General local properties of fuzzy relations and fuzzy multisets used to an algorithm for group decision making

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***Abstract*—Fuzzy relations are compared by membership values and as a consequence new types of local properties of fuzzy relations are introduced. In the new properties of fuzzy relations an arbitrary binary relation is involved. Particularly, a binary aggregation function may be used to define these properties. Connections between the new local properties of fuzzy relations are described. Furthermore, preservation of these properties in aggregation process is considered. Finally, notes on applications of the presented local properties in the context of fuzzy multisets and decision making are provided.**

***Index Terms*—properties of fuzzy relations, aggregation functions, decision making**

I. Introduction

In this paper generalized types of *local properties* for fuzzy relations are examined and compared. Local properties for fuzzy relations were considered in [1] (cf. [2]). These properties are related to an equivalence relation between fuzzy relations, which may play an important role in many fields. Due to the nature of this equivalence, fuzzy relations fulfilling the considered dependence were called *orderly equivalent*. The obtained *ordinal equivalence* classes allow to classify fuzzy relations with respect to the properties that they fulfill. For example, two orderly equivalent fuzzy relations may not be simultaneously reflexive or simultaneously asymmetric. This was the motivation to consider diverse classes of weaker versions of fuzzy relation properties, which are called *local* and which are common for all fuzzy relations in the same equivalence classes. The name is connected with the fact that in definitions of the local properties the influence of the neighboring values of a fuzzy relation on the respected values of this relation are taken into account. This is expressed by considering the adequate supremum and infimum in the notions of fuzzy relations. Orderly equivalent fuzzy relations have the same local properties. Local properties may be useful in decision making problems to reflect preferences of decision makers.

In this paper we consider a more general version of local properties, namely local B-properties, where B may be an arbitrary binary operation, particularly, aggregation functions, which proved to be an effective tool in many areas, may be used [3]. As we have already mentioned, fuzzy relations may represent the preferences of decision-makers over given set of alternatives. The local B-properties proposed in this paper are weaker versions of the properties proposed so far and as a result they may reflect in a better way preferences of decision makers. The local B-properties may be better adjusted to the real-life situations.

The structure of the paper is as follows. Firstly, some basic definitions connected with functions defined on the unit interval will be recalled (Section 2). Then, local B-properties (and local B-properties by row) of fuzzy relations are proposed (Section 3). Moreover, the study of the preservation of local properties in aggregation process is considered (Section 4). Finally, notes on applications of the presented results are indicated (Section 5).

II. Preliminaries

In this section we gather the basic notions applied in the paper. These are some of the properties of functions defined on the unit interval $[0, 1]$. As a special case we may obtain the properties of aggregation functions.

Definition 1 (cf. [4])**.** We say that a function $F : [0, 1]^n \to [0, 1]$ is:

- increasing, if

$$F(x_1, \dots, x_n) \leqslant F(y_1, \dots, y_n)$$

for $x_i, y_i \in [0,1], x_i \leqslant y_i, i = 1, \ldots, n$;

- idempotent, if
$$F(x, \ldots, x) = x \text{ for } x \in [0,1];$$
- averaging, if
$$\min(x_1, \ldots, x_n) \leqslant F(x_1, \ldots, x_n) \leqslant \max(x_1, \ldots, x_n)$$
for $\quad x_1, \ldots, x_n \in [0,1]$.

Proposition 1 (cf. [4])**.** *If $F : [0,1]^n \to [0,1]$ is increasing and idempotent, then it is averaging.*

Proposition 2 ([1])**.** *The unique idempotent function $F : [0,1]^n \to [0,1]$ that fulfils $F \leqslant \min$ (respectively $F \geqslant \max$) is $F = \min$ (respectively $F = \max$).*

Definition 2 ([4])**.** Let $n \in \mathbf{N}$. A function $F : [0,1]^n \to [0,1]$ which is increasing is called an aggregation function if $F(0, \ldots, 0) = 0$ and $F(1, \ldots, 1) = 1$.

Quasi-linear means are examples of aggregation functions
$$F(x_1, ..., x_n) = \varphi^{-1}\left(\sum_{k=1}^{n} w_k \varphi(x_k)\right), \tag{1}$$
where $w_k > 0, \sum_{k=1}^{n} w_k = 1$, where $x_1, \ldots, x_n \in [0,1]$, $\varphi : [0,1] \to \mathbf{R}$ is a continuous, strictly increasing function, are examples of averaging aggregation functions.
For $\varphi = id$ we have $F = A_{wmean}$ the weighted arithmetic mean.
Moreover, we have the following examples of aggregation functions:
- geometric mean $A_{gmean}(x_1, ..., x_n) = \sqrt[n]{x_1 \cdot ... \cdot x_n}$;
- median $A_{median}(x_1, ..., x_n) =$
$$\begin{cases} \frac{1}{2}(x_{\frac{n}{2}} + x_{\frac{n+2}{2}}), & n \textit{ is even} \\ x_{\frac{n+1}{2}}, & \textit{otherwise}. \end{cases}$$

Triangular norms and conorms are well-known examples of aggregation functions, respectively.

Definition 3 (cf. [5])**.** A triangular norm (t-norm) T on $[0,1]$ is an increasing, commutative, associative operation $T : [0,1]^2 \to [0,1]$ with a neutral element 1.
A triangular conorm (t-conorm) S on $[0,1]$ is an increasing, commutative, associative operation $S : [0,1]^2 \to [0,1]$ with a neutral element 0.

Triangular norms are examples of conjunctions and triangular conorms are examples of disjunctions (cf. [5]). An operation $C : [0,1]^2 \to [0,1]$ is called a fuzzy conjunction (respectively disjunction) if it is increasing and $C(1,1) = 1,\ C(0,0) = C(0,1) = C(1,0) = 0$ (respectively $C(0,0) = 0,\ C(1,1) = C(0,1) = C(1,0) = 1$).

Example 1. The following are the most frequently used examples of t-norms and t-conorms:

$T_M(x,y) = \min(x,y),\ \ S_M(x,y) = \max(x,y),$
$T_P(x,y) = xy,\ \ S_P(x,y) = x + y - xy,$
$T_L(x,y) = \max(0, x+y-1),\ \ S_L(x,y) = \min(1, x+y),$
$$T_D(x,y) = \begin{cases} x, & y = 1 \\ y, & x = 1 \\ 0, & otherwise \end{cases},$$
$$S_D(x,y) = \begin{cases} x, & y = 0 \\ y, & x = 0 \\ 1, & otherwise \end{cases}$$
for $x, y \in [0,1]$.

In the sequel we will also use notations: $\min = \wedge$ and $\max = \vee$.

Definition 4 (cf. [6])**.** Let $m, n \in \mathbf{N}$. A function $F : [0,1]^m \to [0,1]$ commutes with a function $G : [0,1]^n \to [0,1]$ (or F and G are commuting), if for all $a_{ik} \in [0,1]$, with $i \in \{1, ..., m\}$ and $k \in \{1, ..., n\}$ we have
$$F(G(a_{11}, \ldots, a_{1n}), \ldots, G(a_{m1}, \ldots, a_{mn})) = \tag{2}$$
$$G(F(a_{11}, \ldots, a_{m1}), \ldots, F(a_{1n}, \ldots, a_{mn})).$$
If in (2) $F = G$, then we get the bisymmetry property.

Example 2. Weighted geometric means (which are special cases of quasi-linear means) $F(t_1, \ldots, t_n) = \prod_{i=1}^{n} t_i^{w_i}$, where $w_k > 0$, commute with the product T_P.

Other examples and many interesting results concerning commuting operations can be found in [6]. Important properties in the next considerations are the following.

Definition 5 ([7])**.** Let $J \neq \emptyset$. We say that a binary function $F : [0,1]^2 \to [0,1]$ is:

- infinitely left distributive with respect to $\vee$ (left $\vee$-preserving), if
$$F(\bigvee_{j \in J} x_j, y) = \bigvee_{j \in J} F(x_j, y), \text{ for } x_j, y \in [0,1], j \in J;$$
- infinitely right distributive with respect to $\vee$ (right $\vee$-preserving), if
$$F(y, \bigvee_{j \in J} x_j) = \bigvee_{j \in J} F(y, x_j), \text{ for } x_j, y \in [0,1], j \in J;$$
- infinitely left distributive with respect to $\wedge$ (left $\wedge$-preserving), if
$$F(\bigwedge_{j \in J} x_j, y) = \bigwedge_{j \in J} F(x_j, y), \text{ for } x_j, y \in [0,1], j \in J;$$
- infinitely right distributive with respect to $\wedge$ (right $\wedge$-preserving), if
$$F(y, \bigwedge_{j \in J} x_j) = \bigwedge_{j \in J} F(y, x_j), \text{ for } x_j, y \in [0,1], j \in J.$$

A binary function $F : [0,1]^2 \to [0,1]$ is $\vee$-preserving, (respectively $\wedge$-preserving) if it is left and right $\vee$-preserving (respectively $\wedge$-preserving).

Example 3 (cf. [8]). One of the first known examples of a $\vee$-preserving t-norms (which is equivalent to the left-continuity), is nilpotent minimum denoted by T^{nM} and defined by

$$T^{nM}(x,y) = \begin{cases} 0, & if\ x+y \leqslant 1, \\ min(x,y), & otherwise. \end{cases}$$

Drastic product T_D is $\wedge$-preserving (which is equivalent to the right-continuity). Let us notice that none of the functions are continuous functions are not continuous. Other basic t-norms presented in Example 1 are continuous, i.e. left and right-continuous so they are also $\vee$-preserving and $\wedge$-preserving.

Results for triangular conorms may be obtained dually, i.e. $S(x,y) = 1 - T(1-x, 1-y)$ for any $x, y \in [0,1]$. A t-norm T is left-continuous if and only if its dual t-conorm is right-continuous, and vice versa.

III. Fuzzy relations and their generalized properties

A notion of a fuzzy relation is a particular case of a fuzzy set (cf. [9]). We consider fuzzy relations in a set $X \neq \emptyset$.

Definition 6 (cf. [10]). A fuzzy relation in X is an arbitrary function $R: X \times X \to [0,1]$. The family of all fuzzy relations in X is denoted by $FR(X)$. The converse to $R \in FR(X)$ is the relation $R^{-1} \in FR(X)$,

$$R^{-1}(x,y) = R(y,x),\ x,y \in X.$$

Let us recall some facts connected with the family $FR(X)$:
• $(FR(X), \leqslant)$ is a partially ordered set, where for $R, S \in FR(X)$

$$R \leqslant S \Leftrightarrow \underset{x,y\in X}{\forall} R(x,y) \leqslant S(x,y).$$

• $(FR(X), \vee, \wedge)$ is a lattice, where for $R, S \in FR(X)$ and $x, y \in X$

$$(R \vee S)(x,y) = \max(R(x,y), S(x,y)),$$
$$(R \wedge S)(x,y) = \min(R(x,y), S(x,y)).$$

• Let card $X = n$. Relation $R \in FR(X)$ may be represented by a matrix $R = [r_{ij}]$, $R(x_i, x_j) = r_{ij}$, $i,j = 1,...,n$.

There are many particular properties of fuzzy relations. In [1] the basic ones (cf. [11], [9]) and the connections of these properties with the considered ordinal equivalence relation were presented. As a consequence the families of local properties, namely the properties more compatible with the ordinal equivalence relation were introduced. Moreover, fuzzy relation properties were often defined with the use of $\max$ or $\min$ operations and later they were generalized to t-norm or t-conorm based properties [11]. In [12], [13] even more general notions were used, namely arbitrary binary operation in the unit interval was applied to define fuzzy relation properties. The mentioned B-properties are listed below.

Definition 7 ([12]). Let $B, B_1, B_2 : [0,1]^2 \to [0,1]$ be binary operations. We say that the relation $R \in FR(X)$ is:
• reflexive, if $\underset{x\in X}{\forall} R(x,x) = 1$,
• irreflexive, if $\underset{x\in X}{\forall} R(x,x) = 0$,
• totally B-connected, if
$\underset{x,y\in X}{\forall} B(R(x,y), R(y,x)) = 1$,
• B-connected, if
$\underset{x,y\in X, x\neq y}{\forall} B(R(x,y), R(y,x)) = 1$,
• B-asymmetric, if
$\underset{x,y\in X}{\forall} B(R(x,y), R(y,x)) = 0$,
• B-antisymmetric, if
$\underset{x,y\in X, x\neq y}{\forall} B(R(x,y), R(y,x)) = 0$,
• B-transitive, if
$\underset{x,y,z\in X}{\forall} B(R(x,y), R(y,z)) \leqslant R(x,z)$,
• negatively B-transitive, if
$\underset{x,y,z\in X}{\forall} B(R(x,y), R(y,z)) \geqslant R(x,z)$,
• B_1-B_2-Ferrers, if
$\underset{x,y,z,w\in X}{\forall} B_1(R(x,y), R(z,w)) \leqslant B_2(R(x,w), R(z,y))$,
• B_1-B_2-semitransitive, if
$\underset{x,y,z,w\in X}{\forall} B_1(R(x,w), R(w,y)) \leqslant B_2(R(x,z), R(z,y))$.

In the next section we consider different classes of properties with a binary operation B involved.

IV. Local B-properties of fuzzy relations

We will consider asymmetry and connectedness properties in the weaker versions, namely using operation B instead of $\vee$ and $\wedge$ (cf. [1]). In this way we obtain the so called local B-properties. Our approach follows from practical point of view. For example, in the area of multicriteria decision support, different approaches to determine relation of preferences between variants are known (compare for example methods from the Electre family [14], [15]). If the preferences between the variants are described by a fuzzy relation, for example the requirement of antisymmetry of such relation (according to the pattern given in Definition 7) is too restrictive. Instead of completely abandoning this property, a suitable option may be to weaken the property considering local B-antisymmetry (defined in Definition 8).

Definition 8. Let $B : [0,1]^2 \to [0,1]$. A fuzzy relation $R \in FR(X)$ is called:
• locally B-asymmetric, if for $Q = B(R, R^{-1})$

$$\underset{x,y\in X}{\forall} (Q(x,y) = \bigwedge_{z\in X} Q(x,z)\ \ and\ \ Q(x,y) = \bigwedge_{z\in X} Q(z,y)), \tag{3}$$

• locally B-antisymmetric, if for $Q = B(R, R^{-1})$

$$\underset{x,y\in X, x\neq y}{\forall} (Q(x,y) = \bigwedge_{z\in X, z\neq x} Q(x,z)\ \ and \tag{4}$$

$$Q(x,y) = \bigwedge_{z\in X, z\neq y} Q(z,y)),$$

• locally totally B-connected, if for $V = B(R, R^{-1})$

$$\underset{x,y\in X}{\forall} (V(x,y) = \bigvee_{z\in X} V(x,z)\ \ and\ \ V(x,y) = \bigvee_{z\in X} V(z,y)), \tag{5}$$

• locally B-connected, if for $V = B(R, R^{-1})$

$$\underset{x,y\in X, x\neq y}{\forall} \left(V(x,y) = \bigvee_{z\in X, z\neq x} V(x,z) \ \ and \right. \tag{6}$$

$$\left. V(x,y) = \bigvee_{z\in X, z\neq y} V(z,y)\right).$$

Example 4. Let card $X = 3, R, S \in FR(X)$ be presented by matrices:

$$R = \begin{bmatrix} 0.5 & 0.625 & 0.3125 \\ 0.4 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0.5 \end{bmatrix},$$

$$S = \begin{bmatrix} 0.5 & 0.375 & 0.6 \\ 0.6 & 0.5 & 0.5 \\ 0.375 & 0.5 & 0.5 \end{bmatrix}.$$

R is locally B-asymmetric (locally B-antisymmetric) but it is not locally irreflexive. S is locally totally B-connected (locally B-connected) and it is not locally reflexive, where

$$Q = B(R, R^{-1}) = \begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 \end{bmatrix},$$

$$V = B(S, S^{-1}) = \begin{bmatrix} 0.75 & 0.75 & 0.75 \\ 0.75 & 0.75 & 0.75 \\ 0.75 & 0.75 & 0.75 \end{bmatrix}.$$

Q fulfils (3) with $B = T_P$ and V fulfils (5) with $B = S_P$.

Local properties (cf. [1]) are special cases of local B-properties.

Proposition 3. *If $B = \min$ in (3) (respectively in (4)), then $R \in FR(X)$ is locally asymmetric (respectively locally antisymmetric). If $B = \max$ in (5) (respectively in (6)), then $R \in FR(X)$ is locally totally connected (respectively locally connected).*

The following statement is immediate by Definition 8.

Proposition 4. *We have the following dependencies:*

- *local B-asymmetry implies local B-antisymmetry;*
- *local total B-connectedness implies local B-connectedness.*

Proposition 5. *Let $B : [0,1]^2 \to [0,1]$ be idempotent:*

- *local B-asymmetry implies local irreflexivity;*
- *local total B-connectedness implies local reflexivity.*

Proof. Let $R \in FR(X)$, $x, y \in X$ and B be idempotent. If R is locally B-asymmetric and $Q = B(R, R^{-1})$, then $Q(x,x) = B(R(x,x), R(x,x)) = R(x,x)$, so

$$R(x,x) = \bigwedge_{z\in X} R(x,z) \wedge \bigwedge_{z\in X} R(z,x).$$

This implies that

$$R(x,x) = \bigwedge_{z\in X} R(x,z) \text{ and } R(x,x) = \bigwedge_{z\in X} R(z,x),$$

which proves local irreflexivity of R. The second property may be proven similarly. □

Remark 1. Although we consider given local B-properties in the most general versions, i.e. with operations $B : [0,1]^2 \to [0,1]$, it may be a natural generalization to use fuzzy conjunction B in local B-asymmetry (local B-antisymmetry) property, fuzzy disjunction B in local B-connectedness properties. This approach enables us to keep the meaning of properties which follows both from the crisp cases and standard versions of properties. However, we do not only restrict ourselves to such examples.

Let us notice that if we would like to consider fuzzy relations having properties listed in Proposition 5 (B idempotent) and conjunctions or disjunctions B with neutral element, i.e. $B \leqslant \min$ or respectively $B = \max$ (cf. Remark 1), then only $B = \min$ or $B = \max$ may fulfil such combination of conditions on B (cf. Proposition 2).

We may characterize fuzzy relations which are locally totally B-connected (locally B-connected) or locally B-asymmetric (locally B-antisymmetric). Firstly, we will pay attention to the local total B-connectedness.

Proposition 6. *Let $B : [0,1]^2 \to [0,1]$. $R \in FR(X)$ is locally totally B-connected if and only if $B(R(x,y), R(y,x)) = a$ for any $x, y \in X$, $a \in [0,1]$.*

Proof. Let $R \in FR(X)$.
Sufficiency. If R and B fulfil the given assumptions, then for $V = B(R, R^{-1})$ we get $V(x,y) = a$ for any $x, y \in X$ so conditions in (5) are fulfilled.
Necessity. Suppose that R is locally totally connected and there exists $(x,y) \in X \times X$ such that $B(R(x,y), R(y,x)) \neq a$. As a result we have the following cases:
(1) $B(R(x,y), R(y,x)) > a$,
(2) $B(R(x,y), R(y,x)) < a$.
Then in the first case we get
$V(x,y) = V(y,x) = B(R(x,y), R(y,x)) > a$ and

$$\underset{(u,v)\neq(x,y)\in X}{\exists} \left(V(u,v) < \bigvee_{z\in X} V(u,z) \ \ or \ \ V(u,v) < \right.$$

$$\left. \bigvee_{z\in X} V(z,v)\right).$$

In the second case we have
$V(x,y) = V(y,x) = B(R(x,y), R(y,x)) < a$
and we see that

$$V(x,y) < \bigvee_{z\in X} V(x,z) \ \ or \ \ V(x,y) < \bigvee_{z\in X} V(z,y).$$

This contradicts with assumption that R is locally totally B-connected and finishes the proof. □

Analogously, we obtain the following

Proposition 7. *Let $B : [0,1]^2 \to [0,1]$. Relation $R \in FR(X)$ is locally B-connected if and only if $B(R(x,y), R(y,x)) = a$ for any $x, y \in X, x \neq y$, where $a \in [0,1]$.*

Dually, we get characterization for local B-asymmetry and local B-antisymmetry.

Proposition 8. *Let $B : [0,1]^2 \to [0,1]$. Relation $R \in FR(X)$ is locally B-asymmetric if and only if $B(R(x,y), R(y,x)) = a$ for any $x, y \in X,\ a \in [0,1]$.*

Proposition 9. *Let $B : [0,1]^2 \to [0,1]$. Relation $R \in FR(X)$ is locally B-antisymmetric if and only if $B(R(x,y), R(y,x)) = a$ for any $x, y \in X,\ x \neq y$, where $a \in [0,1]$.*

Let us notice that if we consider local B-properties with no additional assumptions we get the same conditions for characterizations of local total B-connectedness and local B-asymmetry (respectively, local B-connectedness and local B-antisymmetry).

V. Preservation of local B-properties in aggregation process and notes on applications

Now we will examine preservation of local B-properties in aggregation process which may be useful in decision making problems. As we have already mentioned, fuzzy relations may represent the preferences of decision-makers over given set of alternatives. The local B-properties proposed in this paper are more general versions of usually considered properties and as a result they may reflect in a better way preferences of decision makers. In multicriteria decision making a decision maker has to choose among the alternatives with respect to a set of criteria $K = \{k_1, \ldots, k_n\}$. Let $R_1, \ldots, R_n$ be fuzzy relations corresponding to each criterion represented by matrices, where $R_k : X \times X \to [0,1]$, $k = 1, \ldots, n$, $n \in \mathbf{N}$, $R_k(x_i, x_j) = r^k_{ij}$, $1 \leqslant i, j \leqslant m$ and $X = \{x_1, \ldots, x_m\}$ be a set of alternatives for $m \in \mathbf{N}$. With the use of a function $F : [0,1]^n \to [0,1]$ (usually an aggregation function, i.e. an increasing function such that $F(0,...,0) = 0$ and $F(1,...,1) = 1$, cf. [3], [4]), we aggregate given fuzzy relations $R_1, \ldots, R_n \in FR(X)$, where $n \in \mathbf{N}$. We obtain an aggregated fuzzy relation $R_F \in FR(X)$ (cf. [11], [16], [17], [18], [19]) which helps to find the solution alternative from the set X, where

$$R_F(x,y) = F(R_1(x,y), \ldots, R_n(x,y)),\ x, y \in X \tag{7}$$

and

$$(R_F)^{-1}(x,y) = F(R_1^{-1}(x,y), ..., R_n^{-1}(x,y)), \quad x, y \in X.$$

In the context of preference relations the value $R(x,y)$ may represent the intensity of preference x over y. We consider in this context the introduced in this paper local B-properties.

Theorem 1. *Let $R, S \in FR(X)$, operations $F, B : [0,1]^2 \to [0,1]$ be commuting and F be $\vee$-preserving. If R, S are locally totally B-connected (locally B-connected), then F preserves local total B-connectedness (local B-connectedness).*

Proof. Let R, S be locally totally connected, $x, y \in X$. We will show that under given assumptions $R_F = F(R,S)$ also is locally totally connected.

$$\begin{gathered}
B(F(R(x,y), S(x,y)), F(R^{-1}(x,y), S^{-1}(x,y))) = \\
F(B(R(x,y), R^{-1}(x,y)), B(S(x,y), S^{-1}(x,y))) = \\
F(\bigvee_{z\in X} B(R(x,z), R^{-1}(x,z)), \bigvee_{t\in X} B(S(x,t), S^{-1}(x,t))) = \\
\bigvee_{z\in X} \bigvee_{t\in X} F(B(R(x,z), R^{-1}(x,z)), B(S(x,t), S^{-1}(x,t))) \geqslant \\
\bigvee_{z\in X} F(B(R(x,z), R^{-1}(x,z)), B(S(x,t), S^{-1}(x,t))) = \\
\bigvee_{z\in X} B(F(R(x,z), S(x,t)), F(R^{-1}(x,z), S^{-1}(x,t))) \geqslant \\
B(F(R(x,y), S(x,y)), F(R^{-1}(x,y), S^{-1}(x,y))).
\end{gathered}$$

The second condition in (5) may be proven similarly. Thus F preserves local total B-connectedness. Analogously we may consider local B-connectedness property. ☐

Dually we may prove the result for local asymmetry.

Theorem 2. *Let $R, S \in FR(X)$, operations $F, B : [0,1]^2 \to [0,1]$ be commuting and F be $\wedge$-preserving. If R, S are locally B-asymmetric (locally B-antisymmetric), then F preserves locally asymmetry (local antisymmetry).*

Remark 2. If in Theorems 1 and 2 we have $F = B$, the assumption of commuting property may be replaced with the assumption of bisymmetry.

Now, there will be given examples of functions F preserving local properties.

Example 5. All continuous aggregation functions, for example quasi-linear means, preserve local reflexivity and local irreflexivity. This is justified by the fact that continuity involves $\vee$-preserving property and $\wedge$-preserving property (cf. Example 3).

Example 6. Each t-norm and each t-conorm is symmetric and associative, so it is bisymmetric (cf. Remark 2). As a result, using the dependencies given in Example 3 we see that for $B \in \{T_M, S_M, T_P, S_P, T_L, S_L\}$, B preserves both local total B-connectedness (local B-connectedness) and local B-asymmetry (local B-antisymmetry). Nilpotent minimum T^{nM} is $\vee$-preserving, so it preserves local total T^{nM}-connectedness and local T^{nM}-connectedness, while the drastic product is $\wedge$-preserving, so it preserves local T_D-asymmetry and local T_D-antisymmetry.

Example 7. Weighted geometric means preserve both local total T_P-connectedness (local T_P-connectedness) and local T_P-asymmetry (local T_P-antisymmetry). This is justified by the fact that weighted geometric means commute with T_P (cf. Example 2) and they are continuous, which involves $\vee$-preserving property and $\wedge$-preserving property (cf. Example 3).

VI. An algorithm for decision making

We first introduce some theoretical notions in order to fix the notation for this section. Let O^n be the set of n-elements on $[0,1]$, namely, the set $O^n = \{\mathbf{x} = (x(1),...,x(n)) \in [0,1]^n\}$. We recall that there is a natural partial order $\leq$ on $O^n \subseteq R^n$ given by

$(x(1),...,x(n)) \leq (y(1),...,y(n))$ if and only if $x(i) \leq y(i)$,

$1 \leq i \leq n$.
In this way, $(O^n, \leq)$ is a complete lattice and $(0,...,0)$ and $(1,...,1)$ are the bottom and top elements of the partial order, respectively.

Fuzzy multisets are generalizations of fuzzy sets which were defined in [20] by Yager. Like many other generalizations, the aim of these sets lies on the formalization of a representation to deal with imprecision, inexactness, ambiguity, or uncertainty intrinsic to many problems. In particular, in the case of fuzzy multisets, a fixed number n of membership values is assigned to each element. Taking into account that in a group decision making problem we have as many evaluations as decision makers, fuzzy multisets are suitable models for these problems.

Definition 9 ([20])**.** Let U be a nonempty set usually called a universe. A fuzzy multiset A over U is given by $A: U \to O^n$, where $A(u)$ denotes the membership degree of the element $u \in U$ to A.

To order the fuzzy multisets we propose the following method inspired on [21].

Algorithm. Selection of Multisets.

Let $F = (F_1, ...F_n)$ be a sequence of n-aggregation functions, $F_i : [0,1]^n \to [0,1]$.
For $\mathbf{x}_1, ...\mathbf{x}_k \in O^n$ we calculate for each $1 \leq i,j \leq k$ the measure of connectivity for pairs of values $\mathbf{x}_i$ and $\mathbf{x}_j$, $i \neq j$, $CON : O^n \times O^n \to \mathbb{R}$

$$CON(\mathbf{x}_i, \mathbf{x}_j) = \sum_{1 \leq l \leq n} (F_l(\mathbf{x}_i(t)) - F_l(\mathbf{x}_j(t))), \quad 1 \leq t \leq n.$$

Then
Step Selection. For each $1 \leq i,j \leq k$ we find

$$\max_{1 \leq i,j \leq k} CON(\mathbf{x}_i, \mathbf{x}_j) = CON(\mathbf{x}_z, \mathbf{x}_w), \ \ 1 \leq z, w \leq k;$$

The element $\mathbf{x}_z$ is the best.
We repeat Step Selection by omitting the wining values $\mathbf{x}_z$ in the next iteration.
If

$$\max_{1 \leq i,j \leq k} CON(\mathbf{x}_i, \mathbf{x}_j) = CON(\mathbf{x}_z, \mathbf{x}_w) = CON(\mathbf{x}_{z'}, \mathbf{x}_{w'}),$$

$1 \leq z, z', w, w' \leq k$,
then we find

$$C1 := \sum_{1 \leq w \leq k} (\max(0, CON(\mathbf{x}_z, \mathbf{x}_w)))$$

and

$$C2 := \sum_{1 \leq w' \leq k} (\max((0, CON(\mathbf{x}_{z'}, \mathbf{x}_{w'}))),$$

if $C1 > C2$, then the element $\mathbf{x}_z$ is the best, else the element $\mathbf{x}_{z'}$ is the best, else $\mathbf{x}_z \equiv \mathbf{x}_{z'}$.

We obtain the sequence:

$$\mathbf{x}_z \succeq ...\mathbf{x}_{z'} \succeq ....$$

Especially, if we compare only two multisets $\mathbf{x}, \mathbf{y}$, then $\mathbf{x} \succeq \mathbf{y}$ iff $CON(\mathbf{x}, \mathbf{y}) > CON(\mathbf{y}, \mathbf{x})$, else if $CON(\mathbf{x}, \mathbf{y}) < CON(\mathbf{y}, \mathbf{x})$, then $\mathbf{x} \preceq \mathbf{y}$, else they are equivalent.

An algorithm for group decision making

We use modification of the method presented in [21], but we omit the ordering of preference values of experts and we create locally B-asymmetric versions of preference relations. As we have already noticed, antisymmetrization is sometimes a required step in decision making. This is why we include such step (step 3) in the proposed algorithm. We apply more general version of this property (local B-asymmetry) since it is closer to real-life situations. To create local B-asymmetric relation we may use the characterization given in Proposition 9.

AlgorithmDM

1) Input data (k matrices R of the size $m \times n$, where k is the number of an expert, m is the number of an alternative, n is the number of a criteria);
2) Determination of a preference relation $P(m \times m)$ between alternatives for each criteria c according to an expert e (we obtain $n \times k$ matrices) by the rules:
   For $1 \leq t \leq k$
   For $1 \leq l \leq n$
   For $1 \leq j \leq m$

   If $R_{tl}(a_j, c_l) > R_{tl}(a_{j+1}, c_l)$, **then**

   $$P_{tl}(j, j+1) = \min(1, 0.5 + |a_j - a_{j+1}|)$$

   Else
   If $R_{tl}(a_j, c_l) < R_{tl}(a_{j+1}, c_l)$, **then**

   $$P_{tl}(j, j+1) = \max(0, 0.5 - |a_j - a_{j+1}|)$$

   Else

   $$P_{tl}(j, j+1) = 0.5;$$

3) Conversion of the relations P_{tl} to locally B-asymmetric P^*_{tl}.
4) For the given expert we use aggregation $Agg : [0,1]^n \to [0,1]$ matrices of the criteria - reduction of the matrix with respect to the criteria K according to the rule:

For $1 \leq t \leq k$ we calculate
For $1 \leq i, j \leq m$

$$\mathbf{P}_t^*(i,j) = Agg_{1 \leq l \leq n} P_{tl}^*(i,j);$$

We have k matrices adequate to the number of experts.

5) We create matrix with elements as the multisets of each values created by adequate experts

$$\mathbf{P}'(i,j) = \{\mathbf{P}_1^*(i,j), ...\mathbf{P}_k^*(i,j)\};$$

6) Application of the weighted arithmetic mean A_{wmean} : $O^{k^m} \to O^k$ (we ignore order of experts)
We use $w = [0.35, 0.25, 0.4]$ - coefficient vector of weights in aggregation process: For $1 \leq i \leq m$ we calculate

$$Alternative\ i \ \rightarrow \ A_{wmean}(\mathbf{P}'(i,1), ...\mathbf{P}'(i,m));$$

7) Selection of the best alternative - Algorithm *Selection of Multisets*.

The obtained ranking list contains the expert data in the decreasing order in terms of quality.

VII. Illustrative example

We generate the decision makers' evaluations accordingly to the data in [22] and later examined in [21].

$$e1| \begin{bmatrix} 0.5 & 0.4 & 0.2 \\ 0.4 & 0.7 & 0.6 \\ 0.4 & 0.5 & 0.6 \\ 0.7 & 0.6 & 0.3 \end{bmatrix}, \ e2| \begin{bmatrix} 0.4 & 0.6 & 0.1 \\ 0.6 & 0.6 & 0.7 \\ 0.6 & 0.5 & 0.5 \\ 0.6 & 0.6 & 0.4 \end{bmatrix},$$

$$e3| \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.7 & 0.7 & 0.4 \\ 0.3 & 0.6 & 0.6 \\ 0.8 & 0.7 & 0.3 \end{bmatrix}.$$

Then according to the point 2 of the AlgorithmDM we obtain the following preference relations:

$$e1c1| \begin{bmatrix} 0.5 & 0.6 & 0.6 & 0.3 \\ 0.4 & 0.5 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 & 0.2 \\ 0.7 & 0.8 & 0.5 & 0.5 \end{bmatrix}, \ e1c2| \begin{bmatrix} 0.5 & 0.2 & 0.4 & 0.3 \\ 0.8 & 0.5 & 0.7 & 0.6 \\ 0.6 & 0.3 & 0.5 & 0.4 \\ 0.7 & 0.4 & 0.6 & 0.5 \end{bmatrix},$$

$$e1c3| \begin{bmatrix} 0.5 & 0.1 & 0.1 & 0.4 \\ 0.9 & 0.5 & 0.5 & 0.8 \\ 0.9 & 0.5 & 0.5 & 0.8 \\ 0.6 & 0.2 & 0.2 & 0.5 \end{bmatrix},$$

$$e2c1| \begin{bmatrix} 0.5 & 0.3 & 0.3 & 0.3 \\ 0.7 & 0.5 & 0.5 & 0.5 \\ 0.7 & 0.5 & 0.5 & 0.5 \\ 0.7 & 0.5 & 0.5 & 0.5 \end{bmatrix}, \ e2c2| \begin{bmatrix} 0.5 & 0.5 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 & 0.5 \\ 0.4 & 0.4 & 0.5 & 0.4 \\ 0.5 & 0.5 & 0.6 & 0.5 \end{bmatrix},$$

$$e2c3| \begin{bmatrix} 0.5 & 0 & 0.1 & 0.2 \\ 1 & 0.5 & 0.7 & 0.8 \\ 0.9 & 0.3 & 0.5 & 0.6 \\ 0.8 & 0.2 & 0.4 & 0.5 \end{bmatrix},$$

$$e3c1| \begin{bmatrix} 0.5 & 0.1 & 0.5 & 0 \\ 0.9 & 0.5 & 0.9 & 0.4 \\ 0.5 & 0.1 & 0.5 & 0 \\ 1 & 0.6 & 1 & 0.5 \end{bmatrix}, \ e3c2| \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0.2 \\ 0.8 & 0.5 & 0.6 & 0.5 \\ 0.7 & 0.4 & 0.5 & 0.4 \\ 0.8 & 0.5 & 0.6 & 0.5 \end{bmatrix},$$

$$e3c3| \begin{bmatrix} 0.5 & 0.4 & 0.2 & 0.5 \\ 0.6 & 0.5 & 0.3 & 0.6 \\ 0.8 & 0.7 & 0.5 & 0.8 \\ 0.5 & 0.4 & 0.2 & 0.5 \end{bmatrix}.$$

Then we use the AlgorithmDM (steps one after another). In the step 3 we use the following conversion of the relations P_{tl} to locally B-asymmetric P_{tl}^* (with B=min) for all $1 \leq t \leq k$, $1 \leq l \leq n$.
For each P_{tl} we determine the minimal element: $a = min_{1 \leq i,j \leq m} P_{tl}(i,j)$.
For $i = 1$ do m
For $j = 1$ do m
If $i = j$ then $P_{tl}^*(i,j) = a$ else (If $P_{tl}(i,j) \geq P_{tl}(j,i)$, then $P_{tl}^*(i,j) = P_{tl}(i,j)$ else $P_{tl}^*(i,j) = a$). Moreover, in the step 4 we use $Agg = T_L$ and finally in the step 7 we use the following aggregation functions: A_{mean}, A_{gmean} and A_{median} for each Alternative i obtained in the previous step. Then we get a decreasing sequence:
$CON(Alternative\ 2, Alternative\ 1) = 2.2575$,
$CON(Alternative\ 4, Alternative\ 1) = 1.5333333333$,
$CON(Alternative\ 3, Alternative\ 1) = 1.1504166667$,
$CON(Alternative\ 2, Alternative\ 3) = 1.1070833333$,
$CON(Alternative\ 2, Alternative\ 4) = 0.7241666667$,
$CON(Alternative\ 4, Alternative\ 3) = 0.3829166667$,
....

Thus we obtain the following order of alternatives $Alternative\ 2 \succeq Alternative\ 4 \succeq Alternative\ 3 \succeq Alternative\ 1$.

As a result we obtained the same order of alternatives as in [21], but by observing the multisets in the step 5 of the algorithm, we can infer additional influence of individual experts on decisions (the experts are not anonymous).

VIII. Conclusions

The families of local properties of fuzzy relations considered in this paper do not depend on boundary conditions with values 0 and 1 and moreover they do not depend on operations min and max but they are defined with the use of binary operations B. This approach to properties of fuzzy relations seems to be adequate in fuzzy environment, since such versions are more adjusted to real-life situations. This is why the presented results are promising in considerations of decision making and preference relations which represent the uncertainty of data and information. Moreover, we proposed an algorithm for decision making involving one of the new notions of local B-properties.

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