Generalizing the GMC-RTOPSIS Method using CT-integral Pre-aggregation Functions

Jonata C. Wieczynski
Programa de Pós-Graduação em Computação
Universidade Federal do Rio Grande
Rio Grande, Brazil
jwieczynski@furg.br

Graçaliz P. Dimuro
Centro de Ciências Computacionais
Universidade Federal do Rio Grande
Rio Grande, Brazil
Departamento de Estadística, Informática y Matemáticas
Universidad Publica de Navarra
Pamplona, Spain
gracalizdimuro@furg.br

Eduardo N. Borges, Hélida S. Santos
Centro de Ciências Computacionais
Universidade Federal do Rio Grande
Rio Grande, Brazil
{eduardoborges,helida}@furg.br

Giancarlo Lucca
Programa de Pós-Graduação em Modelagem Computacional
Universidade Federal do Rio Grande
Rio Grande, Brazil
giancarlo.lucca@furg.br

Rodolfo Lourenzutti
Department of Computer Science
University of British Columbia
Vancouver, Canada
rodolfo.lourenzutti@gmail.com

Humberto Bustince
Departamento de Estadística, Informática y Matemáticas
Universidad Publica de Navarra
Pamplona, Spain
bustince@unavarra.es

***Abstract*—In Multi-Criteria Decision Making, one of the most used algorithm designed to deal with decision making is the Technical Order by Preference to Ideal Solution (TOPSIS), which is based on finding a solution that is close to the best possible solution and distant from the worst possible solution. The Group Modular Choquet Random TOPSIS (GMC-RTOPSIS) is a generalization of the TOPSIS method capable of dealing with multiple and heterogeneous data types and interaction among criteria by means of the discrete Choquet integral. On the other hand, CT-integrals are a generalization of the Choquet integral using t-norms, which are more flexible than the standard Choquet integral. CT-integrals are pre-aggregation functions, which means that we do not require them to be monotonic in the whole domain, just in some specific directions, that is, they are directionally monotonic. Due to the excellent performance of CT-integrals in classification and multimodal fuzzy fusion decision problems, the objective of this paper is to generalize the GMC-RTOPSIS by using CT-integrals and to analyze the results provided by the use of five different t-norms in an example of a decision making problem.**

***Index Terms*—Choquet integral, CT-integrals, pre-aggregation function, GMC-RTOPSIS, Multi-Criteria Decision Making**

I. Introduction

Decision making is strongly linked with daily life, not only concerning with individuals, but also with businesses and industries, in general. From choosing something as simple as a shoe to a more complex decision of selecting an appropriated supplier, decisions must be taken to reach a good equilibrium between what will be gained and lost in conflicting criteria.

One algorithm designed to deal with decision making is the Technical Order by Preference to Ideal Solution (TOPSIS) [1], [2]. It falls into a class of problems called Multi-Criteria Decision Making (MCDM) [3], [4] and is based on finding a solution that is closer to the best possible solution (Positive Ideal Solution (PIS)) and distant from the worst possible solution (Negative Ideal Solution (NIS)).

Lourenzutti et al. [5] introduced the Group Modular Choquet Random TOPSIS (GMC-RTOPSIS), which is a generalization of the TOPSIS method capable of dealing with multiple and heterogeneous data types (in particular, they use intuitionistic fuzzy data [6]) and interaction among criteria by means of the discrete Choquet integral [7], proposing a method based on Particle Swarm Optimization (PSO) [8] to learn the fuzzy measure.

Observe that the discrete Choquet integral, as a special aggregation function [9], is a fuzzy integral that can be used to integrate a function with respect to a (non-additive) fuzzy measure [10], and, thus, it is able to consider the influence of the interaction among the elements to be integrated (that is, to be aggregated) [11], [12]. In fact, the Choquet integral is an averaging aggregation function [13], meaning that its outputs carry information laying in a range from the minimum to the maximum of the information provided by the inputs.

The Choquet integral has been applied in different problems, such as image processing [14], [15], risk evaluation [16], models for robust sustainable development [17], classification [18], ensembles of classifiers [19], and mainly in decision

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making and multi-criteria decision making [5], [20], [21]. See the work by Dimuro et al. [12] for a discussion about the applications of the Choquet integral.

In 2016, Lucca et al. introduced a generalization of the Choquet integral using t-norms $T : [0,1]^2 \to [0,1]$ [22], called C_T-integrals [23], which are more flexible than the standard Choquet integral. First, C_T-integrals define a family of fuzzy integrals (which includes the standard Choquet integral) and, then, one always can choose one of them that best fit to the problem in question. Secondly, C_T-integrals are pre-aggregation functions [23], which means that they are not required to be monotonic in the whole domain, just in some specific directions, that is, they are directionally monotonic [24].

C_T-integrals have presented excellent results when used to perform the aggregation task in the fuzzy reasoning method of Fuzzy Rule-based Classification Systems [23] and also in multimodal fuzzy fusion decision [25]. For further details on the use of directional monotonicity in applications see [12].

Then, the objective of this present work is to generalize the GMC-RTOPSIS by using C_T-integrals and to analyze the results obtained by the use of five different t-norms (namely, the algebraic product – which provides the original method –, the Minimum, the Łukasiewicz t-norm, the Nilpotent product and the Hamacher product) in an example of a decision making problem.

This paper is organized as follows. In Section II it is presented the basic framework to understand the algorithm. In Section III a overview of the GMC-RTOPSIS is presented. Section IV explains the generalization of GMC-RTOPSIS by C_T-integrals. Section V shows the example problem used for simulations followed by an analysis of the obtained results. Finally, Section VI is the Conclusion.

II. Preliminary Concepts

In this section, we recall the preliminary concepts necessary to develop the paper.

A. Intuitionistic Fuzzy Numbers

A fuzzy set [26] a is a set defined on universe X by a membership function $\mu_a : X \to [0,1]$, denoted by

$$a = \{\langle x,\ \mu_a(x)\rangle \mid x \in X\}.$$

A fuzzy set a is a trapezoidal fuzzy number (TFN), denoted by $a = (a_1, a_2, a_3, a_4)$, where $a_1 \leq a_2 \leq a_3 \leq a_4$, if the membership function μ_a is defined on $\mathbb{R}$ as:

$$\mu_a(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & \text{if } a_1 \leq x < a_2 \\ 1, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise.} \end{cases}$$

The distance between two TFNs $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$ is defined as:

$$d(a,\ b) = \sqrt{\frac{1}{4}\sum_{i=1}^{4}(a_i - b_i)^2}.$$

The defuzzified value of a TFN $a = (a_1, a_2, a_3, a_4)$ is given by:

$$m(a) = \frac{a_1 + a_2 + a_3 + a_4}{4}.$$

Definition 2.1: An intuitionistic fuzzy set (IFS) [6] A is defined on a universe X by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $v_A : X \to [0,1]$ such that $\mu_A(x) + v_A(x) \leq 1$, for all $x \in X$, that is:

$$A = \{\langle x,\ \mu_A(x),\ \nu_A(x)\rangle \mid x \in X\}.$$

Let $\tilde{\mu}_A$ and $\tilde{\nu}_A$ be the maximum membership degree and the minimum non-membership degree, respectively, of an intuitionistic fuzzy set (IFS) A.

Definition 2.2: An IFS A is an intuitionistic trapezoidal fuzzy number (ITFN), denoted by

$$A = \langle (a_1, a_2, a_3, a_4),\ \tilde{\mu}_A, \tilde{\nu}_A\rangle,$$

where $a_1 \leq a_2 \leq a_3 \leq a_4$, if μ_A and v_A are given, for all $x \in \mathbb{R}$, by:

$$\mu_A(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}\tilde{\mu}_A, & \text{if } a_1 \leq x < a_2 \\ \tilde{\mu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3}\tilde{\mu}_A, & \text{if } a_3 < x \leq a_4 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$v_A(x) = \begin{cases} \frac{1-\tilde{\nu}_A}{a_1-a_2}(x-a_1)+1, & \text{if } a_1 \leq x < a_2 \\ \tilde{\nu}_A, & \text{if } a_2 \leq x \leq a_3 \\ \frac{1-\tilde{\nu}_A}{a_4-a_3}(x-a_4)+1, & \text{if } a_3 < x \leq a_4 \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.3: The distance between two IFS $A = \langle (a_1, a_2, a_3, a_4),\ \tilde{\mu}_A,\ \tilde{\nu}_A\rangle$ and $B = \langle (b_1, b_2, b_3, b_4),\ \tilde{\mu}_B,\ \tilde{\nu}_B\rangle$ is:

$$d(A,\ B) = \frac{1}{2}\left[d_{\tilde{\mu}}(A,\ B) + d_{\tilde{\nu}}(A,\ B)\right]$$

where

$$d_\kappa(A,B) = \left\{\frac{1}{4}\Big[(a_1-b_1)^2 + (1+(\kappa_A-\kappa_B)^2) \right.$$
$$(1+(a_2-b_2)^2+(a_3-b_3)^2)$$
$$\left. -1+(a_4-b_4)^2\Big]\right\}^{1/2}$$

for $\kappa = \tilde{\mu}, \tilde{\nu}$.

B. Aggregation and Pre-aggregation Functions

Aggregation functions are an important class of functions that combines a n-ary input into a single output following two simple proprieties:

Definition 2.4 (Aggregation function [9]): An aggregation function is a function that maps $n > 1$ arguments onto the unit interval, i.e., $f : [0,1]^n \to [0,1]$ such that:

(A1) $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$; (Boundary conditions)

(A2) If $\boldsymbol{x} \leq \boldsymbol{y}$ then $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$. (Monotonicity)

TABLE I: T-norms used in this paper

t-norms	Definition
Algebraic product	$T_P(x,y)=xy$
Minimum	$T_M(x,y)=\min\{x,y\}$
Łukasiewicz	$T_Ł(x,y)=\max\{0,x+y-1\}$
Nilpotent product	$T_{NM}(x,y)=\begin{cases}\min\{x,y\}, & \text{if } x+y>1\\ 0, & \text{otherwise}\end{cases}$
Hamacher product	$T_{HP}(x,y)=\begin{cases}0, & \text{if } x=y=0\\ \frac{xy}{x+y-xy}, & \text{otherwise}\end{cases}$

An important kind of aggregation function are the t-norms.

Definition 2.5 (Triangular norm [22]): A triangular norm (or t-norm) is a bi-variate aggregation function $T:[0,1]^2\to[0,1]$, such that, for all $x,y,z\in[0,1]$, T satisfies the following properties:

(T1) $T(x,y)=T(y,x)$; (Commutativity)
(T2) $T(x,T(y,z))=T(T(x,y),z)$; (Associativity)
(T3) $T(x,y)\le T(x,z)$ if $y\le z$; (Monotonicity)
(T4) $T(x,1)=x$. (Boundary condition)

The t-norms adopted in this paper are shown in Table I.

In order to generalize earlier work on relaxed monotonicity constraints, to encompass non-monotonic functions within the aggregation framework, Bustince et. al. [24] defined directional monotonicity as follows:

Definition 2.6 (Directional Monotonicity): Let $\boldsymbol{r}$ be a real n-dimensional vector, such that $\boldsymbol{r}\neq\boldsymbol{0}$. A function $f:[0,1]^n\to[0,1]$ is said to be $\boldsymbol{r}$-increasing if, for all $\boldsymbol{x}\in[0,1]^n$ and for all $c>0$ such that $\boldsymbol{x}+c\boldsymbol{r}\in[0,1]^n$, it holds that

$$f(\boldsymbol{x}+c\boldsymbol{r})\ge f(\boldsymbol{x}).$$

Considering the directional monotonicity, Lucca et al. [23] introduced the concept of pre-aggregation functions:

Definition 2.7 (Pre-aggregation function [23]): A function $PA:[0,1]^n\to[0,1]$ is said to be an n-ary pre-aggregation function if the following condition holds:

(PA1) There exists $\boldsymbol{r}\in[0,1]^n$, $\boldsymbol{r}\neq\boldsymbol{0}$, such that PA is $\boldsymbol{r}$-increasing; (Directional Monotonicity)
(PA2) $PA(\boldsymbol{0})=0$ and $PA(\boldsymbol{1})=1$. (Boundary Conditions)

C. Choquet Integral

The Choquet integral is defined considering the concept of fuzzy measure and can be used to model the interaction among its arguments.

Definition 2.8 (Fuzzy measure [27]): Let N be a finite set and 2^N be its power set. Then a function $\varphi:2^N\to[0,1]$ is a fuzzy measure if, for all $X,Y\subseteq N$, the following conditions holds:

1) $\varphi(\emptyset)=0$ and $\varphi(X)=1$;
2) if $X\subseteq Y$, then $\varphi(X)\le\varphi(Y)$.

Definition 2.9 (Choquet integral [7]): Let φ be a fuzzy measure. Then the Choquet integral of $\boldsymbol{x}\in[0,1]^n$ with respect to φ is $\mathcal{C}_\varphi[0,1]^n\to[0,1]$, defined by:

$$\mathcal{C}_\varphi(\boldsymbol{x})=\sum_{i=1}^{n}\left[x_{(i)}-x_{(i-1)}\right]\ \varphi(A_{(i)})$$

where (i) represents a permutation on 2^N such that $x_{(i-1)}\le x_{(i)}$ for all $i=1,\dots,n$, with $x_{(0)}=0$ and $A_{(i)}=\{(1),\dots,(i)\}$.

The product used in the Choquet integral is, in fact, the algebraic product t-norm (see Table I). Lucca et al. [23] generalized the concept of Choquet integral by substituting this operator for others from the t-norm family, which created a new class of fuzzy integral, named C_T-integrals by Dimuro et al. [12]:

Definition 2.10 (C_T-integral [12], [23]): Let φ be a fuzzy measure and $T:[0,1]^2\to[0,1]$ a t-norm. Then a C_T-integral is defined as $\mathcal{C}_\varphi^T:[0,1]^n\to[0,1]$, given, for all $\boldsymbol{x}\in[0,1]^n$, by

$$\mathcal{C}_\varphi^T(\boldsymbol{x})=\sum_{i=1}^{n}T\left(x_{(i)}-x_{(i-1)},\ \varphi(A_{(i)})\right)$$

where (i) represents a permutation on 2^N such that $x_{(i-1)}\le x_{(i)}$ for all $i=1,\dots,n$, with $x_{(0)}=0$ and $A_{(i)}=\{(1),\dots,(i)\}$.

In Lucca et al. [23] it was proved that C_T-integrals are pre-aggregation functions that are not, in general, aggregation functions. In fact, C_T-integrals are $(k,\dots,k)$-increasing, for any $k\ge1$, that is, C_T-integrals are weakly monotonic [28], which is a particular case of directional monotonicity.

III. Decision Making and GMC-RTOPSIS

The GMC-RTOPSIS [5] is a method introduced to allow multiple data types to be processed in the decision matrix. It uses the Choquet integral [7] to aggregate the criteria according to their interactions based on the fuzzy measure provided and/or simulated.

The decision matrix for the GMC-RTOPSIS method is:

$$DM^d=\begin{matrix} & C_1 & C_2 & \cdots & C_{n_d}\\ \begin{matrix}A_1\\A_2\\\vdots\\A_m\end{matrix} & \multicolumn{4}{c}{\begin{pmatrix} s^d_{11}(\boldsymbol{Y}^d) & s^d_{12}(\boldsymbol{Y}^d) & \cdots & s^d_{1n_d}(\boldsymbol{Y}^d)\\ s^d_{21}(\boldsymbol{Y}^d) & s^d_{22}(\boldsymbol{Y}^d) & \cdots & s^d_{2n_d}(\boldsymbol{Y}^d)\\ \vdots & \vdots & \ddots & \vdots\\ s^d_{m1}(\boldsymbol{Y}^d) & s^d_{m2}(\boldsymbol{Y}^d) & \cdots & s^d_{mn_d}(\boldsymbol{Y}^d)\end{pmatrix}}\end{matrix}$$

where:

1) Each decision maker $d\in\{1,\dots,k\}$, with $k\in\mathbb{N}$, is represented by one matrix;
2) The set $\boldsymbol{A}=\{A_1,\dots,A_m\}$ is the alternative set, with $m\in\mathbb{N}$, which is the same for all decision makers;
3) The set $\boldsymbol{C}_d=\{C_1,\dots,C_{n_d}\}$, with $n_d\in\mathbb{N}$, is the criteria set for the decision maker d. Additionally, the set $\boldsymbol{C}=\{\boldsymbol{C}_1,\dots,\boldsymbol{C}_k\}=\{C_1,\dots,C_n\}$, where $n=\sum_{d=1}^{k}n_d$, represents the criteria for all decision makers;

4) Each value $s_{ij}^d(\boldsymbol{Y}^d)$, with $1 \leq i \leq m$ and $1 \leq j \leq n_d$, is called the rating of the criterion j for alternative i;
5) The vector $\boldsymbol{Y} = (\boldsymbol{Y}_{rand}, \boldsymbol{Y}_{det})$ is a vector that represents random events and deterministic ones.

After all decision makers have provided their evaluations for a problem in question, then the decision algorithm can be used. The process follows the same structure of the original TOPSIS [1] with one difference: each criterion uses a different distance measure, since each one may contain a different data type. Therefore, each criterion distance is calculated separately and aggregated in the separation measure step using the Choquet integral.

The Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS) are, respectively, the one that is closer to the best possible solution and the one that is distant from the best possible solution.

The algorithm can be summarized as follows [5]:

1) Normalize all matrices;
2) Select the PIS, denoted by $s_j^+(\boldsymbol{Y})$, and the NIS, denoted by $s_j^-(\boldsymbol{Y})$, considering, for each $j \in \{1, \dots, n\}$, respectively:

$$s_j^+(\boldsymbol{Y}) = \begin{cases} \max\limits_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \min\limits_{1 \leq i \leq m}, s_{ij} & \text{if it is a cost/loss criterion,} \end{cases}$$

$$s_j^-(\boldsymbol{Y}) = \begin{cases} \min\limits_{1 \leq i \leq m} s_{ij}, & \text{if it is a benefit criterion,} \\ \max\limits_{1 \leq i \leq m} s_{ij}, & \text{if it is a cost/loss criterion;} \end{cases}$$

3) Calculate the distance measure for each criterion C_j, with $j \in \{1, \dots, n\}$, to the PIS and NIS solutions, that is,

$$d_{ij}^+ = d(s_j^+(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})),$$
$$d_{ij}^- = d(s_j^-(\boldsymbol{Y}), s_{ij}(\boldsymbol{Y})),$$

where $i \in \{1, \dots, m\}$.

4) Calculate the separation measure, for each $i \in \{1, \dots, m\}$, using the Choquet integral as follows:

$$S_i^+(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} \left(d_{i(j)}^+\right)^2 \left(\varphi_{\boldsymbol{Y}}(\boldsymbol{C}_{(j)}^+) - \varphi_{\boldsymbol{Y}}(\boldsymbol{C}_{(j+1)}^+)\right)}$$

$$S_i^-(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} \left(d_{i(j)}^-\right)^2 \left(\varphi_{\boldsymbol{Y}}(\boldsymbol{C}_{(j)}^-) - \varphi_{\boldsymbol{Y}}(\boldsymbol{C}_{(j+1)}^-)\right)}$$

where $d_{i(1)}^+ \leq \dots \leq d_{i(n)}^+$, $d_{i(1)}^- \leq \dots \leq d_{i(n)}^-$, for each $j \in \{1, \dots, n-1\}$, $C_{(j)}^+$ is the criterion correspondent to $d_{i(j)}^+$, $C_{(j)}^-$ is the criterion correspondent to $d_{i(j)}^-$, $\boldsymbol{C}_{(j)}^+ = \{C_{(j)}^+, C_{(j+1)}^+, \dots, C_{(n)}^+\}$, $\boldsymbol{C}_{(j)}^- = \{C_{(j)}^-, C_{(j+1)}^-, \dots, C_{(n)}^-\}$, $\boldsymbol{C}_{(n+1)}^+ = \emptyset$ and $\boldsymbol{C}_{(n+1)}^- = \emptyset$. Note that the separation measures are the square root of the Choquet integral of squared distances and thus is a d-Choquet integral – a Choquet integrals based on dissimilarities – introduced by Bustince et al. [29]. Observe also that the fuzzy measure may be dependent on $\boldsymbol{Y}_{det}$, that is, in each state one may consider a different fuzzy measure.

5) For each $i \in \{1, \dots, m\}$, calculate the relative closeness coefficient to the ideal solution with:

$$CC_i(\boldsymbol{Y}) = \frac{S_i^-(\boldsymbol{Y})}{S_i^-(\boldsymbol{Y}) + S_i^+(\boldsymbol{Y})};$$

6) Since the CC_i values may be a bootstrapped probability distribution we need to use a point cc_i that better describes this distribution. Thus, the rank of the alternatives is done by minimizing a pre-defined risk function:

$$\begin{aligned} \widehat{cc_i} &= \arg\min_c R(c) \\ &= \arg\min_c \int_{\mathbb{R}} L(c,\ CC_i(\boldsymbol{Y}))\ dF(CC_i(\boldsymbol{Y})); \quad (1) \end{aligned}$$

7) Obtain the final estimation $\widehat{cc_i} = f_{x \in \mathcal{X}}(\widehat{cc_i})$, where f is an aggregation function that assembles the $\widehat{cc_i}$ values from all states $x \in \mathcal{X}$.

IV. Generalizing the GMC-RTOPSIS by C_T-integrals

The GMC-RTOPSIS uses the Choquet integral in the separation measure step. This study generalizes this method for working with the generalization of the Choquet integral by t-norms, called C_T-integrals, which are pre-aggregation functions but not aggregation functions. In particular, here we use the t-norms presented in Table I.

Definition 4.1 (T-separation measure): Let $T : [0,1]^2 \to [0,1]$ be a t-norm. A T-separation measure is defined, for each $i \in \{1, \dots, m\}$, by the functions:

$$S_i^+(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} T\Big(\left(d_{i(j)}^+\right)^2, \varphi_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^+\right) - \varphi_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j+1)}^+\right)\Big)}$$

$$S_i^-(\boldsymbol{Y}) = \sqrt{\sum_{j=1}^{n} T\Big(\left(d_{i(j)}^-\right)^2, \varphi_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j)}^-\right) - \varphi_{\boldsymbol{Y}}\left(\boldsymbol{C}_{(j+1)}^-\right)\Big)}$$

where $d_{i(j)}^+$, $d_{i(j)}^-$, $\boldsymbol{C}_{(j)}^+$, $\boldsymbol{C}_{(j)}^-$ and $\varphi_{\boldsymbol{Y}}$ are defined as in item 4 above. Note that the separation measure is the squared root of the C_T-integral, which is a pre-aggregation function as shown in [23]. Additionally, since the square root function is strictly increasing on $[0,1]$, the separation measure is also a pre-aggregation function.

V. Application Example of GMC-RTOPSIS Generalized by C_T-integrals

For the sake of comparison, we use the case study 2 presented in [5].

A. Methodology

The simulations were performed with 10,000 samples from the DM matrix and applied a Particle Swarm Optimization (PSO) [8] algorithm to determine the best fuzzy measure, with 20 iterations and 30 particles. Since the GMC-RTOPSIS used the PSO algorithm and have shown good results, the method

TABLE II: Linguistic variable and their respective trapezoidal fuzzy numbers

Linguistic variables	Trapezoidal fuzzy numbers
Worst (W)	(0, 0, 0.2, 0.3)
Poor (P)	(0.2, 0.3, 0.4, 0.5)
Intermediate (I)	(0.4, 0.5, 0.6, 0.7)
Good (G)	(0.6, 0.7, 0.8, 1)
Excellent (E)	(0.8, 0.9, 1, 1)

presented here also use the same algorithm to learn the fuzzy measure.

Additionally, for ranking the alternatives the squared loss was used:

$$L(cc,\ CC_i) = (cc - CC_i)^2$$

in (1). This gives the mean as a point estimator for the ranking process.

Note that the algorithm complexity is the same as the original GMC-RTOPSIS method for each t-norm.

B. Example Description

Suppose a company is evaluating four suppliers for a determined provision, i.e., A_1, A_2, A_3 and A_4. To analyze those suppliers the company let three managers give their ratings based on their criteria for each of the A_i alternatives.

The budget manager considered the price per batch (in thousands), warranty (in days) and payment conditions (in days), represented as criterion $C_1^{(1)}$, $C_2^{(1)}$ and $C_3^{(1)}$, respectively. Additionally, suppose that the demand of the product is higher in December. To model this into the problem, let τ be a binary variable such that $\tau = 0$ when the month is between January and November and $\tau = 1$ when it is December. The weighting vector given by this manager was

$$\boldsymbol{w}^{(1)} = (0.5,\ 0.25,\ 0.25).$$

The production manager examined the price, delivery time (in hours), production capacity, product quality and the time to respond to a support request (in hours), representing criteria $C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}$ and $C_5^{(2)}$ respectively. Furthermore, this manager wants to account for the reliability of the production process and what a failure in the process can cause in the production capacity of the supplier. So, let P_i be a random variable such that $P_i = 0$ occurs when there are no failures in the production process of the supplier A_i, and $P_i = 1$ when there are failures. Also, when December arrives the production is much higher, so the probability of failure is higher also. Lastly, the higher demand in December and the issues in the production process affect the delivery time, so let it be modeled by the following ITFN:

$$\begin{aligned}
s_{13}^2 &= \big((0.8^{1+P_1}, 0.9^{1+P_1}, 1.0^{1+P_1}, 1.0^{1+P_1}),\ 1.0,\ 0.0\big)\\
s_{23}^2 &= \big((0.8^{1+4P_2}, 0.9^{1+4P_2}, 1.0^{1+4P_2}, 1.0^{1+4P_2}),\ 0.7,\ 0.1\big)\\
s_{33}^2 &= \big((0.6^{1+2P_3}, 0.7^{1+2P_3}, 0.8^{1+2P_3}, 1.0^{1+2P_3}),\ 0.8,\ 0.0\big)\\
s_{43}^2 &= \big((0.5^{1+3P_4}, 0.6^{1+3P_4}, 0.8^{1+3P_4}, 0.9^{1+3P_4}),\ 0.8,\ 0.1\big)
\end{aligned}$$

and consider

$$f_i(x, y) = x\big(1 + y(P_i + \tau)^2\big).$$

The weighting vector for the criteria used by this manager is

$$\boldsymbol{w}^{(2)} = (0.2,\ 0.2,\ 0.2,\ 0.2,\ 0.2).$$

The commercial manager considered the product lifespan (in years), social and environmental responsibility, the quantity of quality certifications and price, representing criteria $C_1^{(3)}$, $C_2^{(3)}$, $C_3^{(3)}$ and $C_4^{(3)}$ respectively. The weighting vector provided by this manager is

$$\boldsymbol{w}^{(3)} = (0.25,\ 0.12,\ 0.23,\ 0.4).$$

Considering all the DMs, we have the following underlying factors: a random component $\boldsymbol{Y}_{rand} = (P_1, P_2, P_3, P_4)$ and a deterministic component $Y_{det} = \tau$ that has two states: S_1 when $\tau = 0$ and S_2 when $\tau = 1$. The underlying factors can be represented by $\boldsymbol{Y} = (\boldsymbol{Y}_{rand},\ \boldsymbol{Y}_{det})$. Since the state S_2 was considered more important, it was given a higher weight in the decision: $w(S_1) = 0.4$ and $w(S_2) = 0.6$. Additionally, the company assigned the weighting vector $\boldsymbol{w} = (0.3,\ 0.4,\ 0.3)$ for the decision makers. The DM matrices for each manager are presented in Table III, where the linguistic variables (W, P, I, G and E) are defined as Table II.

The P_i distribution were determined by historical data of each supplier and are:

- For $\tau = 0$:

$$\begin{aligned}
p(P_1 = 0|S_1) &= 0.98,\\
p(P_2 = 0|S_1) &= 0.96,\\
p(P_3 = 0|S_1) &= 0.97,\\
p(P_4 = 0|S_1) &= 0.95.
\end{aligned}$$

- For $\tau = 1$:

$$\begin{aligned}
p(P_1 = 0|S_2) &= 0.96,\\
p(P_2 = 0|S_2) &= 0.92,\\
p(P_3 = 0|S_2) &= 0.96,\\
p(P_4 = 0|S_2) &= 0.90.
\end{aligned}$$

Furthermore, the company wanted to include some interaction between the criteria, so a variation of 30% was allowed for each fuzzy measure. This measure is calculated computationally by means of the PSO algorithm [5].

C. Results

The summary of the results using each one of the t-norms is listed in Table V for State 1 and Table VI for State 2 with the best result being in **boldface**. It can be seen that when using the algebraic product in the Choquet integral we have the following ranking

$$A_4^{Pr}(0.5508) > A_3^{Pr}(0.5475) > A_1^{Pr}(0.4469) > A_2^{Pr}(0.4323).$$

Also, when the minimum t-norm is applied the alternatives maintain the same rank as when using the algebraic product:

$$A_4^{\min}(0.5373) > A_3^{\min}(0.5087) > A_1^{\min}(0.4544) > A_2^{\min}(0.4302).$$

TABLE III: Decision matrices for the managers

(a) Budget manager

Alternatives	$C_1^{(1)}$	$C_2^{(1)}$	$C_3^{(1)}$	
			$\tau = 0$	$\tau = 1$
A_1	$260.00(1 + 0.15\tau)$	90	G	G
A_2	$250.00(1 + 0.25\tau)$	90	P	W
A_3	$350.00(1 + 0.20\tau)$	180	G	I
A_4	$550.00(1 + 0.10\tau)$	365	I	W

(b) Production manager

Alternatives	$C_1^{(2)}$	$C_2^{(2)}$	$C_3^{(2)}$	$C_4^{(2)}$	$C_5^{(2)}$
A_1	260.00	U($f_1(48,\ 0.10),\ f_1(96,\ 0.10)$)	s_{13}^2	I	[24, 48]
A_2	250.00	U($f_2(72,\ 0.20),\ f_2(120,\ 0.20)$)	s_{23}^2	P	[24, 48]
A_3	350.00	U($f_3(36,\ 0.15),\ f_3(72,\ 0.15)$)	s_{33}^2	G	[12, 36]
A_4	550.00	U($f_4(48,\ 0.25),\ f_4(96,\ 0.25)$)	s_{34}^2	E	[0, 24]

(c) Commercial manager

Alternatives	$C_1^{(3)}$	$C_2^{(3)}$	$C_3^{(3)}$	$C_4^{(3)}$
A_1	Exp(3.5)	W	1	260.00
A_2	Exp(3.0)	W	0	250.00
A_3	Exp(4.5)	P	3	350.00
A_4	Exp(5.0)	I	5	550.00

However, when the Łukasiewicz t-norm is substituted in the Choquet integral the top of the ranking switch positions:

$$A_3^{Ł}(0.6028) > A_4^{Ł}(0.5158) > A_1^{Ł}(0.4963) > A_2^{Ł}(0.4628).$$

The same ranking appears when using the nilpotent product t-norm, with

$$A_3^{N}(0.5843) > A_4^{N}(0.5252) > A_1^{N}(0.4798) > A_2^{N}(0.4591)$$

and the Hamacher product with

$$A_3^{H}(0.5411) > A_4^{H}(0.5386) > A_1^{H}(0.4567) > A_2^{H}(0.4345).$$

When analyzing a decision making problem, in order to determine which ranking result to adopt among the different results provided by the use of several t-norms, we propose to calculate the distance between the alternative ranked in the first place and the one ranked in second place. This may indicate a better discrimination between the alternatives, that is, a guarantee that there is no doubt about which is the best alternative to be chosen.

In the example presented before, the t-norm that best discriminates the first two ranked options is the Łukasiewicz t-norm with a distance of 0.0870. Additionally, the t-norm that least discriminates the two first ranked alternatives is the Hamacher product with a distance of 0.0025, that is, the two first ranked alternatives are too close. The distances calculated for each t-norm is shown in Table IV, where the best result is marked in **boldface** and the worst in <u>underline</u>.

TABLE IV: Distance between the first ranked and the second ranked alternative

t-norm	Distance
Algebraic product	0.0033
Minimum	0.0286
Łukasiewicz	**0.0870**
Nilpotent product	0.0591
Hamacher product	<u>0.0025</u>

VI. Conclusion

This paper introduced a generalization of GMC-RTOPSIS by using C_T-integrals, with an example application in decision making process.

The GMC-RTOPSIS method is a generalization of the classical TOPSIS, with capacity of dealing with multiple and heterogeneous data types and, also, interacting criteria using the discrete Choquet integral. For these reasons, the applications of the method are substantial.

Since the GMC-RTOPSIS uses the discrete Choquet integral, it can be broaden to a wide generalization by using C_T-integrals, which has been already applied and shown its efficiency on other problems. The process is done by substituting the original algebraic product on the Choquet integral by different t-norms.

Observe that this generalization provides more flexibility and confidence in the process, since, by varying the used t-norm, one obtains a solution set that confirm the ranking of alternatives. Moreover, one can choose the solution that presents the highest separation between the ranking values. This is the case, for example, of the Łukasiewicz t-norm

TABLE V: Mean and standard deviation values for state 1 (S_1, $\tau = 0$) using different t-norms

CC	**Algebraic product**		**Minimum**		**Łukasiewiecz product**		**Nilpotent product**		**Hamacher product**	
	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
A_1	0.4482	0.0050	0.4311	0.0118	0.4699	0.0312	0.4843	0.0240	0.4493	0.0128
A_2	0.4514	0.0044	0.4363	0.0106	0.4591	0.0140	0.4777	0.0114	0.4487	0.0093
A_3	**0.5584**	0.0212	0.5039	0.0331	**0.5741**	0.0224	**0.5820**	0.0341	0.5429	0.0300
A_4	0.5464	0.0040	**0.5584**	0.0114	0.5346	0.0165	0.5197	0.0099	**0.5462**	0.0099

TABLE VI: Mean and standard deviation values for state 2 (S_2, $\tau = 1$) using different t-norms

CC	**Algebraic product**		**Minimum**		**Łukasiewiecz product**		**Nilpotent product**		**Hamacher product**	
	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev	mean	std.dev
A_1	0.4460	0.0123	0.4699	0.0153	0.5139	0.0089	0.4768	0.0302	0.4617	0.0154
A_2	0.4195	0.0103	0.4261	0.0105	0.4653	0.0129	0.4467	0.0182	0.4251	0.0135
A_3	0.5402	0.0217	0.5119	0.0365	**0.6219**	0.0258	**0.5858**	0.0341	**0.5399**	0.0338
A_4	**0.5537**	0.0096	**0.5233**	0.0135	0.5032	0.0130	0.5289	0.0179	0.5336	0.0133

and the Nilpotent product, which presents the largest and second largest difference between the first and the second rank. Clearly, in this example, one should adopt the ranking result provided by the Łukasiewiecz t-norm.

It is important to note that, currently, does not exist a metric capable of guarantee that the presented method is the best in all possible problems. Although, with the use of multiple Choquet integrals one can use the t-norm which better discriminates the alternatives.

Further work will consider other kinds of generalizations of the Choquet integral, like CC-integrals [30], C_F-integrals [31], $C_{F_1F_2}$-integrals [32] and $gC_{F_1F_2}$-integrals [33]. In particular, a C_F-Integral generalizes the work in this paper, where instead of using t-norms we use a class of function F satisfying certain conditions. For more information on the generalizations of the Choquet integral and their applications, see [12], [34]–[36].

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