Influence of new interval-valued pre-aggregation function on medical decision making

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***Abstract*—In this contribution the new concept of the operator in the interval-valued fuzzy sets, which is a pre-aggregation function, and their application in medical diagnosis is presented. Taking into account the widths of the intervals the type of interval operator measures are proposed in decision making problem. We suggest to apply the new interval-valued operator in decision model of medical diagnosis support.**

***Index Terms*—IV aggregation function, IV pre-aggregation function, interval order, general approximate reasoning**

I. Introduction

Many new methods and theories behaving imprecision and uncertainty have been proposed since fuzzy sets were introduced by Zadeh [1]. Interval-valued fuzzy sets ([2], [3]) like intuitionistic fuzzy sets [4] appeared very useful because of their flexibility in determining the grade of membership/non-membership of elements into the concept under study (see [5]). Moreover, various applications of interval-valued or intuitionistic fuzzy sets for solving real-life problems involving pattern recognition, medical diagnosis, decision-making or image thresholding were successfully proposed. Especially, many authors can be found in the literature who propose ([6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]) different types of operators (interval-valued operators) study and/or used for the decision making problem. Also in classification the use of interval-valued fuzzy sets IVFSs has led to improvements of the performance of some of the state-of-the-art algorithms for fuzzy rule-based classification systems. More precisely, the use of intervals for the creation of the rules gives more flexibility to the algorithms, which leads to better results ([19], [20]). The use of IVFSs together with pre-aggregation functions provide better results than their fuzzy counterparts. This results are comparable to those obtained using the admissible orders defined in ([21], [22]). Similar situations arise in decision making, for instance [23]. Moreover, in the paper [24] we may find interesting dependences with weak monotonicity, where were combined both trends in the theory of aggregation, i.e. a trend towards the relaxation of the axiom of monotonicity and also towards the extension of the definition to other domains besides real numbers, and based on the structure of Riesz spaces, we provide a framework to define directional monotonicity for functions that handle various types of uncertain data coming from different extensions of fuzzy sets.

The motivation to write this paper was to examine the effect on approximate inference has a new interval operator, i.e an interval-valued pre-aggregation function (IV pre-aggregation function). It was used in the generalized composition used in the inference algorithm. Interesting properties, in particular narrowing the range values, of the introduced operator was the main motive for its use. Since the introduced operator is not monotonic due to none of the orders considered in [25], so the directional monotonicity is considered, that leads to the concept of pre-aggregation functions.

This work is composed of the following parts. Firstly, some concepts and results useful in further considerations are recalled (Section 2). Next, we examine the new operator in interval-valued fuzzy setting. Especially, the composition built with the new operator is examined (Sections 3). In Section 4 we propose an interval-valued multi conditional approximate reasoning algorithm. To finish, Section 5 provides an application of our definition of composition to a medical risk prediction problem.

II. Preliminaries

A. Interval-valued fuzzy set theory. Orders in the interval-valued fuzzy sets

Let $L^I = \{[\underline{a}, \overline{a}] : \ \underline{a}, \overline{a} \in [0,1], \ \underline{a} \leqslant \overline{a}\}$ denote a family of all subintervals of the unit interval. Let $X \neq \emptyset$ denote a universe of discourse.

Definition 1 (cf. [2], [3])**.** An interval-valued fuzzy set IVFS F in X is a mapping $F : X \to L^I$ such that $F(x) = [\underline{F}(x), \overline{F}(x)] \in L^I$ for $x \in X$. The well-known classical monotonicity (partial order) for intervals is of the form

$$[\underline{x}, \overline{x}] \leq_{L^I} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \leq \underline{y} \ \ and \ \ \overline{x} \leq \overline{y}).$$

where

$$[\underline{x}, \overline{x}] <_{L^I} [\underline{y}, \overline{y}] \Leftrightarrow$$

$$[\underline{x}, \overline{x}] \leq_{L^I} [\underline{y}, \overline{y}] \ \ and \ \ (\underline{x} < \underline{y} \ \ or \ \ \overline{x} < \overline{y}).$$

The family of all interval-valued fuzzy sets (relations) on the universe X ($X \times Y$) is denoted by $IVFS(X)$ ($IVFR(X \times Y)$).

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The family of all interval-valued fuzzy sets on a given universe X with $\leq_{L^I}$ is partially ordered and moreover it is a lattice. The operations joint and meet are defined respectively

$$[\underline{x},\overline{x}] \vee [\underline{y},\overline{y}] = [\max(\underline{x},\underline{y}),\max(\overline{x},\overline{y})],$$
$$[\underline{x},\overline{x}] \wedge [\underline{y},\overline{y}] = [\min(\underline{x},\underline{y}),\min(\overline{x},\overline{y})].$$

Note that the structure $(L^I, \vee, \wedge)$ is a complete lattice, with the partial order $\leq_{L^I}$, where

$$\mathbf{1} = [1,1] \;\; and \;\; \mathbf{0} = [0,0]$$

are the greatest and the smallest element of $(L^I, \leq_{L^I})$, respectively.

Since in many real-life problems we need a linear order to be able to compare any two intervals, we are interested in extending the partial order $\leq_{L^I}$ to a linear one. The concept of, so called, admissible order would be useful there. We recall the notion of an admissible order, which was introduced in [22] and studied, for example, in [26] and [27]. The linearity of the order is needed in many applications of real problems, in order to be able to compare any two interval data [23].

Definition 2 ([22])**.** An order $\leq_{Adm}$ in L^I is called admissible if it is linear and satisfies that for all $x, y \in L^I$, such that $x \leq_{L^I} y$, then $x \leq_{Adm} y$.

Proposition 1 ([22])**.** *Let $B_1, B_2 : [0,1]^2 \to [0,1]$ be two continuous aggregation functions, such that, for all $x = [\underline{x},\overline{x}], y = [\underline{y},\overline{y}] \in L^I$, the equalities $B_1(\underline{x},\overline{x}) = B_1(\underline{y},\overline{y})$ and $B_2(\underline{x},\overline{x}) = B_2(\underline{y},\overline{y})$ hold if and only if $x = y$. If the order $\leq_{B_{1,2}}$ on L^I is defined by $x \leq_{B_{1,2}} y$ if and only if*

$$B_1(\underline{x},\overline{x}) < B_1(\underline{y},\overline{y}) \quad or$$
$$(B_1(\underline{x},\overline{x}) = B_1(\underline{y},\overline{y}) \;\; and \;\; B_2(\underline{x},\overline{x}) \leq B_2(\underline{y},\overline{y})),$$

then $\leq_{B_{1,2}}$ is an admissible order on L^I.

Example 1 ([22])**.** The following are special cases of admissible linear orders on L^I:

- The Xu and Yager order:
$$[\underline{x},\overline{x}] \leq_{XY} [\underline{y},\overline{y}] \Leftrightarrow \underline{x}+\overline{x} < \underline{y}+\overline{y} \quad \text{or}$$
$$(\overline{x}+\underline{x} = \overline{y}+\underline{y} \;\text{ and }\; \overline{x}-\underline{x} \leq \overline{y}-\underline{y}).$$
- The first lexicographical order (with respect to the first variable), $\leq_{Lex1}$ defined as:
$$[\underline{x},\overline{x}] \leq_{Lex1} [\underline{y},\overline{y}] \Leftrightarrow \underline{x} < \underline{y} \text{ or } (\underline{x} = \underline{y} \;\text{ and }\; \overline{x} \leq \overline{y}).$$
- The second lexicographical order (with respect to the second variable), $\leq_{Lex2}$ defined as:
$$[\underline{x},\overline{x}] \leq_{Lex2} [\underline{y},\overline{y}] \Leftrightarrow \overline{x} < \overline{y} \text{ or } (\overline{x} = \overline{y} \;\text{ and }\; \underline{x} \leq \underline{y}).$$
- The $\alpha\beta$ order, $\leq_{\alpha\beta}$ defined as:
$$[\underline{x},\overline{x}] \leq_{\alpha\beta} [\underline{y},\overline{y}] \Leftrightarrow K_\alpha(\underline{x},\overline{x}) < K_\alpha(\underline{y},\overline{y}) \text{ or}$$
$$(K_\alpha(\underline{x},\overline{x}) = K_\alpha(\underline{y},\overline{y}) \text{ and } K_\beta(\underline{x},\overline{x}) \leq K_\beta(\underline{y},\overline{y})),$$
for some $\alpha \neq \beta \in [0,1]$ and $x, y \in L^I$, where $K_\alpha : [0,1]^2 \to [0,1]$ is defined as $K_\alpha(x,y) = \alpha x + (1-\alpha)y$.

The orders $\leq_{XY}$, $\leq_{Lex1}$ and $\leq_{Lex2}$ are special cases of the order $\leq_{\alpha\beta}$ with $\leq_{0.5\beta}$ (for $\beta > 0.5$), $\leq_{1,0}$, $\leq_{0,1}$, respectively. The orders $\leq_{XY}$, $\leq_{Lex1}$, $\leq_{Lex2}$, and $\leq_{\alpha\beta}$ are admissible linear orders $\leq_{B_{1,2}}$ defined by pairs of aggregation functions, namely weighted means. In the case of the orders $\leq_{Lex1}$ and $\leq_{Lex2}$, the aggregations that are used are the projections P_1, P_2 and P_2, P_1, respectively.

B. *Interval-valued aggregation functions*

Now we recall the concept of an aggregation function in L^I. We consider aggregation with respect both to $\leq_{L^I}$ and $\leq_{Adm}$. In the later part of the paper we use the notation $\leq$ for both partial and admissible linear order, with 0_{L^I} and 1_{L^I} as minimal and maximal element of L^I, respectively.

Definition 3 ([27], [28])**.** An operation $\mathcal{A} : (L^I)^n \to L^I$ is called an interval-valued aggregation function if it is increasing with respect to the order $\leq$ (partial or total) and

$$\mathcal{A}(\underbrace{\mathbf{0},...,\mathbf{0}}_{n\times}) = \mathbf{0}, \;\; \mathcal{A}(\underbrace{\mathbf{1},...,\mathbf{1}}_{n\times}) = \mathbf{1}.$$

A special class of interval-valued aggregation functions is the one formed by the so called representable interval-valued aggregation functions.

Definition 4 ([14], [29])**.** An interval-valued aggregation function $\mathcal{A} : (L^I)^n \to L^I$ is said to be representable if there exist aggregation functions $A_1, A_2 : [0,1]^n \to [0,1]$ such that

$$\mathcal{A}(x_1,\ldots,x_n) = [A_1(\underline{x}_1,\ldots\underline{x}_n), A_2(\overline{x}_1,\ldots,\overline{x}_n)]$$

for all $x_1,\ldots,x_n \in L^I$, provided that

$$A_1(\underline{x}_1,\ldots\underline{x}_n) \leq A_2(\overline{x}_1,\ldots,\overline{x}_n).$$

Example 2. Lattice operations $\wedge$ and $\vee$ on L^I are examples of representable aggregation functions on L^I with respect to the partial order $\leq_{L^I}$, with $A_1 = A_2 = \min$ in the first case and $A_1 = A_2 = \max$ in the second one. However, $\wedge$ and $\vee$ are not interval-valued aggregation functions with respect to $\leq_{Lex1}$, $\leq_{Lex2}$ or $\leq_{XY}$. The following are other examples of representable interval-valued aggregation functions with respect to $\leq_{L^I}$.

- The projections:
$$\mathcal{A}_L([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\underline{x},\overline{x}], \;\; \mathcal{A}_R([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\underline{y},\overline{y}].$$
- The representable product:
$$\mathcal{A}_p([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\underline{xy},\overline{xy}].$$
- The representable arithmetic mean:
$$\mathcal{A}_{mean}([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\frac{\underline{x}+\underline{y}}{2},\frac{\overline{x}+\overline{y}}{2}].$$
- The representable geometric mean:
$$\mathcal{A}_{gmean}([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\sqrt{\underline{xy}},\sqrt{\overline{xy}}].$$
- The representable harmonic mean:
$$\mathcal{A}_H([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = \begin{cases} [0,0], & if\ x = y = [0,0], \\ [\frac{2\underline{xy}}{\underline{x}+\underline{y}},\frac{2\overline{xy}}{\overline{x}+\overline{y}}], & otherwise. \end{cases}$$

- The representable power mean:

$$\mathcal{A}_{power}([\underline{x},\overline{x}],[\underline{y},\overline{y}]) = [\sqrt{\frac{\underline{x}^2+\underline{y}^2}{2}},\sqrt{\frac{\overline{x}^2+\overline{y}^2}{2}}].$$

Representability is not the only possible way to build interval-valued aggregation functions with respect to $\leq_{L^I}$ or $\leq_{Adm}$.

Example 3. Let $A : [0,1]^2 \to [0,1]$ be an aggregation function.

- The function $\mathcal{A}_1 : (L^I)^2 \to L^I$, where

$$\mathcal{A}_1(x,y) = \begin{cases} [1,1], & if\ (x,y) = ([1,1],[1,1]), \\ [0, A(\underline{x},\overline{y})], & otherwise, \end{cases}$$

  is a non-representable interval-valued aggregation function with respect to $\leq_{L^I}$.
- The functions $\mathcal{A}_2, \mathcal{A}_3 : (L^I)^2 \to L^I$ ([25]), where

$$\mathcal{A}_2(x,y) = \begin{cases} [1,1], & if\ (x,y) = ([1,1],[1,1]) \\ [0, A(\underline{x},\underline{y})], & otherwise, \end{cases}$$

$$\mathcal{A}_3(x,y) = \begin{cases} [0,0], & if\ (x,y) = ([0,0],[0,0]) \\ [A(\underline{x},\underline{y}),1], & otherwise, \end{cases}$$

  are non-representable interval-valued aggregation functions with respect to $\leq_{Lex1}$.
- The functions $\mathcal{A}_4, \mathcal{A}_5 : (L^I)^2 \to L^I$ ([25]), where

$$\mathcal{A}_4(x,y) = \begin{cases} [1,1], & if\ (x,y) = ([1,1],[1,1]) \\ [0, A(\overline{x},\overline{y})], & otherwise, \end{cases}$$

$$\mathcal{A}_5(x,y) = \begin{cases} [0,0], & if\ (x,y) = ([0,0],[0,0]) \\ [A(\overline{x},\overline{y}),1], & otherwise, \end{cases}$$

  are non-representable interval-valued aggregation functions with respect to $\leq_{Lex2}$.

- $\mathcal{A}_{mean}$ is an aggregation function with respect to $\leq_{\alpha\beta}$ (cf. [26]).
- The following function

$$\mathcal{A}_{\alpha}(x,y) = [\alpha\underline{x} + (1-\alpha)\underline{y}, \alpha\overline{x} + (1-\alpha)\overline{y}]$$

  is an interval-valued aggregation function on L^I with respect to $\leq_{Lex1}$, $\leq_{Lex2}$ and $\leq_{XY}$ for $x, y \in L^I$ and $\alpha \in [0,1]$ (cf. [27]).

C. Directional monotonicity of IV functions

Now we recall the concept of the directional monotonicity of interval-valued fuzzy functions.

Definition 5 ([30])**.** Let $v = (a_1, b_1, ..., a_n, b_)) \in (R^2)^n$ such that $(a_i, b_i) \neq \overrightarrow{0}$ for some $i \in \{1,...,n\}$. A function $F : (L^I)^n \to L^I$ is said to be v-increasing (resp. v-decreasing) if for all $x \in (L^I)^n$ and $c > 0$ such that $x + cv \in (L^I)^n$, it holds that

$$F(x) \leq_{L^I} F(x+cv)$$

$$(resp.\ F(x) \geq_{L^I} F(x+cv)).$$

In the case that F is simultaneously v-increasing and v-decreasing, F is said to be v-constant.

Theorem 1 ([30])**.** *Let $a, b > 0$ and $u, v \in (R^2)^n \setminus \{0\}$ such that for all $x \in (L^I)^n$ and $c > 0$ that satisfy $x + c(au + bv) \in (L^I)^n$, it holds that either $x + cau \in (L^I)^n$ or $x + cbv \in (L^I)^n$. Then, if a function $F : (L^I)^n \to L^I$ is both u-increasing and v-increasing, then F is $(au + bv)$-increasing.*

Similarly, we can extend the concept of weak monotonicity to the interval-valued setting.

Definition 6 ([30])**.** Let $\overrightarrow{0} \neq (a,b) \in R^2$. A function $F : (L^I)^n \to L^I$ is said to be (a,b)-weakly increasing (resp. (a,b)-weakly decreasing) if for all $x \in (L^I)^n$ and $c > 0$ such that $x + c(a,b,...,a,b) \in (L^I)^n$, it holds that

$$F(x) \leq_{L^I} F(x + c(a,b,...,a,b))$$

$$(resp.\ F(x) \geq_{L^I} F(x + c(a,b,...,a,b))).$$

Example 4. Let $F : (L^I)^2 \to L^I$ be the IV function defined for $[\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}] \in (L^I)^2$ by

$$F([\underline{x_1},\overline{x_1}],[\underline{x_2},\overline{x_2}]) = \frac{1}{2}[\underline{x_1} + \underline{x_2}, \min(\underline{x_1} + \overline{x_2}, \overline{x_1} + \underline{x_2})].$$

Thus, setting $a, b \in R$, F is (a,b)-weakly increasing if and only if $a > 0$ and $a + b \geq 0$, or $a = 0$ and $b > 0$. This is a consequence of the fact that

$$F(([\underline{x_1},\overline{x_1}],[\underline{x_2},\overline{x_2}]) + c(a,b,a,b)) =$$

$$F([\underline{x_1},\overline{x_1}],[\underline{x_2},\overline{x_2}]) + c[a, \frac{a+b}{2}]$$

for $c > 0$.

Moreover, we quote an interval-valued pre-aggregation function.

Definition 7 ([30])**.** A function $F : (L^I)^n \to L^I$ is said to be an IV pre-aggregation function if it satisfies the following conditions:
1) $F([0,0],...,[0,0]) = [0,0]$;
2) $F([1,1],...,[1,1]) = [1,1]$;
3) There exists a vector

$$v = ((a_1,b_1),...,(a_n,b_n)) \in ((R^+)^2)^n,$$

such that F is v-increasing.

III. New operator in interval-valued fuzzy setting

Now we will focus on an important element of this paper, i.e. a new interval operator - weak operator.

Definition 8. Let $n \in \mathbb{N}$, $i \in \{1,\ldots,n\}$ and $[x_i, y_i] \subset [0,1]$ be an intervals. Operator given by the following formula

$$W([x_1,y_1],\ldots,[x_n,y_n]) = \tag{1}$$

$$\left[\frac{\sum_{i=1}^n x_i \ + \ \sum_{i=1}^n y_i(y_i - x_i)}{n \ + \ \sum_{i=1}^n (y_i - x_i)}, \frac{\sum_{i=1}^n y_i \ + \ \sum_{i=1}^n x_i(y_i - x_i)}{n \ + \ \sum_{i=1}^n (y_i - x_i)}.\right]$$

will be called weak operator.

A. Basic properties of weak operator

At the beginning we will consider the directional monotonicity of weak operator with base vectors. This will lead us to a lack of monotonicity. Therefore, in the following part of the paper we will consider other vectors for directional monotonicity.

Example 5. Let $x = [2/8, 3/8]$, $y = [0, 0]$ and $c = 0.08$. Then
$W(x, y) = [0.139, 0.191]$;
$W(x + c[1, 0], y + c[0, 0]) = [0.169, 0.190]$,
i.e. W is not $((1, 0), (0, 0))$ increasing.

Based on the above example, we get

Theorem 2. *Let $W : L^2 \to L$ be a weak operator given by* (1).
- *W is not $((1, 0), (0, 0))$ increasing;*
- *W is not $((0, 1), (0, 0))$ increasing;*
- *W is not $((0, 0), (1, 0))$ increasing;*
- *W is not $((0, 0), (0, 1))$ increasing.*

Thanks to the above, we can get

Corollary 1. *Let $W : L^2 \to L$ be a weak operator given by* (1). *W is not increasing.*

Regarding other properties, we receive

Theorem 3. *Let $W : L^2 \to L$ be a weak operator given by* (1).
- *W is commutative;*
- *W satisfy the boundary conditions, i.e*

$$W(\mathbf{0}, \mathbf{0}) = \mathbf{0} \text{ and } W(\mathbf{1}, \mathbf{1}) = \mathbf{1};$$

- *W is not representable;*
- *W has no neutral element.*

Now we compare the weak operator with $\min$ and $\max$.

Theorem 4. *Let $W : L^2 \to L$ be a weak operator given by* (1). *Then*

$$\underline{\min(x, y)} \le \underline{W(x, y)};$$
$$\overline{W(x, y)} \le \overline{\max(x, y)}.$$

Proof. Let $x, y \in L^I$. Without loss of generality, using commutativity of W, we can assume that $\underline{x} \le \underline{y}$. Therefore

$$\begin{aligned} 0 &\le \underline{y} - \underline{x} \\ &\le \underline{y} - \underline{x} + (\overline{y} - \underline{y})^2 \\ &\le \underline{y} - \underline{x} + (\overline{y} - \underline{x})(\overline{y} - \underline{y}) \\ &\le \underline{y} - \underline{x} + \overline{x}^2 - \underline{x}^2 + \overline{y}(\overline{y} - \underline{y}) - \underline{x}(\overline{y} - \underline{y}) \\ &\le \underline{y} - \underline{x} + \overline{x}(\overline{x} - \underline{x}) + \overline{y}(\overline{y} - \underline{y}) - \underline{x}(\overline{y} - \underline{y}) - \underline{x}(\overline{x} - \underline{x}) \end{aligned}$$

So

$$\min(\underline{x}, \underline{y}) = \underline{x} \le \frac{\underline{x} + \underline{y} + \overline{x}(\overline{x} - \underline{x}) + \overline{y}(\overline{y} - \underline{y})}{2 + (\overline{y} - \underline{y}) + (\overline{x} - \underline{x})}$$

Similarly, we can show the second part of the theorem. □

Example 6. Let $x = [0, 3/8]$, $y = [0, 1/8]$, $u = [0, 1/2]$, $v = [1/2, 1/2]$. Then
$\min(x, y) = [0, 1/8]$, $W(x, y) = [1/16, 1/5]$,
$\min(u, v) = [0, 1/2]$, $W(x, y) = [1/4, 1/3]$.
This mean that W and $\min$ are not comparable.

Similarly, we can get that W and $\max$ are incomparable.

The next examples will refer to other algebraic properties of the weak operator.

Example 7. We will show that the weak operator is not idempotent, i.e. it does not fulfill the condition

$$\forall_{x \in L^I} W(x, x) = x.$$

Let $x = [0, 1]$. Then we get $W(x, x) = [1/2, 1/2]$, which means that W is not idempotent.

Example 8. The next example shows that the weak operator is not bisymmetric. We will check the condition

$$\forall_{x,y,u,v \in L^I} W(W(x, y), W(u, v)) = W(W(x, u), W(y, v)).$$

Let $x = [1/8, 3/8]$, $y = [0, 0]$, $v = [1/4, 3/8]$, $u = [0, 7/8]$. Then we have
$W(W(x, y), W(u, v)) = [0.23076, 0.297538]$,
$W(W(x, u), W(y, v)) = [0.234569, 0.297368]$,
which means that W is not bisymmetric.

Example 9. To verify associativity, consider the condition

$$\forall_{x,y,z \in L^I} W(W(x, y), z) = W(x, W(y, z)).$$

Let $x = [0, 0]$, $y = [0, 0]$, $z = [0, 1/4]$. Then we have
$W(W(x, y), z) = [1/33, 4/33] \neq [5/253, 15/253] = W(x, W(y, z))$, which means that W is not associative.

Example 10. Let $x = [0, 0]$, $y = [0, 1/4]$, $z = [0, 0]$. Then we have
$W(W(x, y), W(y, z)) = W([1/33, 4/33], [1/33, 4/33]) = [5/132, 5/44] \nleq [0, 0] = W(x, z)$, which means that W does not satisfy the condition

$$\forall_{x,y,z \in L^I} W(W(x, y), W(y, z)) \le W(x, z),$$

i.e. W is not transitive.

Corollary 2. *Let $W : L^2 \to L$ be given by* (1).
- *W is not idempotent;*
- *W is not transitive;*
- *W is not associative;*
- *W is not bisymmetric.*

In Corollary 1 we noted that a weak operator is not monotonic. That is why we are investigating other types of monotonicity. Here we present those that are related to the concept pre-aggregation function.

Theorem 5. *Let $W : L^2 \to L$ be weak operator given by* (1). *W is $(a, b, 0, 0)$- increasing for all $a > 0$, $b > 0$ such that $\frac{1}{3}a \le b \le 3a$.*

Proof. At the beginning we will show that W is $(3, 1, 0, 0)$-increasing. First observe, that $0 < c \le \frac{1}{3}$ and $2c \le \overline{x} - \underline{x}$. Using these inequalities we get

$$c(3 - \underline{x} - \overline{x} - 2c) > 0$$

and next

$$\begin{array}{rl} & (2+\overline{x}-\underline{x}+\overline{y}-\underline{y})(c\overline{x}-c\underline{x}-2c\overline{x}-2c^2+3c) \\ \geq & -2(\underline{x}+\underline{y})-2c\overline{x}(\overline{x}-\underline{x})-2c\overline{y}(\overline{y}-\underline{y}) \end{array}$$

and

$$\begin{array}{rl} & (2+\overline{x}-\underline{x}+\overline{y}-\underline{y})(1+3(\overline{x}-\underline{x})-2(\underline{x}+3c)) \\ \geq & -2(\overline{x}+\overline{y}+\underline{x}(\overline{x}-\underline{x})+\underline{y}(\overline{y}-\underline{y})) \end{array}$$

This implies

$$\begin{array}{rl} & \frac{\underline{x}+3c+\underline{y}+(\overline{x}+c)(\overline{x}+c-\underline{x}-3c)+\overline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x}-2c)+(\overline{y}-\underline{y})} \\ \geq & \frac{\underline{x}+\underline{y}+(\overline{x})(\overline{x}-\underline{x})+\overline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x})+(\overline{y}-\underline{y})} \end{array}$$

and

$$\begin{array}{rl} & \frac{\overline{x}+c+\overline{y}+(\underline{x}+3c)(\overline{x}+c-\underline{x}-3c)+\underline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x}-2c)+(\overline{y}-\underline{y})} \\ \geq & \frac{\underline{x}+\underline{y}+(\overline{x})(\overline{x}-\underline{x})+\overline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x})+(\overline{y}-\underline{y})} \end{array}$$

Now we will show that W is $(1,3,0,0)$-increasing. Since $0<c\leq\frac{1}{3}$ and $1\geq\underline{y}+(\overline{y}-\underline{y})\geq\underline{y}+\overline{y}(\overline{y}-\underline{y})$ then

$$\begin{array}{l} 2+4(\overline{x}+3c)+7(\overline{x}-\underline{x})+2\overline{x}(\overline{x}-\underline{x})+6c(\overline{x}-\underline{x})+ \\ 3(\overline{x}-\underline{x})^2+(\overline{y}-\underline{y})(1+2(\overline{x}+3c)+3(\overline{x}-\underline{x}))\geq \\ 2\underline{x}+2\underline{y}+2\overline{x}(\overline{x}-\underline{x})+2\overline{y}(\overline{y}-\underline{y}) \end{array}$$

This implies

$$\begin{array}{rl} & \frac{\underline{x}+c+\underline{y}+(\overline{x}+3c)(\overline{x}-\underline{x}+2c)+\overline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x}+2c)+(\overline{y}-\underline{y})} \\ \geq & \frac{\overline{x}+\overline{y}+(\underline{x})(\overline{x}-\underline{x})+\underline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x})+(\overline{y}-\underline{y})} \end{array}$$

Again, using the fact that $0<c\leq\frac{1}{3}$ and $2\geq\overline{x}+\overline{y}$ we have

$$\begin{array}{l} 6+4(\underline{x}+c)+2(\overline{x}-\underline{x})+(3+2(\underline{x}+c)+ \\ (\overline{x}-\underline{x}))(\overline{x}-\underline{x})+(3+2(\underline{x}+c)+(\overline{x}-\underline{x}))(\overline{y}-\underline{y})\geq \\ 2\overline{x}+2\overline{y}+2\underline{x}(\overline{x}-\underline{x})+2\underline{y}(\overline{y}-\underline{y}) \end{array}$$

This implies

$$\begin{array}{rl} & \frac{\overline{x}+3c+\overline{y}+(\underline{x}+c)(\overline{x}-\underline{x}+2c)+\underline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x}+2c)+(\overline{y}-\underline{y})} \\ \geq & \frac{\overline{x}+\overline{y}+(\underline{x})(\overline{x}-\underline{x})+\underline{y}(\overline{y}-\underline{y})}{2+(\overline{x}-\underline{x})+(\overline{y}-\underline{y})} \end{array}$$

Which in consequence gives, that W is $(1,3,0,0)$-increasing.

Using Theorem 1 we get that W is $(a,b,0,0)$-increasing for all a,b such that $\frac{1}{3}a\leq b\leq 3a$. □

In a similar way we can get the following theorem.

Theorem 6. *Let $W:L^2\to L$ be weak operator given by* (1)*. W is (a,b)-weakly increasing for all $a\geq 0$, $b\geq 0$.*

Corollary 3. *Weak operator $W:L^2\to L$ given by* (1) *is IV pre-aggregation function.*

B. Preservation of basic properties by weak operator W

We will pay attention to selected properties of interval-valued fuzzy relations:

Definition 9. A relation $R\in IVFR(X)$ is called
(i) reflexive if $R(x,x)=\mathbf{1}$,
(ii) irreflexive if $R(x,x)=\mathbf{0}$,
(iii) W-transitive if $W(R(x,z),R(z,y))\leq R(x,y)$.

We can observe that the examined operator preserves mentioned properties.

Proposition 2 (cf. [25])**.** *Let $R_1,...,R_n\in IVFR(X)$ and $W:(L^I)^n\to L^I$ be weak operator for $n\in N$.*
• *If there all $R_1,...,R_n$ are reflexive/irreflexive, then $W(R_1,...,R_n)$ is also reflexive/irreflexive,*
• *If the relations $R_1,...,R_n\in IVFR(X)$ are symmetric, then $W(R_1,...,R_n)$ is also symmetric,*
• *$R\in IVFR(X)$ is W-transitive if and only if R^{-1} is W-transitive and $R^{-1}(x,y)=R(y,x)$.*

C. Composition of interval-valued fuzzy relations

Based on the paper of Goguen [31] we have authors ([32], [33], [34], [35]) study composition of interval-valued fuzzy relations or for fuzzy relations ([36], [37]).

For arbitrary, non-empty sets X,Y,Z we define for $P\in IVFR(X\times Y)$, $R\in IVFR(Y\times Z)$ and $\mathcal{B}$ an interval-valued operation we can consider:
• $\sup-\mathcal{B}$ composition of relations P and R is denoted by $P\circ_{\vee\mathcal{B}}R\in IVFR(X\times Z)$, where

$$(P\circ_{\vee\mathcal{B}}R)(x,z)=\sup_{y\in Y}\mathcal{B}(P(x,y),R(y,z)).$$

• $\inf-\mathcal{B}$ composition of relations P and R is denoted by $P\circ_{\wedge\mathcal{B}}R\in IVFR(X\times Z)$, where

$$(P\circ_{\wedge\mathcal{B}}R)(x,z)=\inf_{y\in Y}\mathcal{B}(P(x,y),R(y,z)).$$

Moreover, more general in respect to operations, but for finite sets $X,Y,Z\neq\emptyset$, interval-valued aggregation $\mathcal{A}:(L^I)^n\to L^I$ and the operation $\mathcal{B}:(L^I)^2\to L^I$ with respect to the same order $\leq$ we have $\mathcal{A}-\mathcal{B}$ composition of relations P and R, denoted by $P\circ_{\mathcal{AB}}R\in IVFR(X\times Z)$, where

$$(P\circ_{\mathcal{AB}}R)_{ik}=\mathcal{A}_{j=1}^{n}(\mathcal{B}(P_{ij},R_{jk})).$$

Basic properties of $\mathcal{A}-W$ composition

Now we consider some basic properties of composition built by studied weak operator, interesting us with respect to the potential influence them to the application, i.e. isotonicity, symmetry or transititvity.

Proposition 3 (cf. [25])**.** *Let $\mathcal{A}:(L^I)^n\to L^I$ be interval-valued aggregation function and $W:(L^I)^2\to L^I$ be weak operator with respect to the same order. Then the operation $\circ_{\mathcal{A}W}$ is increasing (isotonic) with respect to the same order.*

Proposition 4 (cf. [25])**.** *Let $\mathcal{A}:(L^I)^n\to L^I$, be interval-valued aggregation function and $W:(L^I)^2\to L^I$ be weak* *operator with respect to the same order.*

- *If $R, P \in IVFR(X)$ be symmetric relations. Then*

$$R \circ_{\mathcal{A}W} P = (P \circ_{\mathcal{A}W} R)^{-1}.$$

- *If $R \in IVFR(X)$ is W-transitive, then $R \circ_{\mathcal{A}W} R \leq R$.*

IV. Algorithm of general approximate reasoning

Based on [25], we will consider the following modification of the generalized approximate reasoning algorithm for interval-valued fuzzy set theory, where we propose use the new weak operator.

Algorithm. General Approximate Reasoning

Inputs: Premises $\mathcal{D}_1, ..., \mathcal{D}_n, \mathcal{D}' \in IVFS(X)$; Conclusions $\mathcal{E}_1, ..., \mathcal{E}_n \in IVFS(Y)$; The interval-valued aggregation functions $\mathcal{A}, \mathcal{A}_k$, $k \in \{2, 3\}$ and W weak operator given by (1);

Outputs: $\mathcal{E}'$

1) For each rule, the associated interval-valued fuzzy relation R_i is built, where $R_i \in IVFR(X \times Y)$ and

$$R_i(x, y) = \mathcal{A}_2(N_{IV}(\mathcal{D}_i(x)), \mathcal{E}_i(y))$$

for $i = 1, ..., n$ and $\mathcal{A}_2$ is an interval-valued aggregation function and N_{IV} is an interval-valued fuzzy negation;

2) The interval-valued aggregation functions $\mathcal{A}$, $\mathcal{A}_3$ and W are taken;
3) For each rule, it is calculated:

$$\mathcal{E}'_i(y) = \mathcal{A}_{x \in X}(W(\mathcal{D}'(x), R_i(x, y))), \text{ with } i = 1, ..., n;$$

4) It is computed: $\mathcal{E}' = \mathcal{A}_{3 i=1,...,n}(\mathcal{E}'_i)$.

V. Application

In this paper, we apply the theoretical developments in a real-world problem, which consists in predicting the risk of suffering from a Cardio-Vascular Disease (CVD) in ten years. CVDs are generally caused by some problem that hinders the blood flow, which could provoke affections in the heart. These class of diseases are associated with a high risk of suffering from severe illnesses such as heart attacks or thrombosis, which leads to a high death rate in many developed countries. Therefore, it is important to predict the risk of suffering from CVDs so that doctors can appropriately treat their patients. In order to estimate such a risk, Spanish doctors use REGICOR tables that are composed of numbers representing the degree of suffering from a CVD based on some input variables like gender, age, presence or absence of diabetes, systolic and diastolic blood pressure and total or HDL cholesterol, among others. Later, the risk of suffering from a CVD can be categorized (low, medium and high) depending on the magnitude of these numbers. Consequently, this problem can be seen as a regression or a classification problem. In this paper, we tackle this problem as a regression one and, to face it, we propose to use an interval-valued fuzzy rule- based system to automatically predict the value of the risk (a number). Specifically, for the sake of showing the behaviour of the theoretical developments, we only consider two input variables, which are the age and the systolic blood pressure of the patient, which are normalized in the range [0, 1]. The output variable is the risk of suffering from a CVD in the following ten years, which is ranged in [1, 19]. Dataset consists of 898 clinical cases obtained from the records of seven primary care health centers of Pamplona (Navarra, Spain) during 2008. Many of the values/data provided by the doctors have imperfect information associated with them, which has led us to use interval-valued fuzzy sets, as it was done in [25], [33], where were considered this problem by different interval-valued aggregation functions in algorithm of approximate reasoning. Especially, in [25], [33] we present and applied the composition of interval-valued fuzzy relations using interval- valued aggregation functions. Now we present influence presented earlier new IV operator on decision making problem in predicting the risk of suffering from a Cardio-Vascular Disease (CVD) in ten years.

A. Analyzing the behaviour of the different aggregation functions and weak operator when accomplishing the inference process

In order to test the behaviour of the new weak operator W we have tested them and compare with the results presented in [33], where we received optimal results for the composition $\sup -T_L$ (we use W when accomplishing the inference process).

In [33] (Table 2) are presented the results obtained with the seven aggregation functions, when predicting the risk of the 898 patients composing our dataset. Similarly, to mentioned paper we show the error obtained when predicting the risk of patients having low, moderate and high risk, respectively, whereas in the last one we present the error for all the patients. There we can observe that the results obtained with the Łukasiewicz t-norm are the best ones for patients having low and medium risks, moreover, regarding the total error, the fact that there is a larger number of patients having low (615 patients) or medium (252 patients) risk than patients having a high (31 patients) risk implies that the lowest total error of the system is provided when also using Łukasiewicz t-norm. This is the reason why we compare the W only with operator Łukasiewicz t-norm

We have to point out that, in this case, the matrix representing the rules have been obtained using the mamdani method (I(a,b)=min(a,b)) or the implication Kleene-Dienes (I(a,b)=max(1-a,b)) (the point 1 of the algorithm).

The results obtained are presented in the table below (TABLE I).

As we can see, we have received comparable results, comparably low error, so using the new operator W for approximate reasoning seems to be a good solution. And this suggests the validity of examining the impact of the W operator examined

TABLE I
RESULTS OBTAINED WHEN USING THE ŁUKASIEWICZ T-NORM AND THE NEW WEAK OPERATOR W TO PERFORM THE COMPOSITION OPERATION.

Composition	Low	Medium	High	Total
$\sup - T_L$ [33]	13.20	1.33	22.55	10.19
$\sup - W$	13.28	**1.31**	23.13	10.89

in this paper on approximate reasoning for other types of data and/or on other methods of making decisions.

VI. CONCLUSIONS

In this paper some interval-valued operations with respect to the partial, linear, directional or weak isotonicity were presented. But mainly concentrated on study basic properties of new weak operator (IV pre-aggregation function). Moreover, application with new weak operator was proposed, especially in decision model of medical diagnosis support.

In the future, we will examine the impact of the weak operator W on approximate reasoning for other types of data and other methods of decision making.

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