A note on the links between different qualitative integrals

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***Abstract*—Qualitative or equivalently fuzzy integrals are used as qualitative aggregation functions or as L-fuzzy quantifiers. In both cases they are generalisations of Sugeno integrals. The definitions of these fuzzy integrals are quite similar and coincide in particular cases, but surprisingly there is no deeper analysis of their relationship. The paper attempts to fill this gap and provides unified definitions of fuzzy quantifiers on the basis of which various links between these fuzzy integrals are studied. In order to make these links more visible and to emphasise their logical structure, we present them using the graded square and modern square of opposition.**

***Index Terms*—Fuzzy measures, fuzzy integrals, L-fuzzy quantifiers, qualitative integrals, graded square of opposition, modern square.**

I. Introduction

When we want to find out the relevant features from data, the use of numerical aggregation functions is not always natural since generally the collected data are imprecise and there is often no reliable estimate of the rules (e.g., expressed by probability distributions) governing this imprecision. In such a case the application of qualitative aggregation functions seems to be more appropriate. Among the most important qualitative aggregation functions belong the Sugeno integrals [17], [18]. The definitions of these integrals are based on monotonic set functions named fuzzy measures or capacities that aim to represent possible states of nature or importance of sets of criteria, etc. Sugeno integrals provide a global evaluation according to given local evaluations. These integrals also appear in particular definitions of the type $\langle 1 \rangle$ L-fuzzy quantifiers [7] to model the natural language quantifiers like "all", "some", "many", "none".

In both mentioned contexts, the definition of Sugeno integrals is generalised onto qualitative or equivalently fuzzy integrals,[1] where the originally used operations of minimum and maximum are replaced by more general ones. Although the formal definitions of qualitative integrals in both applications are very similar and coincide in particular cases, their major difference consists in the nature of the evaluation scale:

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[1]Both the terms "qualitative" and "fuzzy" are used in the literature to refer the integrals generalizing the Sugeno integrals.

In the context of L-fuzzy quantifiers the structure of truth values is a complete residuated lattice. Fuzzy measures are monotonic set functions defined on some algebra of sets which take their values on the complete residuated lattice.[2] In [7], we can distinguish two types of fuzzy integrals, namely $\otimes$-fuzzy integrals with respect to a fuzzy measure and $\to$-fuzzy integrals with respect to a complementary fuzzy measure. The $\otimes$-fuzzy integrals are used to model quantifiers like "all" and "some", while the $\to$-fuzzy integrals are used to model quantifiers like "no" and "not all" which are negations of the previous quantifiers from the logical perspective. When the complete residuated lattice is an MV-algebra then both $\otimes$-fuzzy integrals and $\to$-fuzzy integrals are linked by negation, and the $\otimes$-fuzzy integrals coincide with the Sugeno integrals.

In the context of qualitative aggregation functions the evaluation scale is a totally ordered set. Sugeno integrals are generalised extending the operation that combines the value of the fuzzy measure on each subset of criteria and the value of the utility function over elements of the subset. Two qualitative integrals are obtained: one considering fuzzy conjunction generally denoted $\otimes$ and another one considering fuzzy implication denoted $\to$. Sugeno integrals correspond then to the case when the fuzzy conjunction is the minimum and the fuzzy implication is Kleene-Dienes implication.

The aim of this paper is to provide a fundamental comparison of the qualitative integrals which are defined in different contexts and become identical in some particular cases. To achieve our goal we formulate all types of qualitative integrals in the same quite general framework, where the truth values are interpreted in a complete residuated lattice. In order to emphasise the logical structure and to highlight the links between all these qualitative integrals, we present them using the graded square and modern square of opposition. Note that the squares of opposition are not basically designed to interpret the links among qualitative integrals, but the use of qualitative integrals in the context of L-fuzzy quantifiers shows that the squares of opposition could provide a useful tool for their simple visualization. We believe that the analysis of elementary relationships among qualitative integrals helps

[2]Note that the fuzzy measures in [7] are defined more generally over an algebra of fuzzy sets, but for the purpose of this paper, we restrict ourselves to algebras of sets, namely, to power sets of finite sets.

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practitioners to better understand the true meaning of various definitions of qualitative integrals which are important in applications as decision making, classification or syllogistic reasoning.

The paper is structured as follows. Section 2 introduces a complete residuated lattice as the algebraic structure of truth values and the set functions: the fuzzy measures and the complementary fuzzy measures. Section 3 provides the definitions of qualitative integrals and desintegrals and Section 4 summarizes their properties and fundamental relationships. Section 5 presents the square of opposition and two its extensions: the graded square of opposition and the modern square of opposition. In Section 6 we propose several squares of opposition with the qualitative integrals or desintegrals at the vertices. Section 7 is the conclusion.

II. Framework and notations

The algebraic structure of truth values is a complete residuated lattice $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where $(L, \wedge, \vee, 0, 1)$ is a complete lattice, $(L, \otimes, 1)$ is a commutative monoid and the adjointness property for $\otimes$ and $\rightarrow$ is satisfied, i.e., $a \leq b \rightarrow c$ if and only if $a \otimes b \leq c$ holds for any $a, b, c \in L$, where $\leq$ is the lattice ordering. The operations $\otimes$ and $\rightarrow$ are called the *multiplication* and *residuum*, respectively. Define $\neg a = a \rightarrow 0$ the negation of a for any $a \in L$. We say that L is *divisible* if $a \otimes (a \rightarrow b) = a \wedge b$ holds for any $a, b \in L$ and satisfies the *law of double negation* if $\neg\neg a = a$ for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called MV-algebra. Throughout this paper we will use the following property satisfied in each residuated lattice

$$(a \otimes b) \rightarrow c = a \rightarrow (b \rightarrow c). \qquad (1)$$

Moreover, if $\{b_i | i \in I\}$ is a non-empty set of elements from L and $a \in L$, the following properties hold:

(a) $a \otimes (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \otimes b_i)$,
(b) $a \rightarrow \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \rightarrow b_i)$,
(c) $(\bigvee_{i\in I} b_i) \rightarrow a = \bigwedge_{i\in I}(b_i \rightarrow a)$,
(d) $a \otimes \bigwedge_{i\in I} b_i \leq \bigwedge_{i\in I}(a \otimes b_i)$,
(e) $\bigvee_{i\in I}(a \rightarrow b_i) \leq a \rightarrow \bigvee_{i\in I} b_i$,
(f) $\bigvee_{i\in I}(b_i \rightarrow a) \leq \bigwedge_{i\in I} b_i \rightarrow a$.

If L is an MV-algebra the above inequalities may be replaced by equalities (for more details on the residuated lattices see we refer to [1]).

The multiplication $\otimes$ is increasing according to both arguments, the residuum $\rightarrow$ is decreasing according to the left argument and increasing according to the right argument. Moreover for all $a, b \in L$, $a \leq b$ if and only if $a \rightarrow b = 1$.

Let $\mathcal{C}$ be a non-empty universe of discourse.

Definition 1. *A fuzzy measure (or a capacity) $\mu : 2^{\mathcal{C}} \rightarrow L$ is a set function such that $\mu(\emptyset) = 0$, $\mu(\mathcal{C}) = 1$, and $A \subseteq B$ implies $\mu(A) \leq \mu(B)$.*

These set functions are classically used uncertainty modelling [6], or multiple criteria aggregation [12] in order to represent the importance of the sets of possible states of nature or sets of criteria.

Definition 2. *A complementary fuzzy measure (or an anti-capacity) $\nu : 2^{\mathcal{C}} \rightarrow L$ is a set function such that $\nu(\emptyset) = 1$, $\nu(\mathcal{C}) = 0$, and $A \subseteq B$ implies $\nu(B) \leq \nu(A)$.*

The complementary fuzzy measures, are used in multiple criteria decision making when the local evaluation is interpreted as a degree of defect or intensity of rejection. In this context they represent the tolerance level assigned to sets of criteria [5].

If μ is a fuzzy measure then a complementary measure may be introduced in two ways:

(a) $\nu = \mu^{\text{com}}$ defined as $\mu^{\text{com}}(A) = \neg\mu(A)$,
(b) $\nu = \mu^{\text{neg}}$ defined as $\mu^{\text{neg}}(A) = \mu(\mathcal{C} \setminus A)$ where $\mathcal{C} \setminus A$ is the complement of A in $\mathcal{C}$.

One can see that if the fuzzy measure μ in the previous two definitions is replaced by a complementary fuzzy measure ν, we obtain dual definitions of fuzzy measures. Moreover, we have $\mu \leq (\mu^{\text{com}})^{\text{com}}$ and $\mu = (\mu^{\text{neg}})^{\text{neg}}$, where the first inequality becomes the equality if the law of double negation is satisfied. Note that the first inequality is a consequence of the adjointness property, particularly, $a \rightarrow 0 \leq a \rightarrow 0$, hence, $a \otimes (a \rightarrow 0) \leq 0$, which implies $a \leq (a \rightarrow 0) \rightarrow 0$ for any $a \in L$.

Definition 3. *The conjugate fuzzy measure of a fuzzy measure μ is a map $\mu^c : 2^C \rightarrow L$ defined by*

$$\mu^c(A) = \mu(\mathcal{C} \setminus A) \rightarrow 0 = \neg\mu(\mathcal{C} \setminus A).$$

Obviously, we have $\mu^c = (\mu^{\text{neg}})^{\text{com}} = (\mu^{\text{com}})^{\text{neg}}$, $(\mu^{\text{com}})^c = \mu^{\text{neg}}$ and $(\nu^{\text{com}})^c = \nu^{\text{neg}}$.

In what follows, we simply write $\bigwedge_A$ $(\bigvee_A)$ instead of $\bigwedge_{A\subseteq\mathcal{C}}$ $(\bigvee_{A\subseteq\mathcal{C}})$ and $\bigwedge_{A\neq\emptyset}$ $(\bigvee_{A\neq\emptyset})$ instead of $\bigwedge_{A\subseteq\mathcal{C},\, A\neq\emptyset}$ $(\bigvee_{A\subseteq\mathcal{C},\, A\neq\emptyset})$

III. Qualitative integrals and desintegrals

This section presents generalisations of Sugeno integrals named qualitative integrals and qualitative desintegrals [4], [5], [7]. Throughout this part, μ is a fuzzy measure, ν is a complementary fuzzy measure and $f : \mathcal{C} \rightarrow L$ is a function.

A. Qualitative integrals

The Sugeno integral of f with respect to μ is defined by:

$$S_\mu(f) = \bigvee_A \big(\mu(A) \wedge \bigwedge_{i\in A} f_i\big) = \bigvee_A \bigwedge_{i\in A} (\mu(A) \wedge f_i).$$

There are two possibilities to generalise the Sugeno integral: one for each expression. Considering the binary operator $\otimes$, the integrals proposed by Dubois, Prade and Rico (DPR) generalise the first expression [4]; the integrals proposed by Dvořák and Holčapek (DH) generalise the second expression [7].

Definition 4.

- $\int^{\otimes}_{\text{DH},\mu} f = \bigvee_{A\neq\emptyset}(\bigwedge_{i\in A}(f_i \otimes \mu(A))$,
- $\int^{\otimes}_{\text{DPR},\mu} f = \bigvee_A \mu(A) \otimes (\bigwedge_{i\in A} f_i)$.

Both integrals coincide if L is an MV-algebra (see, e.g., Theorem 3.9 in [7]). In general, however, we have $\int^{\otimes}_{\text{DH},\mu} f \geq \int^{\otimes}_{\text{DPR},\mu} f$ since $\bigwedge_{i\in I}(a \otimes b_i) \geq a \otimes \bigwedge_{i\in I} b_i$.

With the binary operator $\rightarrow$ Dubois, Prade and Rico in [5] defined the following residuum based integral, which is also refered as cointegral.

Definition 5. $\int_{\text{DPR},\mu}^{\rightarrow} f = \bigwedge_A (\mu^c(A) \rightarrow \bigvee_{i\in A} f_i)$.[3]

B. Qualitative desintegrals

This section deals with desintegrals, which are integrals whose values are decreasing when the function values are increasing. In [4], the desintegrals are defined for decreasing local evaluation scales (i.e., 0 expresses the best evaluation) assuming that the global evaluations have the standard order. To introduce them, the negative scale is reversed considering $\neg f$ and the fuzzy measure μ is replaced by a complementary fuzzy measure ν. A different approach is considered in [7], where the authors define a type of desintegral in the sense that the aggregation function is decreasing according to increasing values of f and the fuzzy measure is replaced by a complementary fuzzy measure. Nevertheless, in contrast to [4], the authors do not consider $\neg f$, since the function is the first argument of the residuum operation and, therefore, the resulting values are decreasing from the definition of residuum.

Definition 6.
- $\int_{\text{DH},\nu}^{\rightarrow} f = \bigwedge_A \bigvee_{i\in A} (f_i \rightarrow \nu(A))$,
- $\int_{\text{DPR},\nu}^{-\otimes} f = \int_{\text{DPR},\neg\nu^c}^{\otimes} \neg f$,
- $\int_{\text{DPR},\nu}^{-\rightarrow} f = \int_{\text{DPR},\neg\nu^c}^{\rightarrow} \neg f$.

IV. Properties between different qualitative integrals and desintegrals

This section presents various relationships between the qualitative integrals and desintegrals introduced in the previous section. Let us start with a result proved in [7].

Theorem 1 (Theorem 3.18)**.** *If L is a complete MV-algebra,*

(i) $\int_{\text{DH},\nu}^{\rightarrow} f = \bigwedge_A (\bigwedge_{i\in A} f_i \rightarrow \nu(A))$.

(ii) $\int_{\text{DH},\mu^{\text{com}}}^{\rightarrow} f = \neg \int_{\text{DH},\mu}^{\otimes} f$,

(iii) $\int_{\text{DH},\nu^{\text{com}}}^{\otimes} f = \neg \int_{\text{DH},\nu}^{\rightarrow} f$.

Similar properties hold for the qualitative integrals defined by Dubois et al. [4].

Theorem 2. *If L is a complete MV-algebra, then*

(i) $\int_{\text{DPR},\mu}^{\rightarrow} f = \bigwedge_A (\bigvee_{i\in A} (\mu^c(A) \rightarrow f_i)$.

(ii) $\int_{\text{DPR},\mu}^{\rightarrow} f = \neg \int_{\text{DPR},\mu^c}^{\otimes} \neg f$,

(iii) $\int_{\text{DPR},\mu^c}^{\otimes} f = \neg \int_{\text{DPR},\mu}^{\rightarrow} \neg f$.

Proof. (i) It immediately follows from
$\bigvee_{i\in I} (a \rightarrow b_i) = a \rightarrow \bigvee_{i\in I} b_i$.

(ii) We have

$$\begin{aligned}
\int_{\text{DPR},\mu}^{\rightarrow} f &= \bigwedge_A \left(\mu^c(A) \rightarrow \bigvee_{i\in A} f_i\right) \\
&= \bigwedge_A \left(\mu^c(A) \rightarrow (\neg\neg \bigvee_{i\in A} f_i)\right) \\
&= \bigwedge_A \neg(\mu^c(A) \otimes \neg(\bigvee_{i\in A} f_i)) \\
&= \neg \left(\mu^c(A) \otimes \bigvee_A \left(\bigwedge_{i\in A} \neg f_i\right)\right) \\
&= \neg \int_{\text{DPR},\mu^c}^{\otimes} \neg f,
\end{aligned}$$

where we used $a \rightarrow b = a \rightarrow (\neg\neg b) = a \rightarrow ((\neg b) \rightarrow 0) = (a \otimes \neg b) \rightarrow 0 = \neg(a \otimes \neg b)$, which holds for any $a, b \in L$ in each MV-algebra.
(iii) It can be proved analogously to (ii). □

The relationship between two types of residuum based qualitative integrals is as follows.

Proposition 1. *Assuming that the law of double negation is satisfied in L, it holds that* $\int_{\text{DH},\nu}^{\rightarrow} f \leq \int_{\text{DPR},\nu^{\text{neg}}}^{\rightarrow} \neg f$.

Proof. The property $\bigvee_{i\in I} (b_i \rightarrow a) \leq \bigwedge_{i\in I} b_i \rightarrow a$ entails $\neg \bigwedge_{i\in A} f_i = \bigvee_{i\in A} \neg f_i$ and using $\neg a \rightarrow \neg b = b \rightarrow a$, we get

$$\int_{\text{DPR},\nu^{\text{neg}}}^{\rightarrow} \neg f = \bigwedge_A ((\nu^{\text{neg}})^c(A) \rightarrow \bigvee_{i\in A} \neg f_i) =$$
$$\bigwedge_A (\neg\nu(A) \rightarrow \neg \bigwedge_{i\in A} f_i) =$$
$$\bigwedge_A (\bigwedge_{i\in A} f_i) \rightarrow \nu(A) \geq \bigwedge_A \bigvee_{i\in A} (f_i \rightarrow \nu(A)) = \int_{\text{DH},\nu}^{\rightarrow} f,$$

where $(\nu^{\text{neg}})^c = (((\nu^{\text{neg}})^{\text{neg}})^{\text{com}} = \nu^{\text{com}} = \neg\nu$. □

It should be noted that the restriction to complete residuated lattices satisfying the law of double negation is essential in the previous proposition, otherwise, there is no general relationship (at least we do not see it). For complete MV-algebras, we get the following coincidence.

Theorem 3. *If L is a complete MV-algebra, then* $\int_{\text{DH},\nu}^{\rightarrow} f = \int_{\text{DPR},\nu^{\text{neg}}}^{\rightarrow} \neg f$.

Proof. Using (i) of Theorem 1, we have

$$\begin{aligned}
\int_{\text{DH},\nu}^{\rightarrow} f &= \bigwedge_A (\bigwedge_{i\in A} f_i \rightarrow \nu(A)) \\
&= \bigwedge_A (\neg\nu(A) \rightarrow \neg \bigwedge_{i\in A} f_i) \\
&= \bigwedge_A (\neg\nu(A) \rightarrow \bigvee_{i\in A} \neg f_i) \\
&= \bigwedge_A (\nu^{\text{com}}(A) \rightarrow \bigvee_{i\in A} \neg f_i) \\
&= \bigwedge_A (((\nu^{\text{com}})^{\text{neg}})^{\text{neg}}(A) \rightarrow \bigvee_{i\in A} \neg f_i) \\
&= \bigwedge_A (((\nu^{\text{neg}})^{\text{com}})^{\text{neg}}(A) \rightarrow \bigvee_{i\in A} \neg f_i) \\
&= ((\nu^{\text{neg}})^c(A) \rightarrow \bigvee_{i\in A} \neg f_i) \\
&= \int_{\text{DPR},\nu^{\text{neg}}}^{\rightarrow} \neg f,
\end{aligned}$$

where we use $a \rightarrow b = \neg b \rightarrow \neg a$. □

Noticing that $\int_{\text{DPR},\nu}^{-\rightarrow} f = \int_{\text{DPR},\nu^{\text{neg}}}^{\rightarrow} \neg f$, it is easy to check that $\int_{\text{DH},\nu}^{\rightarrow} f \leq \int_{\text{DPR},\nu}^{-\rightarrow} f$ when the law of double negation is satisfied and that $\int_{\text{DH},\nu}^{\rightarrow} f = \int_{\text{DPR},\nu}^{-\rightarrow} f$ when L is a complete MV-algebra.

Note that similarly one can define another desintegral replacing a complementary fuzzy measure ν by a fuzzy measure

[3] Note that, although, the conjugate fuzzy measure μ^c is applied in the definition of residuum based integral, the integral is referred to the fuzzy measure μ. The reason is the coincidence of $\int_{\text{DPR},\mu}^{\rightarrow}$ and $\int_{\text{DPR},\mu}^{\otimes}$ with the Sugeno integral for the fuzzy measure μ that holds for the Kleene-Dienes implication and the Kleene-Dienes conjunction.

μ, namely, $\int^{\rightarrow}_{\mathrm{DH},\mu} f$. Then, we obtain $\int^{\rightarrow}_{\mathrm{DH},\mu} f = \int^{\rightarrow}_{\mathrm{DPR},\mu^{\mathrm{neg}}} \neg f$ in a complete MV-algebra. Indeed, by Theorem 1, we have

$$\begin{aligned}
\int^{\rightarrow}_{\mathrm{DH},\mu} f &= \bigwedge_{A}(\bigwedge_{i\in A} f_i \to \mu(A)) \\
&= \bigwedge_{A}(\neg \bigvee_{i\in A} \neg f_i \to \neg\neg\mu(A)) \\
&= \bigwedge_{A}(\neg\mu(A) \to \bigvee_{i\in A} \neg f_i) \\
&= \bigwedge_{A}(\neg(\mu^{\mathrm{neg}})^{\mathrm{neg}}(A) \to \bigvee_{i\in A} \neg f_i) \\
&= \bigwedge_{A}((\mu^{\mathrm{neg}})^{c}(A) \to \bigvee_{i\in A} \neg f_i) = \int^{\rightarrow}_{\mathrm{DPR},\mu^{\mathrm{neg}}} \neg f,
\end{aligned}$$

where we used $(\mu^{\mathrm{neg}})^{\mathrm{neg}} = \mu$ and the properties of a complete MV-algebra.

Theorem 4. *Let L be a complete residuated lattice, and let us assume that the law of double negation is satisfied. Then,*

(i) $\int^{\rightarrow}_{\mathrm{DPR},\mu} f \geq \int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} \neg f$,

(ii) $\int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} f \leq \int^{\rightarrow}_{\mathrm{DPR},\mu} \neg f$,

(iii) if L is an MV-algebra, then $\int^{\rightarrow}_{\mathrm{DPR},\mu} f = \int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} \neg f$.

Proof. (i)

$$\begin{aligned}
\int^{\rightarrow}_{\mathrm{DPR},\mu} f &= \bigwedge_A(\mu^c(A) \to \bigvee_{i\in A} f_i) \\
&= \bigwedge_A(\neg \bigvee_{i\in A} f_i \to \neg\mu^c(A)) \\
&= \bigwedge_A(\bigwedge_{i\in A} \neg f_i \to (\mu^c)^{\mathrm{com}}(A)) \\
&= \bigwedge_A(\bigwedge_{i\in A} \neg f_i \to \mu^{\mathrm{neg}}(A)) \\
&\geq \bigwedge_A \bigvee_{i\in A}(\neg f_i \to \mu^{\mathrm{neg}}(A)) \\
&= \int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} \neg f,
\end{aligned}$$

where we used $a \to b = \neg b \to \neg a$ (due to the law of double negation), $\bigvee_i a_i \to b = \bigwedge_i (a_i \to b)$ (i.e. for $b = 0$, we obtain $\neg \bigvee_i a_i = \bigwedge_i \neg a_i$) which holds in each complete residuated lattice, and $(\mu^c)^{\mathrm{com}}(A) = ((\mu^{\mathrm{neg}})^{\mathrm{com}})^{\mathrm{com}}(A) = \neg\neg\mu^{\mathrm{neg}}(A) = \mu^{\mathrm{neg}}(A)$.

(ii)

$$\begin{aligned}
\int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} f &= \bigwedge_A \bigvee_{i\in A}(f_i \to \mu^{\mathrm{neg}}(A)) \\
&\leq \bigwedge_A(\bigwedge_{i\in A} f_i \to \mu^{\mathrm{neg}}(A)) \\
&= \bigwedge_A(\neg\mu^{\mathrm{neg}}(A) \to \neg \bigwedge_{i\in A} f_i) \\
&= \bigwedge_A(\mu^c(A) \to \bigvee_{i\in A} \neg f_i) \\
&= \int^{\rightarrow}_{\mathrm{DPR},\mu} \neg f,
\end{aligned}$$

where we used the same properties as in case (i).

(iii)

$$\begin{aligned}
\int^{\rightarrow}_{\mathrm{DPR},\mu} f &= \bigwedge_A(\mu^c(A) \to \bigvee_{i\in A} f_i) \\
&= \bigwedge_A(\neg \bigvee_{i\in A} f_i \to \neg\mu^c(A)) \\
&= \bigwedge_A(\bigwedge_{i\in A} \neg f_i \to \mu^{\mathrm{neg}}(A)) \\
&= \int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} \neg f,
\end{aligned}$$

where we used $a \to b = \neg b \to \neg a$ which holds in each MV-algebra and (i) of Theorem 1. □

As a consequence of Theorem 4(iii), we obtain the following corollary.

Corollary 1. *If L be a complete MV-algebra, then it holds that* $\int^{\dashrightarrow}_{\mathrm{DPR},\nu} f = \int^{\rightarrow}_{\mathrm{DH},\nu} f$.

Proof. From the definition of $\int^{\dashrightarrow}_{\mathrm{DPR},\nu} f$, we find that

$$\int^{\dashrightarrow}_{\mathrm{DPR},\nu} f = \int^{\rightarrow}_{\mathrm{DPR},\neg\nu^c} \neg f = \int^{\rightarrow}_{\mathrm{DH},(\neg\nu^c)^{\mathrm{neg}}} \neg\neg f = \int^{\rightarrow}_{\mathrm{DH},\nu} f,$$

where $(\neg\nu^c)^{\mathrm{neg}} = ((\nu^c)^{\mathrm{com}})^{\mathrm{neg}}) = (\nu^c)^c = \nu$, which holds in any MV-algebra. □

V. The graded and modern squares of opposition

The traditional square of opposition was introduced by logicians of Ancient Greeks in Aristotle's time [16] to describe a logical structure between universally and existentially quantified statements using the schema displayed in Figure 1.

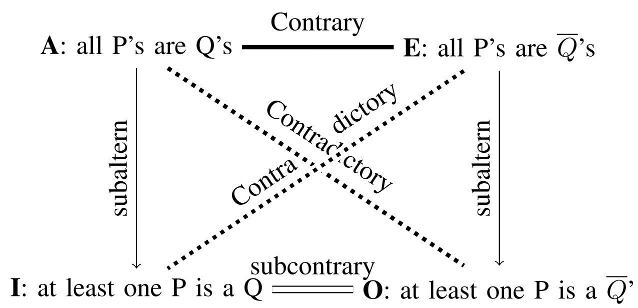

Figure 1. Traditional square of opposition

The vertices of the square are denoted by **AEOI** and satisfy the following relations:

- **A** and **O** are the negation of each other,
- **A** entails **I** and **E** entails **O**,
- together **A** and **E** cannot be true but both maybe false,
- together **I** and **0** cannot be false but both maybe false.

The following subsections present the recently proposed grade version of the traditional square of opposition [2] and the generalisation of the square of opposition called the modern square of opposition [19].

A. Graded square of opposition

In the graded square of opposition we attach four variables $\boldsymbol{\alpha}, \boldsymbol{\epsilon}, \boldsymbol{o}, \boldsymbol{\iota}$ valued on L to vertices A,E,O,I, respectively. The involutive nature of the negation on L is essential to define the graded square of opposition because of the expected symmetry between contradictories. So, we assume that L satisfies the law of double negation, the property, which is satisfied in any MV-algebra.

Let us consider a triplet (i, c, d) of implication, conjunction and disjunction operations on L. Recall that $c, d : L \times L \to L$ are increasing in both arguments, $i : L \times L \to L$ is decreasing in the first argument and increasing in the second argument.

Definition 7. *A graded square of opposition $\boldsymbol{\alpha\epsilon o\iota}$ displayed in Figure 4 respects the following constraints:*

(a) $\boldsymbol{\alpha}$ and $\boldsymbol{o}$ (resp. $\boldsymbol{\epsilon}$ and $\boldsymbol{\iota}$) are each other's negation: $\boldsymbol{o} = \neg(\boldsymbol{\alpha})$ *and* $\boldsymbol{\iota} = \neg(\boldsymbol{\epsilon})$.

(b) a subaltern relationship between $\boldsymbol{\alpha}$ and $\boldsymbol{\iota}$ (resp. $\boldsymbol{\epsilon}$ and $\boldsymbol{o}$): $i(\boldsymbol{\alpha}, \boldsymbol{\iota}) = 1$ *and* $i(\boldsymbol{\epsilon}, \boldsymbol{o}) = 1$.

(c) There is mutual exclusion between $\boldsymbol{\alpha}$ and $\boldsymbol{\epsilon}$, i.e., they cannot be simultaneously equal to 1, but can be both 0: $c(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) = c(\boldsymbol{\epsilon}, \boldsymbol{\alpha}) = 0$.

(d) $\boldsymbol{\iota}$ and $\boldsymbol{o}$ must cover all situations but they can be simultaneously 1: $d(\boldsymbol{\iota}, \boldsymbol{o}) = d(\boldsymbol{o}, \boldsymbol{\iota}) = 1$.

If we assume that

- i and c are mutually definable by semi-duality: $i(x, y) = \neg(c(x, \neg(y))) \iff c(x, y) = \neg(i(x, \neg(y)))$,
- $d : L \times L \to L$ is associated with c by the De Morgan duality: $d(x, y) = \neg(c(\neg(x), \neg(y)))$,

then the properties $i(\boldsymbol{\alpha}, \boldsymbol{\iota}) = 1$, $i(\boldsymbol{\epsilon}, \boldsymbol{o}) = 1$, $c(\boldsymbol{\alpha}, \boldsymbol{\epsilon}) = 0$ and $d(\boldsymbol{\iota}, \boldsymbol{o}) = 1$ are equivalent.

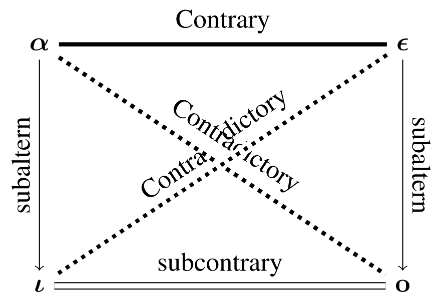

Figure 2. Graded square of opposition

B. Modern square of opposition

In a modern square of opposition the vertices $\boldsymbol{\alpha}$ and $\boldsymbol{o}$ are contradictory as well as $\boldsymbol{\iota}$ and $\boldsymbol{\epsilon}$. In contrast to the traditional square of opposition, the vertices do not fulfill the respective contrary, subcontrary nor subaltern relationships, but the relationships are designed to model different oppositions between quantified sentences. More precisely, the modern square of opposition is formalised using the notation $Q(S, P)$ expressing the formula "Q S are P", where Q is a quantifier, S and P are qualifiers. The modern square of opposition displayed in Figure 3 is defined by the following relationships between vertices:

- the internal negation: the negation of the predicate P,
- the external negation: the negation of the whole sentence,
- the dual relation is defined as the commutative composition of the internal relation and the external negation.

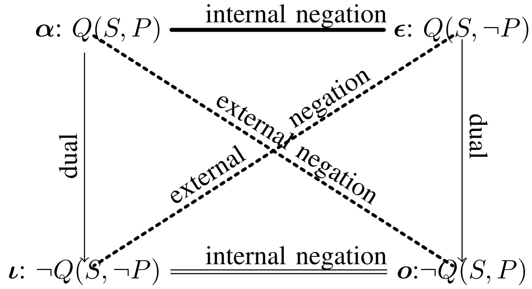

Figure 3. Modern square of opposition

Note that the dual relation is what we name the semi-dual relation. To summarize in a modern square of opposition, there is a semi-duality relation between the vertices $\boldsymbol{\alpha}$ and $\boldsymbol{\iota}$.

The modern square of opposition is used by instance to graphically represent all relations holding between all variants of negations that can be built using the various degrees of freedom existing in fuzzy linguistic summaries [13].

VI. Squares of opposition and qualitative integrals or desintegrals

In this section, we present different squares in which vertices are qualitative integrals or desintegrals. These representations should highlight the common features between both generalisations of the Sugeno integrals.

A. Graded square of opposition with integrals

a) L is a totally ordered set (not necessary a residuated lattice): Let us consider the qualitative aggregation schemes defined as follows. Let $\pi : \mathcal{C} \to L$ be a map such that $\max_{x \in \mathcal{C}} \pi(x) = 1$. Obviously, there exists $x \in \mathcal{C}$ such that $\pi(x) = 1$. In the multicriteria decision making the map π is a possibility distribution of a possibility measure Π on $\mathcal{C}$ defined by $\Pi(A) = \max_{x \in A} \pi(x)$ for any $A \in \mathcal{C}$. The conjugate of a possibility measure is the map $N : \mathcal{C} \to L$ defined by $N(A) = \min_{x \notin A} \neg\pi(x)$ for any $A \in \mathcal{C}$.

For any map $f : \mathcal{C} \to L$ one can define the following qualitative aggregation schemes:

$$\inf_x \pi(x) \to f(x) \quad \text{and} \quad \sup_x \pi(x) \otimes f(x).$$

Adding a binary operation $*$ on L and the transformations

- the residuation of $*$ is $aRes(*)b = sup\{c : a * c \leq b\}$
- the semi-dual of $*$ is $a\mathcal{S}(*)b = \neg(a * \neg b)$.

we get the following result presented in [5].

Proposition 2. *If we suppose that there exists x such that $f(x) = 1$, x' such that $\neg f(x') = 1$, x" such that $\pi(x") = 1$ and y such that $\neg\pi(y) = 1$, then the graded square presented in Fig. 4 is a graded square of opposition whenever we choose*

- *the operations $\neg$, $\otimes$, $\to$ such that :* $1 \otimes a \geq \neg(1 \otimes \neg a)$, $a \otimes 1 \geq \neg(\neg a \otimes 1)$, *and* $\to = \mathcal{S}(\otimes)$,
- *the negation $\neg$, the implication $i = Res(*)$, the conjunction $c = \mathcal{S}(i)$ and d the De Morgan dual of c where $*$ is any conjunction on L such that $a * b \leq \min(a, b)$.*

Note that the previous result does not assume the commutativity of $\otimes$ and 1 need not be the neutral element for $\otimes$.

The properties: $\Pi(A) = \max_{x \in A} \pi(x)$ and $\otimes$ is increasing according to both arguments, entail

$$\int_{\mathrm{DPR},\Pi}^{\otimes} f = \int_{\mathrm{DH},\Pi}^{\otimes} f = \sup_x \pi(x) \otimes f(x).$$

Similarly the properties: $N(A) = \min_{x \notin A} \neg\pi(x)$ and $\to$ is decreasing according to the first argument and increasing according to the second one, entail

$$\int_{\mathrm{DPR},N}^{\to} f = \inf_x \pi(x) \to f(x).$$

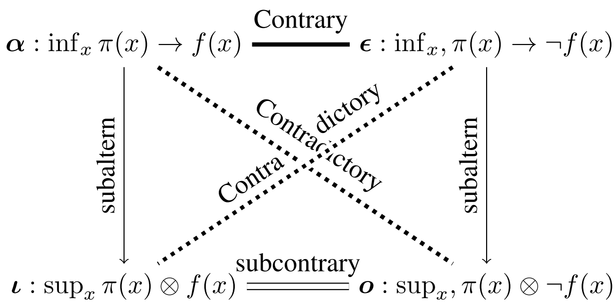

Figure 4. Square of opposition of weighted qualitative aggregations

Hence, using the different equalities between the integrals, the square of opposition in Fig. 4 can be rewritten to the square of opposition displayed in Fig. 5 expressing certain links between various qualitative integrals.

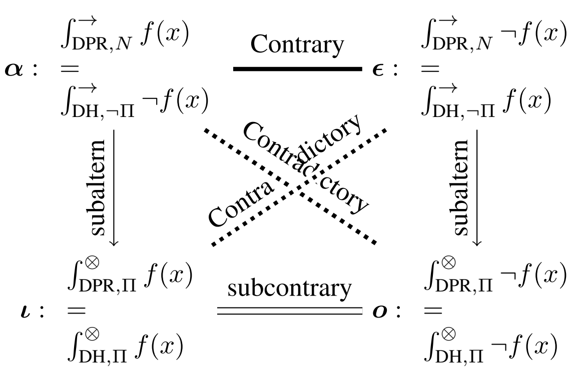

Figure 5. Square of opposition of various qualitative integrals

b) L is a complete residuated lattice:

Recall that $a \to b = 1$ if and only if $a \leq b$ holds in any complete residuated lattice.

Proposition 3. *In a complete residuated lattice satisfying the law of double negation the square presented in Figure 6 is a square of opposition considering $\otimes$, $\to$, $\neg$, conjunction $c = \otimes$, implication $i = \to$ and defining d as the De Morgan duality of $\otimes$.*

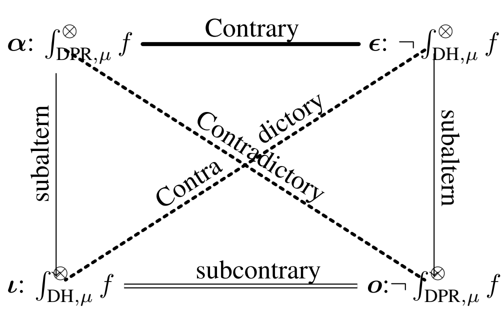

Figure 6. Graded square of opposition with qualitative integrals

Proof. It immediately follows from the inequality $\int^{\otimes}_{\mathrm{DPR},\mu} f \leq \int^{\otimes}_{\mathrm{DH},\mu} f$ mentioned below Definition 4. Indeed, for the sake of simplicity, put $\alpha = \int^{\otimes}_{\mathrm{DPR},\mu} f$, $\iota = \int^{\otimes}_{\mathrm{DH},\mu} f = 1$, $\epsilon = \neg\iota$ and $o = \neg\alpha$. From $\alpha \leq \iota$, we find that $i(\alpha, \iota) = \alpha \to \iota = 1$. Since $\alpha \leq \iota$ implies $\epsilon = \neg\iota \leq \neg\alpha = o$, we get $i(\epsilon, o) = 1$. Hence, we proved the subaltern. The contrary follows from $c(\alpha, \epsilon) = \alpha \otimes \epsilon = \alpha \otimes \neg\iota \leq \alpha \otimes \neg\alpha = 0$, where we used the monotonicity of $\otimes$ and $\neg\iota \leq \neg\alpha$. Note that $\alpha \otimes \neg\alpha = \alpha \otimes (\alpha \to 0) \leq 0$, which holds in any residuate lattice. Finally, the subcontrary follows from $d(\iota, o) = \neg(\neg\iota \otimes \neg o) = \neg(\neg\iota \otimes \alpha) \geq \neg(\alpha \otimes \neg\alpha) = \neg 0 = 1$, where we used the monotonicity of $\otimes$ (i.e. $\neg\iota \otimes \alpha \leq \alpha \otimes \neg\alpha$) and the negation reversing the ordering (i.e. $\neg(\neg\iota \otimes \alpha) \geq \neg(\alpha \otimes \neg\alpha)$). ☐

It should be noted that if the complete residuated lattice in the previous proposition is a complete MV-algebra, then we obtain a degenerated square of opposition with the pairs of identical vertices, namely, $\alpha = \iota$ and $\epsilon = o$.

B. The modern square of opposition

This section presents modern squares with fuzzy quantifiers or qualitative integrals. Let us syart with a generalisation of the modern square with quantifiers presented in Fig. 3.

Let $f : \mathcal{C} \to L$ be a map. Define $f_{\inf}, f_{\sup} : 2^{\mathcal{C}} \to L$ as follows:

$$f_{\inf}(A) = \bigwedge_{i \in A} f_i \text{ and } f_{\sup}(A) = \bigvee_{i \in A} f_i.$$

It is easy to see that if the law of double negation is satisfied, then $\neg f_{\sup} = (\neg f)_{\inf}$ and $\neg f_{\inf} = (\neg f)_{\sup}$. Now, we can introduce a quantifier as follows:

$$Q(\mu, f_{\sup}) = \bigwedge_{A \in 2^{\mathcal{C}}} \mu^c(A) \to f_{\sup}(A), \tag{2}$$

which is defined for any pair of maps from $2^{\mathcal{C}}$ to L, where the first map is a fuzzy measure μ and the second map is the map $f_{\sup}$ determined from a map $f : \mathcal{C} \to L$. One can see that the quantifier Q is modeled as a classical type $\langle 1, 1 \rangle$ fuzzy quantifier "all" defined over the universe $2^{\mathcal{C}}$ (cf. [9]–[11]), where the residuum is used to express the implication between qualifiers. Motivated by the modern square in Fig. 3, we can formulate a modern square for the fuzzy quantifier defined over the universe $2^{\mathcal{C}}$, which is displayed in Fig. 7. It should

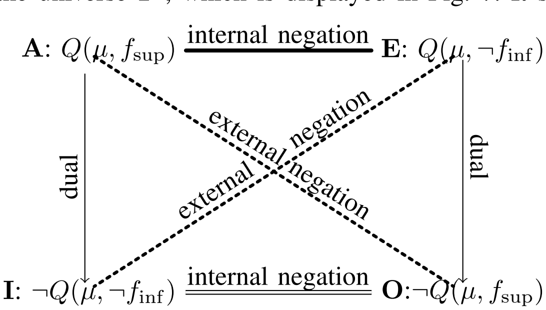

Figure 7. Modern square with fuzzy quantifiers

be noted that we introduce the internal negation of $f_{\sup}$ as the negation of the map f and not $f_{\sup}$, i.e., $(\neg f)_{\sup} = \neg f_{\inf}$. Hence, the double application of the internal negation $f_{\sup}$ gives us the original map $f_{\sup}$, whenever the law of double negation is satisfied, but $c(f_{\sup}, (\neg f)_{\sup}) = 0$ is not true for a conjunction. Therefore, $c(Q(\mu, f_{\sup}), Q(\mu, \neg f_{\inf})) \neq 0$ can occur and Contrary used in the traditional (and also graded) square of opposition fails in general, which is a difference between these two approaches to the squares of opposition. The following example shows that Contrary can be satisfied if one restricts the space of fuzzy measures and maps appropriately.

Example 1. *Assume that $c(Q(\mu, f_{\sup}), Q(\mu, \neg f_{\inf}) = Q(\mu, f_{\sup}) \otimes Q(\mu, \neg f_{\inf})$ and consider a proper filter $\mathcal{F}$ on $2^{\mathcal{C}}$, i.e., $\mathcal{F} \subset 2^{\mathcal{C}}$ such that $\mathcal{F} \neq \emptyset$, $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$ and if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$. Note that the elements of a proper filter $\mathcal{F}$ can be interpreted as big sets in $2^{\mathcal{C}}$. If $f : \mathcal{C} \to L$ and μ is a fuzzy measure such that there exists $X \in \mathcal{F}$ with $\mu(X) = 1$ and $f_{\sup}(X) \otimes \neg f_{\inf}(X) = 0$, then $Q(\mu, f_{\sup}) \otimes Q(\mu, \neg f_{\inf}) = 0$. Indeed, we have*

$$\begin{aligned} &Q(\mu, f_{\sup}) \otimes Q(\mu, \neg f_{\inf}) \\ &\quad = \bigwedge_A \mu^c(A) \to f_{\sup}(A) \otimes \bigwedge_B \mu^c(B) \to \neg f_{\inf} \\ &\quad \leq \bigwedge_A \bigwedge_B \mu^c(A) \otimes \mu^c(B) \to f_{\sup}(A) \otimes \neg f_{\inf} \\ &\quad \leq \mu^c(X) \otimes \mu^c(X) \to f_{\sup}(X) \otimes \neg f_{\inf}(X) = 1 \to 0 = 0. \end{aligned}$$

Hence, a restriction of maps and fuzzy measures can ensure the satisfaction of Contrary in the graded square of opposition. Note that we used a proper filter which could solve a similar issue posted by Murinová and Novák in [14] (see also [15]).

Note that the external negation of Q defined in (2) is the type $\langle 1, 1\rangle$ fuzzy quantifier "Some", where the multiplication is used to express the conjunction between qualifiers. If one accepts the square proposed in Fig. 7 as a modern square of opposition, one can use it for a clear description the relationships among qualitative integrals and desintegrals proposed by Dubois, Prade and Rico in [4]. More precisely, we have the following results.

Proposition 4. *In a complete MV-algebra the square presented in Fig. 8 is a modern square considering $\neg f$ as internal negation and $\neg$ as the external negation.*

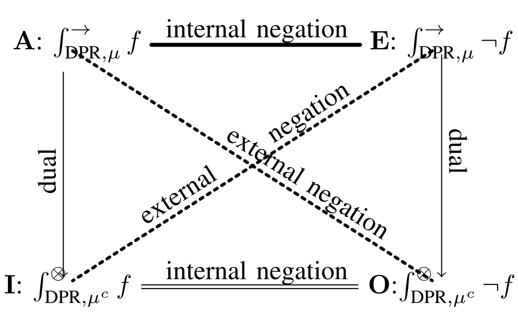

Figure 8. Modern square with qualitative integrals

Proof. It is easy to see that $Q(\mu, f_{\sup}) = \int^{\to}_{\mathrm{DPR},\mu} f$. We prove only the next equality of vertices $\neg Q(\mu, \neg f_{\inf})$ and $\int^{\otimes}_{\mathrm{DPR},\mu^c} f$ in Fig. 7 and Fig. 8, respectively, the remaining equalities can be done analogously. Similarly to the proof of Theorem 2(ii), we have

$$\begin{aligned} \neg Q(\mu, \neg f_{\inf}) &= \neg \textstyle\bigwedge_A \left(\mu^c(A) \to \neg f_{\inf}(A)\right) \\ &= \neg \textstyle\bigwedge_A \left(\mu^c(A) \to (\neg \bigwedge_{i \in A} f_i)\right) \\ &= \neg \textstyle\bigwedge_A \neg \left(\mu^c(A) \otimes (\neg\neg \bigwedge_{i \in A} f_i)\right) \\ &= \textstyle\bigvee_A \mu^c(A) \otimes \bigwedge_{i \in A} f_i \\ &= \textstyle\int^{\otimes}_{\mathrm{DPR},\mu^c} f, \end{aligned}$$

where we used $a \to b = \neg(a \otimes \neg b)$. □

Note that the square obtained in Fig. 8 is a generalisation of the square in Fig. 5. A natural question that arises is whether this square can also possess the properties of graded square of opposition. Assuming that the law of double negation is satisfied, one can see from the comment below Definition 7 that the square in Fig. 8 is a graded square of opposition if

$$\int^{\to}_{\mathrm{DPR},\mu} f \leq \int^{\otimes}_{\mathrm{DPR},\mu^c} f, \tag{3}$$

where we use $a \to b = 1$ if and only if $a \leq b$ for $a, b \in L$. Unfortunately, (3) is not true in general as the following proposition demonstrates.

Proposition 5. *For any $A \subseteq \mathcal{C}$ such that $0 < \mu^c(A) < 1$ and $\mu^c(A) \otimes \neg \mu^c(\overline{A}) < \neg \mu^c(\overline{A})$, then there exists a map $f : \mathcal{C} \to L$ such that $\int^{\to}_{\mathrm{DPR},\mu} f > \int^{\otimes}_{\mathrm{DPR},\mu^c} f$.*

Proof. Let us consider the map $f : \mathcal{C} \to L$ give by

$$f_i = \begin{cases} \neg \mu^c(\overline{A}), & i \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} &\int^{\to}_{\mathrm{DPR},\mu} f = \bigwedge_X \mu^c(X) \to \bigvee_{i \in X} f_i = \\ &(\bigwedge_{X \cap A \neq \emptyset} \mu^c(X) \to \bigvee_{i \in X} f_i) \wedge (\bigwedge_{X \cap A = \emptyset} \mu^c(X) \to \bigvee_{i \in X} f_i) = \\ &(\bigwedge_{X \cap A \neq \emptyset} \mu^c(X) \to \neg \mu^c(\overline{A})) \wedge (\bigwedge_{X \cap A = \emptyset} \mu^c(X) \to 0) = \\ &(\mu^c(\mathcal{C}) \to \neg \mu^c(\overline{A}) \wedge \mu^c(\overline{A}) \to 0 = \\ &(1 \to \neg \mu^c(\overline{A})) \wedge \neg \mu^c(\overline{A}) = \neg \mu^c(\overline{A}), \end{aligned}$$

where we used the antitony of $\to$ in the first argument, i.e., $a \to c \leq b \to c$ if $a \geq b$, , and the monotony of μ, i.e., $\mu(X) \leq \mu(\overline{A})$ for any $X \subseteq \mathcal{C}$ such that $X \cap A = \emptyset$. On the other hand, we have

$$\begin{aligned} &\int^{\otimes}_{\mathrm{DPR},\mu^c} f = \bigvee_X \mu^c(X) \otimes \bigwedge_{i \in X} f_i = \\ &\bigvee_{X \cap A = X} \mu^c(X) \otimes \bigwedge_{i \in X} f_i \vee \bigvee_{X \cap A \neq X} \mu^c(A) \otimes \bigwedge_{i \in X} f_i = \\ &\bigvee_{X \cap A = X} \mu^c(X) \otimes \neg \mu^c(\overline{A}) \vee \bigvee_{X \cap A \neq X} \mu^c(A) \otimes 0 \\ &= \mu^c(A) \otimes \neg \mu^c(\overline{A}), \end{aligned}$$

where we used the monotony of μ. □

From the previous proposition, one can see that $\mu^c(A) \otimes \neg\mu^c(\overline{A}) = \neg\mu^c(\overline{A})$ is a necessary condition to ensure (3) for any map $f : \mathcal{C} \to L$. In a complete residuated lattice satisfying the law double negation, for all $A \subseteq \mathcal{C}$, we have $\neg\mu^c(\overline{A}) = \neg\neg\mu(A) = \mu(A)$. Hence, the previous equality can be rewritten as $\mu^c(A) \otimes \mu(A) = \mu(A)$. Since $\mu^c(A) \otimes \mu(A) \leq \mu^c(A)$, we find that $\mu(A) \leq \mu^c(A)$, i.e., μ is a pessimistic fuzzy measure (see, e.g. [3]). The following proposition shows a sufficient condition for which the desired inequality (3) is satisfied.

Proposition 6. *If we consider a complete residuated lattice satisfying the law double negation, and if we consider a pessimistic fuzzy measure then under the condition that there exists $i \in \mathcal{C}$ such that $\mu(\{i\}) = 1$ the modern square with qualitative integrals presented in Fig. 8 is also a graded square of opposition.*

Proof. Let us consider a pessimistic fuzzy measure μ and let us suppose that there exists $i_0 \in \mathcal{C}$ such that $\mu(\{i_0\}) = 1$. Note that we have also $\mu^c(\{i\}) = 1$. Hence, we have

$$\begin{aligned}\int^{\rightarrow}_{\mathrm{DPR},\mu} f &= \bigwedge_A (\mu^c(A) \to \bigvee_{i\in A} f_i) \leq \bigwedge_A (\mu(A) \to \bigvee_{i\in A} f_i)\\ &\leq \mu(\{i_0\}) \to f_{i_0} = \neg(1 \otimes \neg f_{i_0}) = f_{i_0}\\ &\leq \mu^c(\{i_0\}) \otimes f_{i_0} \leq \bigvee_A \mu^c(A) \otimes \bigwedge_{i\in A} f_i = \int^{\otimes}_{\mathrm{DPR},\mu^c} f,\end{aligned}$$

where we used the fact that μ is a pessimistic measure, i.e., $\mu(A) \leq \mu^c(A)$ for any $A \subseteq \mathcal{C}$. □

A trivial example of the pessimistic fuzzy measure satisfying the sufficient condition of the previous proposition is

$$\mu(A) = \begin{cases} 0, & A = \emptyset,\\ 1, & \text{otherwise.}\end{cases}$$

One can simply verify that $\mu^c = \mu$ and $\int^{\otimes}_{\mathrm{DPR},\mu^c} f = \bigvee_{\mathcal{C}} f_i$ and $\int^{\rightarrow}_{\mathrm{DPR},\mu} f = \bigwedge_{\mathcal{C}} f_i$.

Proposition 7. *In a complete MV-algebra the square presented in Fig. 9 is a modern square.*

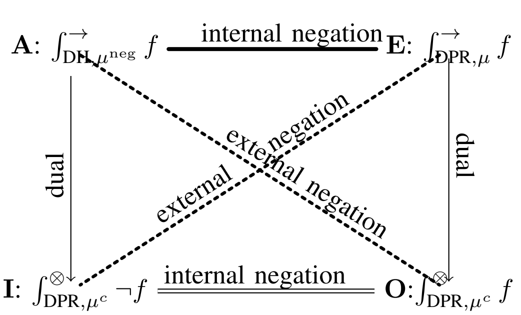

Figure 9. Modern square with qualitative integrals

Proof. It immediately follows from Proposition 4, where f and $\neg f$ are interchanged and $\int^{\rightarrow}_{\mathrm{DPR},\mu} \neg f$ is replace by $\int^{\rightarrow}_{\mathrm{DH},\mu^{\mathrm{neg}}} f$, which is correct according to Theorem 3. □

VII. Conclusion

In this paper, we analyzed the relationships between recently introduced qualitative integrals and desintegrals in literature. For our analysis, we introduced all integrals in the unique framework with complete residuated lattices as the algebraic structure for truth values. In order to highlight these relationships, we used the graded and modern squares of opposition with the integrals or desintegrals at the vertices. Both squares of opposition allow to obtain results from one type of integrals to another one, which can be used in applications as decision making or classification. Moreover, the properties following from the squares of opposition can be applied in syllogistic reasoning with generalized (fuzzy) quantifiers.

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