$$\begin{cases}\widehat{A}_{x,h}=0\\ \widetilde{A}_{x,h}=\left|\ \left\|\sum_{i=1}^{n}(x_i+h_i)c_i\right|-\left|\sum_{i=1}^{n}x_ic_i\right\|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|;\end{cases}$$

on the other hand, from the fact that $\widehat{A}_{x,h}=0$, it is $\left\|A_{x,h}\right\|_H=\widetilde{A}_{x,h}$ with

$$\widetilde{A}_{x,h}=\left|\ \left|\ \left|\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}h_ic_i\right|-\left|\sum_{i=1}^{n}x_ic_i\right|\ \right|\ -\ \left|\sum_{i=1}^{n}h_ic_i\right|\ \right|$$

so that

$$\frac{\left\|\left(F(x+h)\ominus_{gH}F(x)\right)\ominus_{gH}F(h)\right\|_H}{\|h\|}=\frac{\widetilde{A}_{x,h}}{\|h\|}.$$

Considering that the vector $x\in\mathbb{R}^n$ is fixed, we have three cases for the sign of the quantity $\sum_{i=1}^{n}x_ic_i$ (not depending on h):

1) if $\sum_{i=1}^{n}x_ic_i=0$, then

$$\widetilde{A}_{x,h}=\left|\ \left|\ \left|0+\sum_{i=1}^{n}h_ic_i\right|-|0|\right|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|=0$$

so that $\frac{\widetilde{A}_{x,h}}{\|h\|}=0$ for all $h\neq0$;

2) if $\sum_{i=1}^{n}x_ic_i\geq\rho>0$, then for sufficiently small $\|h\|$, it will be $\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}h_ic_i\geq0$ so that

$$\begin{aligned}\widetilde{A}_{x,h}&=\left|\ \left|\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}h_ic_i-\sum_{i=1}^{n}x_ic_i\right|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|\\&=\left|\ \left|\sum_{i=1}^{n}h_ic_i\right|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|=0;\end{aligned}$$

3) if $\sum_{i=1}^{n}x_ic_i\leq\rho<0$, then for sufficiently small $\|h\|$, it will be $\sum_{i=1}^{n}x_ic_i+\sum_{i=1}^{n}h_ic_i\leq0$ so that

$$\begin{aligned}\widetilde{A}_{x,h}&=\left|\ \left|-\sum_{i=1}^{n}x_ic_i-\sum_{i=1}^{n}h_ic_i+\sum_{i=1}^{n}x_ic_i\right|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|\\&=\left|\ \left|-\sum_{i=1}^{n}h_ic_i\right|-\left|\sum_{i=1}^{n}h_ic_i\right|\ \right|=0.\end{aligned}$$

Then, in all cases, $\widetilde{A}_{x,h}=0$ for all $h\in\mathbb{R}^n$ with sufficiently small $\|h\|$ and for all $x\in\mathbb{R}^n$; we conclude that $\lim\limits_{\|h\|\longrightarrow0}\frac{\left\|\left(F(x+h)\ominus_{gH}F(x)\right)\ominus_{gH}F(h)\right\|_H}{\|h\|}=0$ and F is Fréchet gH-differentiable with differential $D_{gH}F(x)(h)=F(h)$ for all $h\in\mathbb{R}^n$.

Using the same concept of gH-linear function, the definition of Gateaux derivative is the following:

Definition 7: Let $F:K\subseteq\mathbb{R}^n\rightarrow\mathcal{K}_C$, $F(x)=(\widehat{F}(x);\widetilde{F}(x))=[\widehat{F}(x)-\widetilde{F}(x),\widehat{F}(x)+\widetilde{F}(x)]$ and let $x^{(0)}\in K$ such that $x^{(0)}+h\in K$. We say that F is Gateaux gH-differentiable at $x^{(0)}$ if and only if there exist a (continuous) gH-linear function $L:\mathbb{R}^n\rightarrow\mathcal{K}_C$, depending only on $x^{(0)}$ such that, for all $h\in\mathbb{R}^n$ the following limit exists:

$$lim_{t\rightarrow0}\frac{\left(F(x^{(0)}+th)\ominus_{gH}F(x^{(0)}\right)}{t}=L(h).\tag{7}$$

The gH-linear interval-valued function $L:h\longrightarrow L(h)$ on $\mathbb{R}^n$ is called the Gateaux gH-derivative of F at $x^{(0)}$. Clearly, if it exists, it is unique (by unicity of the limit for all h).

In a recent paper, the following definition of gH-differentiability has been given; it implicitly suggest how to define linearity in the context of interval-valued functions.

Definition 8: [16] Let $F:K\subseteq\mathbb{R}^n\rightarrow\mathcal{K}_C$, $F(x)=(\widehat{F}(x);\widetilde{F}(x))=[\widehat{F}(x)-\widetilde{F}(x),\widehat{F}(x)+\widetilde{F}(x)]$ and let $x^{(0)}\in K$ such that $x^{(0)}+h\in K$, for all $h\in\mathbb{R}^n$ with $\|h\|<\delta$ for a given $\delta>0$. We say that F is gH-differentiable at $x^{(0)}$ if and only if there exist two vectors $\widehat{w},\widetilde{w}\in\mathbb{R}^n$, $\widehat{w}=(\widehat{w}_1,...,\widehat{w}_n)$, $\widetilde{w}=(\widetilde{w}_1,...,\widetilde{w}_n)$ and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\underset{h\rightarrow0}{lim}\widehat{\varepsilon}(h)=\underset{h\rightarrow0}{lim}\widetilde{\varepsilon}(h)=0$, such that, for all $h\neq0$,

$$\widehat{F}(x^{(0)}+h)-\widehat{F}(x^{(0)})=\sum_{j=1}^{n}h_j\widehat{w}_j+\|h\|\widehat{\varepsilon}(h),\tag{8}$$

$$\left|\widetilde{F}(x^{(0)}+h)-\widetilde{F}(x^{(0)})\right|=\left|\sum_{j=1}^{n}h_j\widetilde{w}_j+\|h\|\widetilde{\varepsilon}(h)\right|.\tag{9}$$

The interval-valued function $D_{gH}F(x^{(0)}):\mathbb{R}^n\rightarrow\mathcal{K}_C$ defined, for $h=(h_1,...,h_n)\in\mathbb{R}^n$, by

$$D_{gH}F(x^{(0)})(h)=\left(\sum_{j=1}^{n}h_j\widehat{w}_j;\left|\sum_{j=1}^{n}h_j\widetilde{w}_j\right|\right)\tag{10}$$

is called the gH-differential (or the total gH-derivative) of F at $x^{(0)}$.

In [16] we proved that if there exist two vectors $\widehat{w^*},\widetilde{w^*}$, and two functions $\widehat{\varepsilon^*}(h)$, $\widetilde{\varepsilon^*}(h)$ with $\underset{h\rightarrow0}{lim}\widehat{\varepsilon^*}(h)=\underset{h\rightarrow0}{lim}\widetilde{\varepsilon^*}(h)=0$, verifying equations (8) and (9), then $\widehat{w^*}=\widehat{w}$ and $\widetilde{w^*}=\widetilde{w}$ or $\widetilde{w^*}=-\widetilde{w}$.

Remark 9: The authors of [6] define a "linearity" concept for a function $L:\mathcal{X}\rightarrow\mathcal{K}_C$, with $L(x)=[\underline{L}(x),\overline{L}(x)]$ for $x\in\mathcal{X}$, $\mathcal{X}$ being a linear subspace of $\mathbb{R}^n$, by requiring that

(i) $L(\lambda x)=\lambda L(x)$ for all $x\in\mathcal{X}$ and $\lambda\in\mathbb{R}$;

(ii) for all $x,y\in\mathcal{X}$,

(ii.a) either $L(x+y)=L(x)\oplus_M L(y)$,

(ii.b) or $L(x+y)$ and $L(x)\oplus_M L(y)$ are LU-incomparable.

Consider that a homogeneous $F(x)$ such that $F(x+y)$ and $F(x)\oplus_M F(y)$ have the same midpoint value always satisfy conditions (i)-(ii), as, for example, $F(x_1,x_2)=[\underline{F}(x_1,x_2),\overline{F}(x_1,x_2)]$ with

$$\begin{aligned}\underline{F}(x_1,x_2)&=a_1x_1+a_2x_2-\alpha\sqrt{b_1x_1^2+b_2x_2^2}\\ \overline{F}(x_1,x_2)&=a_1x_1+a_2x_2+\alpha\sqrt{b_1x_1^2+b_2x_2^2}\end{aligned}$$

where $a_i,b_i\in\mathbb{R}$, $b_i>0$, $i=1,2$ and $\alpha>0$. $\underline{F}(x_1,x_2)$ and $\overline{F}(x_1,x_2)$ are both positively homogeneous, it is $F(x+y)\subseteq$

$F(x) \oplus_M F(y)$ and, using midpoint notation, it is $\widehat{F}(x+y) = \widehat{F}(x) + \widehat{F}(y)$ (i.e. $\widehat{F}$ is linear) and $\widetilde{F}(x+y) \le \widetilde{F}(x) + \widetilde{F}(x)$ (i.e., $\widetilde{F}$ is subadditive). Furthermore, it is possible to verify that functions which are linear according to definition in [6] do not necessarily satisfy the limit condition (6), as is the case, e.g., of $F(x_1, x_2) = x_1[\alpha-1, \alpha+1] + x_2[\beta-1, \beta+1]$ and $x = (0, 1)$ for which the limit (6) does not exist for $\|(h_1, h_2)\| \longrightarrow 0$ with $(h_1, h_2) \longrightarrow (0^+, 0^-)$.

III. General construction of gH-linear functions using intervals

Note that the expression (10) valued for vectors $\widehat{w}, \widetilde{w}$ and for vectors $\widehat{w^*}, \widetilde{w^*}$ is the same. This means that the gH-differential of F at $x^{(0)}$ is unique and does not depend on the election of $\widehat{w}, \widetilde{w}$. But for each one of these pairs of vectors we define the signed gH-differential, as follows.

To this end, we introduce a concept of a generator set of $\mathcal{K}_C$.

Definition 10: Given $r \in \mathbb{N}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_r\}$, with $G_i = (\widehat{g_i}; \widetilde{g_i}) \in \mathcal{K}_C$ for $i = 1, \ldots, r$, $\mathcal{G}$ is said to be a generator set of $\mathcal{K}_C$ if for any $A \in \mathcal{K}_C$ there exist $l_i \in \mathbb{R}$, $i = 1, \ldots, r$, such that

$$A = (\widehat{a}; \widetilde{a}) = \left(\sum_{i=1}^{r} l_i \widehat{g_i}; \left| \sum_{i=1}^{r} l_i \widetilde{g_i} \right| \right).$$

Proposition 11: The minimal numbers of elements in a generator set of $\mathcal{K}_C$ is two.

Proof 3: Clearly, $\mathcal{K}_C$ cannot be generated by a set with only one interval; we can provide $G_1, G_2 \in \mathcal{K}_C$ such that $\mathcal{G} = \{G_1, G_2\}$ is a generator set of $\mathcal{K}_C$. Indeed, consider $G_1 = (2; 1)$ and $G_2 = (0; 1)$. Considering any $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_C$, we calculate l_1, l_2 such that the two equalities $\widehat{a} = 2l_1$ and $\widetilde{a} = |l_1 + l_2|$ are satisfied by $l_1 = \frac{\widehat{a}}{2}$, $\quad l_2 = \widetilde{a} - \frac{\widehat{a}}{2}$. That is

$$A = (\widehat{a}; \widetilde{a}) = \left(\frac{\widehat{a}}{2} \widehat{g}_1 + \left(\widetilde{a} - \frac{\widehat{a}}{2} \right) \widehat{g}_2; \left| \frac{\widehat{a}}{2} \widetilde{g}_1 + \left(\widetilde{a} - \frac{\widehat{a}}{2} \right) \widetilde{g}_2 \right| \right).$$

In consequence, $\mathcal{G} = \{G_1, G_2\}$ is a generator set of $\mathcal{K}_C$.

Definition 12: If $\mathcal{G} = \{G_1, G_2\}$, $G_i = (\widehat{g_i}; \widetilde{g_i})$, $i = 1, 2$, is a generator set of $\mathcal{K}_C$, then $\mathcal{G}$ is said to be a base of $\mathcal{K}_C$. Given $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_C$, then l_1 and l_2 are said to be the coordinates of A with respect to the base $\mathcal{G}$ if $A = (l_1 \widehat{g_1} + l_2 \widehat{g_2}; |l_1 \widetilde{g_1} + l_2 \widetilde{g_2}|)$. This is denoted as $A = (l_1, l_2)_{\mathcal{G}}$.

We could introduce something like a *canonical base*.

Proposition 13: The set $\mathcal{E} = \{E_1, E_2\} = \{(1; 0), (0; 1)\}$ is a base of $\mathcal{K}_C$ and for any $A = (\widehat{a}; \widetilde{a})$, $\widetilde{a} \ge 0$, we have

$$A = (\widehat{a}, \widetilde{a})_{\mathcal{E}} = (\widehat{a}, -\widetilde{a})_{\mathcal{E}}.$$

Definition 14: The base $\mathcal{E} = \{E_1, E_2\} = \{(1; 0), (0; 1)\}$ is said to be the *canonical base* of $\mathcal{K}_C$.

Proposition 13 and Definition 14 imply that, given an interval $A = (u, v)_{\mathcal{E}}$ with respect to the canonical base, it follows that $A = (u; |v|)$. We say that A is written in *positive signed representation* with respect to the canonical base if $v \ge 0$; and in *negative signed representation* if $v \le 0$. Then, we say that an interval A can be represented as $A^{\oplus}$ and $A^{\ominus}$, respectively. So, $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_C$, $\widehat{a}, \widetilde{a} \in \mathbb{R}$, $\widetilde{a} \ge 0$, can be represented by a (positive or negative) **signed notation** $A^s = (u, v)_{\mathcal{E}} \in \{A^{\oplus}, A^{\ominus}\}$ with $A^{\oplus} = (\widehat{a}, \widetilde{a})_{\mathcal{E}}$, $A^{\ominus} = (\widehat{a}, -\widetilde{a})_{\mathcal{E}}\}$, i.e., $s \in \{\oplus, \ominus\}$.

We denote by $\mathcal{K}_C^S$ as the family of signed representations $(u, v)_{\mathcal{E}}$, for any $u, v \in \mathbb{R}$, i.e., for all $A = (\widehat{a}; \widetilde{a}) \in \mathcal{K}_C$ it is $A^{\oplus}, A^{\ominus} \in \mathcal{K}_C^S$ and for all $A^s = (u, v)_{\mathcal{E}} \in \mathcal{K}_C^S$ it is $A = (u; |v|) \in \mathcal{K}_C$.

Two arithmetic operations are natural in $\mathcal{K}_C^S$.

Definition 15: Given $A^s = (u, v)_{\mathcal{E}}$, $A_1^{s_1} = (u_1, v_1)_{\mathcal{E}}$, $A_2^{s_2} = (u_2, v_2)_{\mathcal{E}} \in \mathcal{K}_C^S$, the multiplication by a scalar $\tau \in \mathbb{R}$ and the addition are defined as

$$\begin{aligned} \tau A^s &= (\tau u, \tau v)_{\mathcal{E}} \\ A_1^{s_1} + A_2^{s_2} &= (u_1 + u_2, v_1 + v_2)_{\mathcal{E}}. \end{aligned}$$

We define $-A^s = (-1)A^s = (-u, -v)_{\mathcal{E}}$, so that $-(-A^s) = A^s$.

More generally, given $r \in \mathbb{N}$, $A_i^{s_i} = (u_i, v_i)_{\mathcal{E}} \in \mathcal{K}_C^S$, their linear combination with scalar coefficients $\tau_i \in \mathbb{R}$ for $i = 1, \ldots, r$, is

$$\sum_{i=1}^{r} \tau_i A_i^{s_i} = \tau_1 A_1^{s_1} + \tau_2 A_2^{s_2} + \cdots + \tau_r A_r^{s_r} \tag{11}$$

$$= \left(\sum_{i=1}^{r} \tau_i u_i, \sum_{i=1}^{r} \tau_i v_i \right)_{\mathcal{E}}. \tag{12}$$

Definition 16: The (canonical) *projection* of $\mathcal{K}_C^S$ onto $\mathcal{K}_C$ is the mapping $\boldsymbol{P} : \mathcal{K}_C^S \to \mathcal{K}_C$ defined by $\boldsymbol{P}((u, v)_{\mathcal{E}}) = (u; |v|)$ and the interval $(u; |v|)$ is said to be the projection of the signed $(u, v)_{\mathcal{E}}$. The inverse mapping is $\boldsymbol{P}^{-1}((\widehat{a}; \widetilde{a})) = \{(\widehat{a}, \widetilde{a})_{\mathcal{E}}, (\widehat{a}; -\widetilde{a})_{\mathcal{E}}\}$.

We have that any gH-linear function $L : \mathbb{R}^n \longrightarrow \mathcal{K}_C$ can be obtained in terms of appropriate signed representations of n intervals.

Proposition 17: Let $L : \mathbb{R}^n \longrightarrow \mathcal{K}_C$ with $L(x) = \left(\widehat{L}(x); \widetilde{L}(x) \right)$ be any gH-linear interval-valued function. Then there exist n intervals $A_1, A_2, ..., A_n \in \mathcal{K}_C$ and there exist a selection of their signed representations $A_1^{s_1}, A_2^{s_2}, ..., A_n^{s_n} \in \mathcal{K}_C^S$ with such that, for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$,

$$L(x) = \boldsymbol{P} \left(\sum_{i=1}^{n} x_i A_i^{s_i} \right) \tag{13}$$

More precisely, if $A_i^{s_i} = (u_i, v_i)_{\mathcal{E}}$ with $u_i, v_i \in \mathbb{R}$ for $i = 1, 2, ..., n$ and with $v_i \ge 0$ if $s_i = \oplus$, $v_i \le 0$ if $s_i = \ominus$, then

$$\widehat{L}(x) = \sum_{i=1}^{n} x_i u_i \text{ and } \widetilde{L}(x) = \left| \sum_{i=1}^{n} x_i v_i \right|.$$

Proof 4: The proof follows immediately from the fact that any linear functions of n variables $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ has the form $f(x) = \sum_{i=1}^{n} x_i a_i$ with appropriate coefficients $a_i \in \mathbb{R}$ and all linear functions can be obtained in that way. On the other hand, the components $u_i, v_i \in \mathbb{R}$ of $(u_i, v_i)_{\mathcal{E}} \in \mathcal{K}_C^S$ can have any real value (consider the obvious bijection between $\mathbb{R}^2$ and $\mathcal{K}_C^S$ by the correspondence $(u, v) \leftrightarrow (u, v)_{\mathcal{E}}$) so that both linear functions $\widehat{L}(x)$ and $W(x)$ such that $\widetilde{L}(x) = |W(x)|$ can be obtained by choosing the intervals $A_i = (\widehat{a}_i; \widetilde{a}_i)$ with $\widehat{a}_i = u_i$ and $\widetilde{a}_i = v_i$ or $\widetilde{a}_i = -v_i$ according to the value of $s_i \in \{\oplus, \ominus\}$.

Remark 18: For any set of n intervals $A_1, A_2, ..., A_n \in \mathcal{K}_C$, the number of possible selections of their signed representations, i.e., the number of possible n-tuples $(s_1, s_2, ..., s_n)$ with $s_i \in \{\oplus, \ominus\}$, is in general 2^n (eventually reduced if $v_i = 0$ for some i) so that the construction of gH-linear functions has a combinatorial nature. We can also remark that obviously, from the presence of the absolute value in $\widetilde{L}(x)$, the gH-linear function $L(x)$ does not change for all x if the signed representations $A_i^{s_i} = (u_i, v_i)_{\mathcal{E}}$ are all reverted by changing each $s_i = \oplus$ to $\ominus$ and each $s_i = \ominus$ to $\oplus$.

Proposition 17 provides a characterization for gH-linear interval-valued functions, which is clearly a natural extension of the classical linear functions in vector spaces. In general, the functions pointed out in Remark 9 do not fulfil the gH-linearity defined in this work, as expected, although they fulfil the linearity definition given in [6]. Furthermore, it is also expected that if one defines a class of linear interval-valued functions, then this class is contained into the class of convex functions that one defines too, such as we have proved for the gH-linear functions.

Definition 19: Let $x^{(0)} \in K$, $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$ gH-differentiable at $x^{(0)}$, two vectors $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, $\widehat{w} = (\widehat{w}_1, ..., \widehat{w}_n)$, $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)$, and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\lim_{h\to 0}\widehat{\varepsilon}(h) = \lim_{h\to 0}\widetilde{\varepsilon}(h) = 0$, such that (8) and (9) are fulfilled. The signed interval-valued function $D^S_{gH}F(x^{(0)}) : \mathbb{R}^n \to \mathcal{K}^S_C$, defined, for $h = (h_1, ..., h_n) \in \mathbb{R}^n$, by

$$D^S_{gH}F(x^{(0)})(h) = \sum_{j=1}^{n} h_j(\widehat{w}_j, \widetilde{w}_j)_{\mathcal{E}} \tag{14}$$

is called the signed gH-differential of F with respect to $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$ at $x^{(0)}$.

A first relationship between the gH-differential and the signed gH-differential is that the signed gH-differential of an interval-valued function at a point $x^{(0)}$ is a signed interval-valued function representation of the gH-differential at $x^{(0)}$, such as the following two results establish.

Proposition 20: Let $x^{(0)} \in K$, $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$ gH-differentiable at $x^{(0)}$, two vectors $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, $\widehat{w} = (\widehat{w}_1, ..., \widehat{w}_n)$, $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)$, and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\lim_{h\to 0}\widehat{\varepsilon}(h) = \lim_{h\to 0}\widetilde{\varepsilon}(h) = 0$, such that (8) and (9) are fulfilled. Then, for every $h = (h_1, ..., h_n) \in \mathbb{R}^n$,

$$D_{gH}F(x^{(0)})(h) = \boldsymbol{P}(D^S_{gH}F(x^{(0)})(h)). \tag{15}$$

Proof 5: From the definitions 8 and 19, it is derived:

$$D^S_{gH}(x^{(0)})(h) = \sum_{j=1}^{n} h_j(\widehat{w}_j, \widetilde{w}_j)_{\mathcal{E}}.$$

Then,

$$\begin{aligned}\boldsymbol{P}((\sum_{j=1}^{n} h_j\widehat{w}_j, \sum_{j=1}^{n} h_j\widetilde{w}_j)_{\mathcal{E}}) &= \left(\sum_{j=1}^{n} h_j\widehat{w}_j; \left|\sum_{j=1}^{n} h_j\widetilde{w}_j\right|\right) \\ &= D_{gH}F(x^{(0)})(h),\end{aligned}$$

for all $h = (h_1, ..., h_n) \in \mathbb{R}^n$.

On the other hand, we have the following representation result.

Proposition 21: Let $x^{(0)} \in K$, $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$ be gH-differentiable at $x^{(0)}$, two vectors $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, $\widehat{w} = (\widehat{w}_1, ..., \widehat{w}_n)$, $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)$, and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\lim_{h\to 0}\widehat{\varepsilon}(h) = \lim_{h\to 0}\widetilde{\varepsilon}(h) = 0$, such that (8) and (9) are fulfilled. Then, for every $h = (h_1, ..., h_n) \in \mathbb{R}^n$,

$$\boldsymbol{P}^{-1}(D_{gH}F(x^{(0)})(h)) = \left\{D^{s_1}_{gH}F(x^{(0)})(h), D^{s_2}_{gH}F(x^{(0)})(h)\right\}, \tag{16}$$

where $D^{s_1}_{gH}F(x^{(0)})(h)$ is the signed interval-valued gH-differential of F at $x^{(0)}$ with respect to h that corresponds to $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, and $D^{s_2}_{gH}F(x^{(0)})(h)$ is the signed interval-valued gH-differential of F at $x^{(0)}$ with respect to h that correspond to $\widehat{w}, -\widetilde{w} \in \mathbb{R}^n$.

Proof 6: From definitions 16, 8 and Definition 19, it is

$$\begin{aligned}\boldsymbol{P}(D^S_{gH}F(x^{(0)})(h)) &= \boldsymbol{P}(\sum_{j=1}^{n} h_j(\widehat{w}_j, \widetilde{w}_j)_{\mathcal{E}}) \\ &= \boldsymbol{P}((\sum_{j=1}^{n} h_j\widehat{w}_j, \sum_{j=1}^{n} h_j\widetilde{w}_j)_{\mathcal{E}})) \\ &= \left(\sum_{j=1}^{n} h_j\widehat{w}_j; \left|\sum_{j=1}^{n} h_j\widetilde{w}_j\right|\right) \\ &= D_{gH}F(x^{(0)})(h),\end{aligned}$$

for all $h = (h_1, ..., h_n) \in \mathbb{R}^n$.

Proposition 20 shows that the interval-valued differential of F at $x^{(0)}$ with respect to h does not depend on the election of the signed interval-valued function representation of the gH-differential at $x^{(0)}$. Also,

Theorem 22: Let $x^{(0)} \in K$, $F : K \subseteq \mathbb{R}^n \to \mathcal{K}_C$ gH-differentiable at $x^{(0)}$, two vectors $\widehat{w}, \widetilde{w} \in \mathbb{R}^n$, $\widehat{w} = (\widehat{w}_1, ..., \widehat{w}_n)$, $\widetilde{w} = (\widetilde{w}_1, ..., \widetilde{w}_n)$, and two functions $\widehat{\varepsilon}(h)$, $\widetilde{\varepsilon}(h)$ with $\lim_{h\to 0}\widehat{\varepsilon}(h) = \lim_{h\to 0}\widetilde{\varepsilon}(h) = 0$, such that (8) and (9) are fulfilled. Then, the interval-valued function $D_{gH}F(x^{(0)}) : \mathbb{R}^n \to \mathcal{K}_C$ with the signed interval-valued function $D^S_{gH}F(x^{(0)}) : \mathbb{R}^n \to \mathcal{K}^S_C$ is gH-linear. Furthermore, considering the intervals $W_j = (\widehat{w}_j; |\widetilde{w}_j)|)$ with signed representations $W_j^{s_j} = (\widehat{w}_j, \widetilde{w}_j)_{\mathcal{E}}$ for $j = 1, \ldots, n$, we have

$$D_{gH}F(x^{(0)})(h) = \boldsymbol{P}\left(\sum_{j=1}^{n} h_j(W_j^{s_j})_{\mathcal{E}}\right) \tag{17}$$

for all $h = (h_1, ..., h_n) \in \mathbb{R}^n$.

A final interesting result is that any Minkowski combination of n intervals $A_j = (\widehat{a}_j; \widetilde{a}_j)$, $j = 1, ..., n$, seems to be Fréchet gH-differentiable only at points $x^{(0)} \in \mathbb{R}^n$ devoid of null components, i.e., $x_j^{(0)} \neq 0$ for all $j = 1, ..., n$ (the case $n = 1$ is well known, see [16]).

Proposition 23: Let $A_j = (\widehat{a}_j; \widetilde{a}_j)$, $j = 1, 2, ..., n$, $n > 1$, be given intervals and consider the interval-valued function $F : \mathbb{R}^n \to \mathcal{K}_C$ defined by $F(x) = \sum_{j=1}^{n} x_j A_j = \left(\sum_{j=1}^{n} x_j\widehat{a}_j;\ \sum_{j=1}^{n} \left|x_j\right|\widetilde{a}_j\right)$. Let $x^{(0)} \in \mathbb{R}^n$ with $x^{(0)} =$

$(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})$ and $x_j^{(0)} = 0 \Longrightarrow \widetilde{a}_j = 0$ for all j; then F is Fréchet gH-differentiable at $x^{(0)}$ with

$$D_{gH}F(x^{(0)})(h) = L_{x^{(0)}}(h) \text{ for all } h \in \mathbb{R}^n$$

where the gH-linear function $L_{x^{(0)}} : \mathbb{R}^n \to \mathcal{K}_C$ (depending on $x^{(0)}$) is given by

$$L_{x^{(0)}}(h) = \left(\sum_{j=1}^{n} h_j\widehat{a}_j; \left|\sum_{j=1}^{n} h_j\widetilde{w}_j\right|\right) \text{ with}$$

$$\widetilde{w}_j = \begin{cases} \widetilde{a}_j & \text{if } x_j^{(0)} \geq 0 \\ -\widetilde{a}_j & \text{if } x_j^{(0)} < 0 \end{cases}, \ j = 1, ..., n.$$

Proof 7: We have $F(x^{(0)} + h) = \left(\sum_{j=1}^n (x_j^{(0)} + h_j)\widehat{a}_j;\ \sum_{j=1}^n \left|x_j^{(0)} + h_j\right|\widetilde{a}_j\right)$ so that $\left(F(x^{(0)} + h) \ominus_{gH} F(x^{(0)})\right)$ equals $\left(\sum_{j=1}^n h_j\widehat{a}_j;\ \left|\sum_{j=1}^n \left(\left|x_j^{(0)} + h_j\right| - \left|x_j^{(0)}\right|\right)\widetilde{a}_j\right|\right)$; then, for $h \neq 0$ and sufficiently small $\|h\|$ we have that $\varphi(h) = \frac{\left\|\left(F(x^{(0)}+h)\ominus_{gH}F(x^{(0)})\right)\ominus_{gH}L_{x^{(0)}}(h)\right\|_H}{\|h\|}$ is $\varphi(h) = \frac{\left\|\sum_{j=1}^n\left(\left|x_j^{(0)}+h_j\right|-\left|x_j^{(0)}\right|\right)\widetilde{a}_j\right|-\left|\sum_{j=1}^n h_j\widetilde{w}_j\right\|}{\|h\|}$; on the other hand, for small $\|h\|$,

$$\left(\left|x_j^{(0)} + h_j\right| - \left|x_j^{(0)}\right|\right)\widetilde{a}_j = = \begin{cases} h_j\widetilde{a}_j = h_j\widetilde{w}_j, & x_j^{(0)} > 0 \\ |h_j|\widetilde{a}_j, & x_j^{(0)} = 0 \\ -h_j\widetilde{a}_j = h_j\widetilde{w}_j, & x_j^{(0)} < 0 \end{cases}$$

and, under the assumption that $x_j^{(0)} = 0 \Longrightarrow \widetilde{a}_j = 0$, the numerator of $\varphi(h)$ is $\left\|\sum_{j=1}^n h_j\widetilde{w}_j\right| - \left|\sum_{j=1}^n h_j\widetilde{w}_j\right\| = 0$.

Remark 24: Consider $n = 2$, $x_1^{(0)} = 0$, $x_2^{(0)} = 1$, $\widetilde{a}_1, \widetilde{a}_2 > 0$; with $\varphi(h)$ as in the proof of Proposition 23, the two limits $\lim_{h_1\to 0^-, h_2\to 0^+} \varphi(h)$ and $\lim_{h_1\to 0^+, h_2\to 0^-} \varphi(h)$, if they exist for some $\widetilde{w}_1, \widetilde{w}_2$, cannot be equal; we conclude that $\lim_{\|h\|\to 0} \varphi(h)$ does not exist. This shows, in particular, that the definition of Fréchet (or Gateaux) differentiability suggested in [6], based on Minkowski linear combinations, is such that a linear function in not differentiable at all points.

IV. The fuzzy case and some conclusions

The concepts of Gateaux and Fréchet gH-differentiability to the fuzzy case has been recently addressed, e.g., in [5], [12]. The extension of interval gH-linearity to the level-wise fuzzy setting is an immediate by-product of the results introduced in the previous sections and are simply obtained:

Definition 25: A fuzzy-valued function $F : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$, having α-cuts $[F(x)]_\alpha = \left(\widehat{F}_\alpha(x); \widetilde{F}_\alpha(x)\right)$ for all $\alpha \in [0, 1]$, is Fréchet-LgH-differentiable (or Gateaux-LgH-differentiable) at a point $x_0 \in K$ if and only if all interval-valued functions $x \longrightarrow \left(\widehat{F}_\alpha(x); \widetilde{F}_\alpha(x)\right)$ for $\alpha \in [0, 1]$ satisfy Definition 4 (or Definition 7, respectively) at x_0.

Clearly, if F satisfies the definition above, it is possible that the corresponding families of interval gH-linear functions $\{L_\alpha|\alpha \in [0, 1]\}$ do not define the level-sets of fuzzy numbers (in particular, they may not satisfy the nesting property); we then introduce the following definitions for the fuzzy case:

Definition 26: A fuzzy-valued function $L : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$ is said to be gH-linear if all its α-cuts have the form $[L(x)]_\alpha = \left(\widehat{L}_\alpha(x); |W_\alpha(x)|\right)$ with linear functions $\widehat{L}_\alpha(x)$ and $W_\alpha(x)$ for all $\alpha \in [0, 1]$.

Definition 27: A fuzzy-valued function $F : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$ is said to be Fréchet-gH-differentiable at $x_0 \in K$ if it is Fréchet-LgH-differentiable at x_0 with an associated gH-linear fuzzy-valued function $L : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$.

Definition 28: A fuzzy-valued function $F : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$ is said to be Gateaux-gH-differentiable at $x_0 \in K$ if it is Gateaux-LgH-differentiable at x_0 with an associated gH-linear fuzzy-valued function $L : \mathbb{R}^n \longrightarrow \mathbb{R}_{\mathcal{F}}$.

We remark explicitly that the concepts and results on fuzzy total and directional gH-derivatives and LgH-derivatives, as obtained in [16], can be immediately applied to characterize Gateaux and Fréchet LgH-differentiability. In a forthcoming paper, we analyse and detail these facts and their usefulness in applications.

References

[1] G. Alefeld, and J. Herzberger, "Introduction to Interval Computations", Academic Press, New York, 1983.

[2] M. Arana-Jiménez, "Nondominated solutions in a fully fuzzy linear programming problem", Math. Methods Appl. Sci., vol. 41, 2018, pp. 7421–7430.

[3] M. Arana-Jiménez, and C. Sánchez-Gil, "On generating the set of nondominated solutions of a linearprogramming problem with parameterized fuzzy numbers", Journal of Global Optimization, https://doi.org/10.1007/s10898-019-00841-7

[4] Y. Chalco-Cano, W.A. Lodwick, and B. Bede, "Singel level constraint interval arithmetic", Fuzzy Sets and Systems, vol. 257, 2014, pp. 146-168.

[5] E. Esmi, F. Santo Pedro, L. Carvalho de Barros, W. Lodwick, Fréchet derivative for linearly correlated fuzzy function, Information Sciences, 435 (2018) 150-160.

[6] D. Ghosh, R.S. Chauhan, R. Mesiar, A.K. Debnath, Generalized Hukuhara Gâteaux and Fréchet derivatives of interval-valued functions and their application in optimization with interval-valued functions, Information Sciences, 510 (2020) 317-340.

[7] M.L. Guerra, and L. Stefanini. "A comparison index for interval based on generalized Hukuhara difference", Soft. Comput., vol. 16, 2012, pp. 1931-1943.

[8] Z. Kulpa, Diagrammatic representation for interval arithmetic, Linear Algebra and its Applications, 324 (2001) 55–80.

[9] S. Markov, "Calculus for interval functions of real variable", Computing, vol. 22, 1979, pp. 325-337.

[10] R.E. Moore, "Interval Analysis Prentice-Hall", Englewood Cliffs, NJ, 1966.

[11] R.E. Moore, "Method and Applications of Interval Analysis", SIAM, Philadelphia, 1979.

[12] F. Santo Pedro, E. Esmi, L. Carvalho de Barros, Calculus for linearly correlated fuzzy function using Fréchet derivative and Riemann integral, Information Sciences, 512 (2020) 219-237.

[13] L. Stefanini, "A generalization of Hukuhara difference". In D. Dubois et Al. (Eds.), Soft Methods for Handling Variability and Imprecision, Springer, 2008, pp. 205-210.

[14] L. Stefanini, and B. Bede, "Generalized Hukuhara differentiability of interval-valued functions and interval differential equations", Nonlinear Anal., vol. 71, 2009, pp. 1311-1328.

[15] L. Stefanini, "A generalization of Hukuhara difference and division for interval and fuzzy arithmetic", Fuzzy Sets and Systems, vol. 161, 2010, pp. 1564-1584.

[16] L. Stefanini, and M. Arana-Jiménez, "Karush-Kuhn-Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability", Fuzzy Sets and Systems, vol. 362, 2019, pp. 1-34.

[17] H.C. Wu , Duality theory for optimization problems with interval-valued objective function, J. Optim. Theory Appl. 144 (3), 2010, pp. 615–628.