Automorphism groups of Lindenbaum algebras of some propositional many-valued logics with locally finite algebraic semantics

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***Abstract*—We characterize the structure of the automorphism groups of finitely generated free algebras in locally finite varieties constituting the algebraic semantics of well-known many-valued propositional logics, such as Gödel logic and the logic of Gödel hoops, Nilpotent Minimum logic, n-valued Łukasiewicz logic, Drastic product logic. We introduce the subalgebras of automorphism invariant elements of the free algebras, and study their structure in the case of Gödel algebras.**

***Index Terms*—automorphism group, Lindenbaum algebra, free algebra, algebraic semantics of many-valued logics.**

I. Introduction

In this paper we characterize the automorphism groups of finitely generated free algebras in varieties forming the algebraic semantics of some well-known many-valued propositional logics. Further, we introduce the subalgebras of automorphism invariant elements of finitely generated free Gödel algebras.

A variety, (or, equivalently, an equational class) $\mathbb{V}$ of algebras is *locally finite* iff its finitely generated free algebras are finite. In this paper we deal with propositional many-valued logics L having a locally finite variety $\mathbb{L}$ as equivalent algebraic semantics, which means that the Lindenbaum algebra of formulas of L is the free $\mathbb{L}$-algebra over ω generators. In the same way, for each natural $n \geq 0$, the Lindenbaum algebra of formulas of L built using only the first n propositional letters $x_1, x_2, \ldots, x_n$ is isomorphic with the free n-generated $\mathbb{L}$-algebra.

A *substitution* σ over $\{x_1, \ldots, x_n\}$ is displayed as

$$x_1 \mapsto \varphi_1, \ldots, x_n \mapsto \varphi_n$$

for $\varphi_1, \ldots, \varphi_n$ formulas built over $\{x_1, \ldots, x_n\}$, with the obvious meaning that $\sigma(x_i) = \varphi_i$. The substitution σ extends naturally to each formula over $\{x_1, \ldots, x_n\}$, via the following inductive definition:

$$\sigma(*(\psi_1, \ldots, \psi_k)) = *(\sigma(\psi_1), \ldots, \sigma(\psi_k))$$

for each k-ary connective $*$ and k-tuple of formulas $(\psi_1, \ldots, \psi_k)$. As it is clear that if $\varphi \equiv \psi$ then $\sigma(\varphi) \equiv \sigma(\psi)$, then the substitution σ can be identified with an endomorphism of the n-generated free algebra:

$$\sigma \colon \mathbf{F}_n(\mathbb{L}) \to \mathbf{F}_n(\mathbb{L})\,.$$

The set of all substitutions over $\{x_1, \ldots, x_n\}$, equipped with functional composition, forms the *monoid of endomorphisms* $\mathbf{End}(\mathbf{F}_n(\mathbb{L}))$ of $\mathbf{F}_n(\mathbb{L})$, having the identity $id\colon x_i \mapsto x_i$ as neutral element. The bijective endomorphisms in $\mathbf{End}(\mathbf{F}_n(\mathbb{L}))$ are clearly the same as isomorphisms of $\mathbf{F}_n(\mathbb{L})$ onto itself, and form the *group of automorphisms* $\mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$ of $\mathbf{F}_n(\mathbb{L})$. In terms of substitutions, $\mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$ is the group of invertible substitutions over $\{x_1, \ldots, x_n\}$, that is, those σ such that there exists a substitution σ^{-1} such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = id$.

Notice, that in Boolean propositional logic, if σ is an automorphism, then φ and $\sigma(\varphi)$ are satisfied by exactly the same number of truth-value assignments, and, on the other hand, if two formulas φ and ψ are satisfied by exactly the same number of truth-value assignments, then there is an automorphism σ such that $\psi \equiv \sigma(\varphi)$. The last connection is lost in the many-valued logics considered in this paper: automorphisms preserve more information than just the number of satisfying truth-value assignments, which pieces of additional information are preserved depending on the chosen logic.

In an earlier co-authored work [13] we have characterised the automorphism group of finite Gödel algebras — the algebraic semantics of propositional Gödel logic — by means of a dual categorical equivalence. In this paper we shall apply and generalise those techniques to a bunch of many-valued propositional logics and their algebraic semantics. We take here the opportunity to correct a nasty mistake in the introduction of [13]: for a quirk of carelessness, there we erroneously declared that automorphisms preserve logical equivalence, which is clearly not the case (this mistake does not invalidate any technical result in the paper). In that paper we had in mind the more algebraic notion of equivalence given in the following Proposition, enucleating the fact that automorphisms preserve all relevant algebraic information of logical equivalence between the formulas.

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Proposition 1: Let $\mathbb{L}$ be a locally finite variety constituting the algebraic semantics of a logic L. Then σ is an automorphism of $\mathbf{F}_n(\mathbb{L})$ if and only if for all pair of formulas φ, ψ:

$$\varphi \equiv \psi \qquad \text{if and only if} \qquad \sigma(\varphi) \equiv \sigma(\psi)\,.$$

(Or, equivalently for the logics considered, $\models \varphi \leftrightarrow \psi$ iff $\models \sigma(\varphi) \leftrightarrow \sigma(\psi)$.)

Proof: Clearly the property holds for automorphisms. Pick then $\sigma \in \mathbf{End}(\mathbf{F}_n(\mathbb{L})) \setminus \mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$. Since $\mathbb{L}$ is locally finite, we assume σ is not injective. Then, there are $\varphi \not\equiv \psi$ such that $\sigma(\varphi) \equiv \sigma(\psi)$. ■

Actually, in classical Boolean propositional logic, those formulas which are logically equivalent with their images under any automorphism σ:

$$\models \varphi \leftrightarrow \sigma(\varphi)\,,$$

are an interesting class, as they form the set which is the union of tautologies and contradictions, that is, exactly those formulas whose behaviour is independent from truth-value assignments. As we shall see, in the logics that we are going to study, the class of formulas which are logically equivalent to their images under all automorphisms form a more structured, and nuanced, subalgebra of the corresponding Lindenbaum algebra.

II. Preliminaries

A. $\mathbb{MTL}$-*algebras and the like*

A *t-norm* is an operator $*\colon [0,1]^2 \to [0,1]$ which is associative, commutative, monotonically non-decreasing in each argument, and having 0 and 1, as, respectively, absorbent and neutral elements. The t-norm $*$ is left-continuous (in the euclidean topology) if and only if it admits an associate residuum $\Rightarrow\colon [0,1]^2 \to [0,1]$, that is an operator satisfying:

$$x * z \leq y \qquad \text{if and only if} \qquad z \leq x \Rightarrow y\,. \tag{1}$$

Esteva and Godo's *monoidal t-norm based logic* MTL [19] is proved in [22] to be the logic of all left-continuous t-norms and their residua. MTL can be axiomatized with a Hilbert style calculus: for the reader's convenience, we list the axioms of MTL:

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$
$$(\varphi \& \psi) \to \varphi$$
$$(\varphi \& \psi) \to (\psi \& \varphi)$$
$$(\varphi \wedge \psi) \to \varphi$$
$$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$$
$$(\varphi \& (\varphi \to \psi)) \to (\psi \wedge \varphi)$$
$$(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$$
$$((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$$
$$((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$$
$$\bot \to \varphi$$

As inference rule we have modus ponens:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

Usually introduced derived connectives are $\neg\varphi := \varphi \to \bot$, $\top := \neg\varphi$, $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$ and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

Clearly, t-norms are not the only algebraic models of MTL: as a matter of fact, the algebraic semantics of MTL is the variety of $\mathbb{MTL}$-algebras.

A system $(A, *, \Rightarrow, \wedge, \vee, 0, 1)$ is in $\mathbb{MTL}$ if and only if:

- $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice with minimum 0 and maximum 1.
- $(A, *, 1)$ is a commutative monoid;
- $(*, \Rightarrow)$ forms a residuated pair: that is (1) holds for them;
- prelinearity holds: $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$.

A formula φ is a tautology of MTL ($\models \varphi$) if and only if the identity $\varphi^{\mathbf{A}} = \top^{\mathbf{A}}$ holds in any $\mathbb{MTL}$-algebra $\mathbf{A} = (A, \&^{\mathbf{A}}, \to^{\mathbf{A}}, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \bot^{\mathbf{A}}, \top^{\mathbf{A}})$. Equivalently, if φ is written over the variables $x_1, \ldots, x_n$, then $\models \varphi$ if and only if $\varphi \in \top/\equiv$ in the Lindenbaum $\mathbb{MTL}$-algebra of formulas over $x_1, \ldots, x_n$. Recall that the latter algebra is the algebra whose universe is the set of classes of logically equivalent formulas, where $\varphi \equiv \psi$ iff $\models \varphi \leftrightarrow \psi$, equipped with the operations given by the connectives. In equivalent terms, The Lindenbaum $\mathbb{MTL}$-algebra of formulas over $x_1, \ldots, x_n$ is the free $\mathbb{MTL}$-algebra over n free generators $\mathbb{F}_n(\mathbb{MTL})$.

A variety is *locally finite* iff its finitely generated free algebras are finite. In this paper we shall deal with some locally finite subvarieties of $\mathbb{MTL}$-algebras, as, in those cases, the automorphism groups of finitely generated free algebras are finite, too, and we can describe their structure via combinatorial means. The general problem of describing those automorphism groups can be really tough. See the paper [14] for such a result for a particular class of $\mathbb{MTL}$-algebras (actually, a class of $\mathbb{MV}$-algebras), which is not locally finite.

The usual signature for $\mathbb{MTL}$-algebras is $(A, *, \Rightarrow, \wedge, 0)$ as the other operations are definable from the selected ones.

The subvariety of *Gödel* algebras $\mathbb{G}$ is formed by idempotent $\mathbb{MTL}$-algebras, that is those satisfying the identity $x = x * x$. As in this case $* = \wedge$, we drop $*$ from the signature when dealing with $\mathbb{G}$-algebras.

Gödel hoops are the 0-free subreducts of Gödel algebras. They form a variety of algebras, denoted $\mathbb{GH}$, which clearly is not a subvariety of $\mathbb{MTL}$, but it is a subvariety of *prelinear semihoops* which in turn are the 0-free subreducts of $\mathbb{MTL}$-algebras. The signature used for Gödel hoops is $\wedge, \Rightarrow, 1$.

The subvariety of *Nilpotent Minimum algebras* ([8], [12]) $\mathbb{NM}$ is formed by the $\mathbb{MTL}$-algebras satisfying $(x \Rightarrow 0) \Rightarrow 0 = x$ and $((x * y) \Rightarrow 0) \vee ((x \wedge y) \Rightarrow (x * y)) = 1$.

Basic Logic algebras $\mathbb{BL}$ form the algebraic semantics of Hájek's Basic fuzzy logic [21], that is, the logic of all *continuous t* norms and their residua. They are obtained as those $\mathbb{MTL}$-algebras further satisfying $x * (x \Rightarrow y) = x \wedge y$. $\mathbb{MV}$-algebras form the algebraic semantics of Łukasiewicz logic [17], and are those $\mathbb{BL}$ algebras satisfying $(x \Rightarrow 0) \Rightarrow 0 = x$. $\mathbb{MV}$ and $\mathbb{BL}$ are not locally finite varieties (see resp. [17] and [4] for a description of free algebras in these varieties), but their n-contractive members —those satisfying $x^{n+1} = x^n$ — form locally finite subvarieties.

Finally, we shall consider also the subvariety of $\mathbb{MTL}$ given by *Drastic Product* algebras, $\mathbb{DP}$ [3]. whose members satisfy the identity $x \vee ((x * x) \Rightarrow 0) = 1$.

A *filter* F of an $\mathbb{MTL}$-algebra $\mathbf{A}$ is an upward closed subset A, that is closed under $*$, too. Filters are in bijection with congruences via $\Theta_F = \{(a,b) \mid a \Leftrightarrow b \in F\}$ and $F_\Theta = \{a \mid (a,1) \in \Theta\}$. A filter $\mathfrak{p}$ of $\mathbf{A}$ is *prime* iff it is proper ($\mathfrak{p} \neq A$), and $a \Rightarrow b \in \mathfrak{p}$ or $b \Rightarrow a \in \mathfrak{p}$ for all $a, b \in A$. The set of prime filters of $\mathbf{A}$ is called the *prime spectrum* of $\mathbf{A}$, written $Spec\,\mathbf{A}$.

B. Automorphism groups

With each algebraic structure $\mathbf{A}$ we can associate its *monoid of endomorphisms* $\mathbf{End}(\mathbf{A}) = (\{f \colon \mathbf{A} \to \mathbf{A}\}, \circ, id)$, having as universe the set of all homomorphisms of $\mathbf{A}$ into itself, where $\circ$ is functional composition, and $id \colon a \mapsto a$ for each $a \in A$ is the identity. The invertible elements of $\mathbf{End}(\mathbf{A})$, that is, those f such that there exists $f^{-1} \in \mathbf{End}(\mathbf{A})$ with the property $f \circ f^{-1} = id = f^{-1} \circ f$, constitute the universe of the *group of automorphisms* $\mathbf{Aut}(\mathbf{A})$ of $\mathbf{A}$.

Let $\mathbf{Sym}(n)$ denote the symmetric group over n elements, that is, the group of all permutations of an n-element set.

Let $\mathbb{B}$ denote the variety of Boolean algebras. We prove the well-known fact stated in Proposition 3 by means of a dual categorical equivalence, since we will constantly be using this approach throughout the paper. We start stating the following:

Fact 2: The category $\mathbb{B}_{fin}$ of finite Boolean algebras and their homomorphisms is dually equivalent to the category Set_{fin} of finite sets and functions between them.

Proof: This is just the restriction to finite objects of the well known Stone's duality between Boolean algebras and Stone spaces. ■

Let us call $\mathrm{Sub} \colon \mathsf{Set}_{fin} \to \mathbb{B}_{fin}$ and $\mathrm{Spec} \colon \mathbb{B}_{fin} \to \mathsf{Set}_{fin}$ the functors implementing the equivalence. It is folklore that $\mathrm{Sub}\, S$ is the Boolean algebra of the subsets of S, and $\mathrm{Spec}\,\mathbf{A}$ is the set of maximal filters of $\mathbf{A}$. On arrows, Sub and Spec are defined by taking preimages.

Clearly, for each Boolean algebra $\mathbf{A}$, $\mathbf{Aut}(\mathbf{A}) \cong \mathbf{Aut}(\mathrm{Spec}\,\mathbf{A})$.

Proposition 3: $\mathbf{Aut}(\mathbf{F}_n(\mathbf{B})) \cong \mathbf{Sym}(2^n)$.

Proof: Just recall that $\mathrm{Spec}\,\mathbf{F}_n(\mathbf{B})$ is the set of 2^n elements, and an automorphism of a finite set is just a permutation of its elements. ■

From now on, we shall write $\mathcal{P}(n)$ to denote $\mathrm{Spec}\,\mathbf{F}_n(\mathbf{B})$.

To deal with the structure of the automophism groups of Lindenbaum algebras of other logics we shall introduce some constructions from group theory. We refer to [23] for background.

Definition 4: Given two groups $\mathbf{H}$ and $\mathbf{K}$ and a group homomorphism $f \colon k \in \mathbf{K} \mapsto f_k \in \mathbf{Aut}(\mathbf{H})$, the *semidirect product* $\mathbf{H} \rtimes_f \mathbf{K}$ is the group obtained equipping $H \times K$ with the operation:

$$(h,k) * (h',k') = (hf_k(h'), kk')\,.$$

Theorem 5: Let $\mathbf{G}$ be a group with identity e and let $\mathbf{H}, \mathbf{K}$ be two subgroups of $\mathbf{G}$. If the following hold:

- $\mathbf{K} \lhd \mathbf{G}$ ($\mathbf{K}$ is a normal subgroup of $\mathbf{G}$);
- $G = H \times K$;
- $H \cap K = \{e\}$,

then $\mathbf{G}$ is isomorphic with the semidirect product of $\mathbf{H}$ and $\mathbf{K}$ with respect to the homomorphism $f : k \in \mathbf{K} \to f_k \in \mathbf{Aut}(\mathbf{H})$ where for each $h \in \mathbf{H}$, $f_k(h) = khk^{-1}$. Hence $|\mathbf{G}| = |\mathbf{H}| \cdot |\mathbf{K}|$.

In the following, we shall simply write $\mathbf{H} \rtimes \mathbf{K}$ instead of $\mathbf{H} \rtimes_f \mathbf{K}$, as in any usage we assume f is as in Theorem 5.

C. Dual categorical equivalences

As we have done in Proposition 3, to describe automorphism groups of Lindenbaum algebras we shall make use of dual categorical equivalences. We recall that two categories C and D are dually equivalent iff there exists a pair of contravariant functors $F \colon \mathsf{C} \to \mathsf{D}$ and $G \colon \mathsf{D} \to \mathsf{C}$ whose compositions FG and GF are naturally isomorphic with the identities in D and C.

In this paper we shall consider categories dual to the algebraic categories $\mathbb{L}_{fin}$, whose objects are the finite $\mathbb{L}$-algebras and whose arrows are the homomorphisms between them.

For sake of convention, we shall name $\mathrm{Sub} \colon \mathbb{L}^{op}_{fin} \to \mathbb{L}_{fin}$ and $\mathrm{Spec} \colon \mathbb{L}_{fin} \to \mathbb{L}^{op}_{fin}$ the pair of functors implementing the desired equivalence, as, in all our cases, Spec maps an algebra to its prime spectrum, suitably enriched, and Sub makes an $\mathbb{L}$-algebra out of subparts of the enriched prime spectrum.

Clearly, given an object $X \in \mathbb{L}^{op}_{fin}$ we can speak of its group of automorphisms $\mathbf{Aut}(X)$, and, by duality it obviously holds that for each algebra $\mathbf{A} \in \mathbb{L}$,

$$\mathbf{Aut}(\mathbf{A}) \cong \mathbf{Aut}(\mathrm{Spec}\,\mathbf{A})\,.$$

We shall use this fact without further notice. Further, Sub and Spec are defined contravariantly on the arrows of $\mathbb{L}$ and $\mathbb{L}^{op}$. Usually they are defined by taking preimages, or slight variants thereof. In this paper, however, we are only interested in the object part of the dualities. We shall detail the full definition of the functors Sub and Spec only for the cases of Gödel algebras and Gödel hoops (see next Section). In the other Sections we only refer to the existing literature, and we just use *black box* the fact that *there are* such pair of contravariant functors implementing the desired dual equivalence.

We are particularly interested in automorphism groups of free algebras. To this purpose we recall that, in any variety $\mathbb{L}$, the n-generated free algebra $\mathbf{F}_n(\mathbb{L})$ is the nth copower of $\mathbf{F}_1(\mathbb{L})$. By duality,

$$\mathrm{Spec}\,\mathbf{F}_n(\mathbb{L}) \cong (\mathrm{Spec}\,\mathbf{F}_1(\mathbb{L}))^n\,.$$

Whence, in the following Sections we shall focus on the structure of the objects dual to finitely generated free algebras in the varieties considered.

III. Automorphisms of Gödel algebras and of Gödel hoops

In [13] we described the automorphism groups of finite Gödel algebras, using the following dual categorical equivalence.

A *forest* $F = (F, \leq)$ is a poset such that the *downset* $\downarrow x = \{y \in F \mid y \leq x\}$ is totally ordered by $\leq$. A map $f\colon F \to G$ between finite forests is order-preserving if $x \leq_F y$ implies $f(x) \leq_G f(y)$ and it is open if $y \leq_G f(x)$ implies that there is $z \in F$, with $z \leq_F x$, such that $f(z) = y$. Open maps carry downward closed sets to downward closed sets. Let F_{fin} denote the category of *finite forests* and *order-preserving, open* maps between them.

Proposition 6: The category $\mathbb{G}_{fin}$ of finite Gödel algebras and homomorphisms between them is dually equivalent to F_{fin}.

Proof: The functors implementing the dual equivalence act on objects as follows. $\operatorname{Spec}\mathbf{A} = (Spec\,\mathbf{A}, \supseteq)$ (the prime spectrum of $\mathbf{A}$, equipped with reverse inclusion), and $\operatorname{Sub} F = (\{G \subseteq F \mid G = \downarrow G\}, \cap, \Rightarrow, \emptyset)$, where $X \Rightarrow Y = F \setminus \uparrow (X \setminus Y)$, for all downward closed subsets X, Y of F. On arrows, $\operatorname{Spec} h\colon \operatorname{Spec}\mathbf{B} \to \operatorname{Spec}\mathbf{A}$ is given by $(\operatorname{Spec} h)(\mathfrak{p}) = h^{-1}[\mathfrak{p}]$ for all $\mathfrak{p} \in \operatorname{Spec}\mathbf{B}$, and analogously $\operatorname{Sub} f\colon \operatorname{Sub} G \to \operatorname{Sub} F$ is given by $(\operatorname{Sub} f)(X) = f^{-1}[X]$ for all $X \in \operatorname{Sub} G$. For details see, for instance [10]. ■

An endomorphism $f\colon F \to F$ is called an *order preserving permutation* iff f is bijective and $x \leq y$ implies $f(x) \leq f(y)$.

Lemma 7: For each algebra $\mathbf{A} \in \mathbb{G}_{fin}$, $\mathbf{Aut}(A)$ is the group of order preserving permutations of $\operatorname{Spec}\mathbf{A}$.

Proof: See [13]. ■

To elucidate the structure of $\mathbf{Aut}(\mathbf{F}_n(\mathbb{G}))$ we have to recall the structure of $\operatorname{Spec}\mathbf{F}_n(\mathbb{G})$, and how to determine order preserving permutations over it. We start recalling some properties of F_{fin} that will be useful throughout the paper.

A *tree* T is a forest with minimum, called *root* of T. Given any forest F, we write $F_\perp$ for the tree obtained appending to F a fresh root. Clearly, each tree T can be thought of as $F_\perp$ for a uniquely determined forest $F = T \setminus \{\min T\}$. We denote T_{fin} the full subcategory of F_{fin} whose objects are trees.

Lemma 8:

1) The singleton, denoted $\mathbf{1}$, is the terminal object in F_{fin}; the empty forest $\emptyset$ is the initial one. In T_{fin}, $\mathbf{1}$ is both initial and terminal.
2) The coproduct $F + G$ of two forests $F, G \in \mathsf{F}_{fin}$ is the disjoint union of F and G. The coproduct of two trees $F_\perp, G_\perp \in \mathsf{T}_{fin}$ is given by $(F +_{\mathsf{F}_{fin}} G)_\perp$.
3) In F_{fin}, product distributes over coproduct: $F \times (G + H) \cong (F \times G) + (F \times H)$.
4) In both F_{fin} and T_{fin}, $F_\perp \times G_\perp \cong ((F_\perp \times G) + (F \times G) + (F \times G_\perp))_\perp$, where products and coproducts in the right hand side are computed in F_{fin}.

Proof: See [9]. ■

Lemma 8 allows to compute recursively the object in the product of any two finite forests, and of any two finite trees. See [9] for the description of the associated projection maps.

Proposition 9: The category $\mathbb{GH}_{fin}$ of finite Gödel hoops and homomorphisms between them is dually equivalent to T_{fin}.

Proof: The functors implementing the dual equivalence act on objects as follows. $\operatorname{Spec}\mathbf{A} = (Spec\,\mathbf{A} \cup \{A\}, \supseteq)$ and $\operatorname{Sub} F = (\{\emptyset \neq G \subseteq F \mid G = \downarrow G\}, \cap, \Rightarrow, F)$, where $X \Rightarrow Y = F \setminus \uparrow (X \setminus Y)$, for all non-empty downward closed subsets X, Y of F. On arrows, Spec and Sub are defined by taking preimages, as in Proposition 6. For details, see, for instance [7]. ■

The following results in this section are taken from [13] for what regards $\mathbb{G}$. They are here straightforwardly adapted also to the case $\mathbb{GH}$.

Lemma 10: Let F and G be forests. Then the following hold.

1) $\mathbf{Aut}(F_\perp) \cong \mathbf{Aut}(F)$.
2) If $F_\perp$ and $G_\perp$ are non-isomorphic trees then $\mathbf{Aut}(F_\perp + G_\perp) \cong \mathbf{Aut}(F) \times \mathbf{Aut}(G)$.
3) $\mathbf{Aut}(n(F_\perp)) \cong \mathbf{Sym}(n) \rtimes (\mathbf{Aut}(F))^n$.

We now define by recurrence a family of finite forests. We set

$$H_0 = \emptyset, \qquad H_n = \sum_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp .$$

We further define

$$G_n = H_n + (H_n)_\perp .$$

We observe that $\operatorname{Spec}\mathbf{F}_1(\mathbb{G}) \cong G_1$ and $\operatorname{Spec}\mathbf{F}_1(\mathbb{GH}) \cong (H_1)_\perp$.

Theorem 11: For any integer $n \geq 0$,

$$\operatorname{Spec}\mathbf{F}_n(\mathbb{GH}) \cong (H_n)_\perp \qquad \text{and} \qquad \operatorname{Spec}\mathbf{F}_n(\mathbb{G}) \cong G_n .$$

Further,

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{GH})) \cong \mathbf{Aut}(H_n) \cong$$

$$\cong \prod_{i=1}^{n-1} \mathbf{Sym}\left(\binom{n}{i}\right) \rtimes (\mathbf{Aut}(H_i))^{\binom{n}{i}} .$$

and

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{G})) \cong (\mathbf{Aut}(\mathbf{F}_n(\mathbb{GH})))^2 .$$

Given integers $a_1, a_2, \ldots, a_m \geq 0$, their *multinomial coefficient*, written

$$\binom{a_1 + a_2 + \cdots + a_m}{a_1, a_2, \ldots, a_m} ,$$

is the quantity

$$\frac{(a_1 + a_2 + \cdots + a_m)!}{a_1! a_2! \cdots a_m!} .$$

For any $i \leq n$, let b_i^n be the set of all multinomial coefficients of the form

$$\binom{n}{i_1, i_2, \ldots, i_m} ,$$

for $i_1 = i$ and $i_h > 0$ for all $h \in \{1, 2, \ldots, m\}$.

Theorem 12: For every integer $n \geq 0$,

$$|\mathbf{Aut}(H_n)| = \prod_{i=1}^{n-1} \binom{n}{i}! \, |\mathbf{Aut}(H_n)|^{\binom{n}{i}} ,$$

and,

$$|\mathbf{Aut}(\mathbf{F}_n(\mathbb{GH}))| = \prod_{i=1}^{n-1} \left(\binom{n}{i}! \prod_{h \in b_i^n} \left(\prod_{j=1}^{i-1} \binom{i}{j}! \right)^h \right) .$$

IV. AUTOMORPHISMS OF NILPOTENT MINIMUM ALGEBRAS

A *finite labeled tree* is a pair (T, j) where $T \in \mathsf{T}_{fin}$ and $j \in \{0,1\}$. Let $\{(T_i, j_i) \mid i = 1, \ldots, k\}$ be a set of finite labeled trees. Then its associated *finite labeled forest* is the pair (F, b) where $F = \sum_{i=1}^{k} T_i$ and $b\colon \{T_1, \ldots, T_k\} \to \{0,1\}$ maps T_i to j_i. A morphism of finite labeled trees $f\colon (T, i) \to (S, j)$ is a map $f\colon T \to S$ in T_{fin}, provided that $i \le j$. A morphism of finite labeled forests $g\colon (F, b) \to (G, c)$, where $F = \sum_{i=1}^{k} T_i$ and $G = \sum_{i=1}^{h} U_i$, is a map $g\colon F \to G$ in F_{fin}, provided that $b(T_i) \le c(U_j)$, where (i, j) is the pair of indices determined by the fact that g maps T_i into U_j. Let LF_{fin} be the category of finite labeled forests with the described morphisms.

In [8] we prove the following.

Theorem 13: $\mathbb{NM}_{fin}$ is dually equivalent to the category LF_{fin}.

Clearly, automorphisms of finite labeled forests must preserve the labels: given such a forest (F, b) with $F = \sum_{i=1}^{k} T_i$, then $f\colon (F, b) \to (F, b)$ is in $\mathbf{Aut}((F, b))$ if and only if $f \in \mathbf{Aut}(F)$ and $b(T_i) = b(T_j)$ for (i, j) be determined by the fact that f maps T_i onto T_j.

Now, $\operatorname{Spec} \mathbf{F}_1(\mathbb{NM})$ is the labeled forest $(2(H_1)_\perp + H_1, b)$, with $b(H_1) = 0$ and $b((H_1)_\perp) = 1$ for both copies of $(H_1)_\perp$.

By duality, for any integer $n > 0$, the dual of the n-generated free $\mathbb{NM}$-algebra $\operatorname{Spec} \mathbf{F}_n(\mathbb{NM})$ is the nth power of $\operatorname{Spec} \mathbf{F}_1(\mathbb{NM})$. That is:

Lemma 14:

$$\operatorname{Spec} \mathbf{F}_n(\mathbb{NM}) \cong \left(\sum_{i=0}^{n} 2^i \binom{n}{i} (H_i)_\perp, b_n \right),$$

where $b_n((H_n)_\perp) = 1$, for all 2^n copies of $(H_n)_\perp$, while $b_n((H_i)_\perp) = 0$ for all 2^i copies of $(H_i)_\perp$ for each $0 \le i < n$.

Proof: In LF_{fin}, $(F, b) \times (G, c) \cong (F \times_{\mathsf{F}_{fin}} G, b \cdot c)$, where $b \cdot c$ is pointwise bit multiplication, that is $(b \cdot c)(T_i \times U_j) = b(T_i) \cdot c(U_j)$. ■

Lemma 14 shows that in considering automorphisms of free $\mathbb{NM}$-algebras the condition on the labels is automatically granted, as $H_i \ncong H_j$ if $i \ne j$. Whence:

Theorem 15: For any integer $n \ge 0$,

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{NM})) \cong \prod_{i=0}^{n} \mathbf{Sym}\left(2^i \binom{n}{i}\right) \rtimes (\mathbf{Aut}(H_i))^{2^i \binom{n}{i}}.$$

and $|\mathbf{Aut}(\mathbf{F}_n(\mathbb{NM}))| =$

$$= \prod_{i=0}^{n} \left(2^i \binom{n}{i}\right)! \left(\prod_{h \in b_i^n} \left(\prod_{j=1}^{i-1} \binom{i}{j}! \right)^{h 2^i} \right).$$

As we have seen the categorical dualities that allow us to deal with Gödel hoops and with Nilpotent Minimum algebras are minor variants of F_{fin}. There are several other varieties either in $\mathbb{MTL}$, or in other ways related to many-valued logics, which offer dual categorical equivalences that can be considered minor variants of F_{fin} as well. In particular there are varieties that are dually equivalent to F_{fin} itself, the only difference at dual level being the structure of the dual of the free 1-generated algebra in each of these varieties. See [11] for a thorough investigation of this topic. In the paper [7], we provide a dual categorical equivalence for the variety of Gödel Nelson paraconsistent residuated lattices whose objects are pairs made of a tree and one of its subtrees. In [16] the authors provide a duality for $\mathbb{RDP}$-algebras, and in [15] the locally finite variety of $\mathbb{EMTL}$-algebras is introduced. In these cases we are in a position to describe the group of automorphisms of free finitely generated algebras studying the automorphisms of the corresponding dual objects. The case of the large locally finite variety of $\mathbb{WNM}$-algebras may be harder to take, due to the sheer complexity of the structure of its free algebras [6].

V. AUTOMORPHISMS OF DRASTIC PRODUCT ALGEBRAS

In [1] e [2] the authors prove categorical dualities for the class of finite Gödel$_\Delta$ algebras and of finite Drastic Product algebras.

Let MC_{fin} be the category whose objects are finite multisets of nonempty finite chains, and whose arrows satisfy the following constraint: Let $C = \{C_1, \ldots, C_m\}, D = \{D_1, \ldots, D_n\} \in \mathsf{MC}_{fin}$. Then $h\colon C \to D$ is given as $h = \{h_i \mid i = 1, \ldots, m\}$, where each h_i is an order preserving surjection $h_i\colon C_i \twoheadrightarrow D_j$ for some $j = 1, \ldots, n$.

Let further $\mathsf{MC}_{fin}^{\top}$ be the (non-full) subcategory of C_{fin} which has the same objects of C_{fin}, but whose arrows $h\colon C \to D$ satisfy additionally: for each $i = 1, \ldots, m$ if the codomain D_j of h_i is not isomorphic with $\mathbf{1}$ then $h_i^{-1}(\max D_j) = \{\max C_i\}$.

Theorem 16: $\mathbb{DP}_{fin}$ is dually equivalent to $\mathsf{MC}_{fin}^{\top}$.

It is clear that for any multiset $C = \{C_1, \ldots, C_m\} \in \mathsf{MC}$ or $\in \mathsf{MC}^{\top}$, the morphism $f = \{f_i \mid i = 1, \ldots, m\}\colon C \to C$ is an automorphism of C if and only if for all $i = 1, \ldots, m$, $f_i\colon C_i \twoheadrightarrow C_j$ is such that $|C_j| = |C_i|$ and if $i_1 \ne i_2$ then the codomains of f_{i_1} and f_{i_2} are distinct chains of C. Let us express $C = \{C_1, \ldots, C_m\} = \bigcup_{h \in \omega} C^{(h)}$, where $C^{(h)}$ is the sub-multiset of C formed by the chains with h elements. Then, if k is the maximum cardinality of chains in C:

Lemma 17:

$$\mathbf{Aut}(C) \cong \prod_{i=1}^{k} \mathbf{Sym}(|C^{(i)}|),$$

and

$$|\mathbf{Aut}(C)| = \prod_{i=1}^{k} |C^{(i)}|!.$$

For each integer $n > 0$, let $\mathbf{n}$ denote the chain of n elements.

Proposition 18: In $\mathsf{MC}^{\top}$ the coproduct $C + D$ of two finite multisets of chains is the disjoint union of C and D. The terminal object is the singleton multiset with the one-element chain $\{\mathbf{1}\}$. Furthermore, products distribute over coproducts: $C \times (D + E) \cong (C \times D) + (C \times E)$. Moreover, let $C^{\top}$ be the chain obtained adding a fresh top element to C. Then, for all $i > 0$:

- $\{\mathbf{i}\} \times \{\mathbf{1}\} \cong \{\mathbf{i}\}$ and $\{\mathbf{i}\} \times \{\mathbf{2}\} \cong \{\mathbf{i+1}\}$.

- $\{\mathbf{i+2}\} \times \{\mathbf{j+2}\} \cong (\{\mathbf{i+2}\} \times \{\mathbf{j+1}\}) + (\{\mathbf{i+1}\} \times \{\mathbf{j+1}\}) + (\{\mathbf{i+1}\} \times \{\mathbf{j+2}\}))^{\top}$.

The dual of the free 1-generated $\mathbb{DP}$-algebra is

$$\mathrm{Spec}\,\mathbf{F}_1(\mathbb{DP}) \cong \{\mathbf{1},\mathbf{3},\mathbf{2},\mathbf{1}\}\,.$$

Then (see [3], [2]):

Theorem 19: For each integer $n \geq 0$: $\mathrm{Spec}\,\mathbf{F}_n(\mathbb{DP}) \cong$

$$2^n\{\mathbf{1}\} + (3^n - 2^n)\{\mathbf{2}\}+ + \sum_{h=3}^{n+2}\left(\sum_{i=0}^{h-2}(-1)^i\binom{h-2}{i}(h+i-1)^n\right)\{\mathbf{h}\}\,.$$

Whence, by Lemma 17,

Theorem 20: For each integer $n \geq 0$:

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{DP})) \cong \mathbf{Sym}(2^n) \times \mathbf{Sym}(3^n - 2^n)\times \times \prod_{h=3}^{n+2}\mathbf{Sym}\left(\sum_{i=0}^{h-2}(-1)^i\binom{h-2}{i}(h+i-1)^n\right)\,,$$

and $|\mathbf{Aut}(\mathbf{F}_n(\mathbb{DP}))| =$

$$(2^n)!\,(3^n-2^n)!\prod_{h=3}^{n+2}\left(\sum_{i=0}^{h-2}(-1)^i\binom{h-2}{i}(h+i-1)^n\right)!\,.$$

In mathematical fuzzy logic is customary to consider the extension/expansion of algebraic models by adding the Δ operator, which can be axiomatised (see [1], [2]) in such a way that on totally ordered algebras its behaviour is to *crispify* values, that is $\Delta 1 = 1$, $\Delta a = 0$ for all $a \neq 1$. Adding Δ to Gödel algebras provides us with the locally finite variety of $\mathbb{G}_\Delta$ algebras, which is closely related to $\mathbb{DP}$. As a matter of fact $(\mathbb{G}_\Delta)_{fin}$ is dually equivalent to MC_{fin}, and $\mathbb{DP}$ is equivalent with a non-full subcategory of $(\mathbb{G}_\Delta)_{fin}$. We are in a position to adapt the content of this Section to $\mathbb{G}_\Delta$ algebras and derive the structure and cardinality of the group of automorphisms of free finitely generated $\mathbb{G}_\Delta$-algebras. We shall elaborate on this topic elsewhere.

VI. Automorphisms of n-valued $\mathbb{MV}$-algebras

The variety $\mathbb{MV}$ of MV-algebras constitutes the algebraic semantics of propositional Łukasiewicz logic [17]. $\mathbb{MV}$ is not locally finite, but the k-contractive $\mathbb{MV}$-algebras form a locally finite subvariety of $\mathbb{MV}$. Here we consider the subvariety $\mathbb{MV}_k$ of k-valued $\mathbb{MV}$-algebras, which constitutes the algebraic semantics of k-valued Łukasiewicz logic. $\mathbb{MV}_k$ is axiomatised by imposing k-contractivity: $x^k = x^{k+1}$, and Grigolia's axioms [20] $k(x^h) = (h(x^{h-1}))^k$ for every integer $2 \leq h \leq k-2$ that does not divide $k-1$.

For any integer $d > 1$ let $\mathrm{Div}(d)$ be the set of coatoms in the lattice of divisors of d, and for any finite set of natural numbers X, let $\gcd(X)$ be the greatest common divisor of the numbers in X. Then let $\alpha(0,1) = 1$, $\alpha(0,d) = 0$ for all $d > 1$, and for all $n \geq 1$,

$$\alpha(n,d) = (d+1)^n + \sum_{\emptyset \neq X \subseteq \mathrm{Div}(d)} (-1)^{|X|}(\gcd(X)+1)^n\,.$$

Then $\alpha(n,d)$ counts the number of points in $[0,1]^n$ whose denominator is d. It is known that

$$\mathbf{F}_n(\mathbb{MV}_k) \cong \prod_{d|(k-1)} \text{Ł}_{d+1}^{\alpha(n,d)}\,,$$

where Ł_m is the $\mathbb{MV}$-chain of cardinality m.

Let $\mathsf{MN_k}_{fin}$ be the category whose objects are finite multisets of natural numbers dividing $k-1$ and whose arrows $f\colon M \to N$ are functions from M to N such that $f(x)$ divides x for any $x \in M$.

Then $\mathsf{MN_k}_{fin}$ is dually equivalent to $\mathbb{MV}_k$. In particular,

$$\mathrm{Spec}\,\mathbf{F}_n(\mathbb{MV}_k) \cong \bigcup_{d|(k-1)} \biguplus_{i=1}^{\alpha(n,d)} \{d\}\,,$$

where $\biguplus_{i=1}^{m}\{t\}$ denotes the multiset formed by m copies of t.

It is clear that an automorphism $f\colon M \to M$ in $\mathsf{MN_k}_{fin}$ must be a bijection such that each copy of $x \in M$ is mapped to a copy of $x \in M$. Then

Theorem 21:

$$\mathbf{Aut}(\mathbf{F}_n(\mathbb{MV}_k)) \cong \prod_{d|(k-1)} \mathbf{Sym}(\alpha(n,d))\,,$$

and

$$|\mathbf{Aut}(\mathbf{F}_n(\mathbb{MV}_k))| = \prod_{d|(k-1)} (\alpha(n,d))!\,.$$

In [5] the authors introduce a category dually equivalent to finite Grigolia $\mathbb{BL}$-algebras, denoted here $(\mathbb{BL}_k)_{fin}$. These are $\mathbb{BL}$-algebras further satisfying k-contractivity and the Grigolia's axioms. Actually, the chains in $(\mathbb{BL}_k)_{fin}$ are ordinal sums of a finite number of copies of chains in $(\mathbb{MV}_k)_{fin}$. The category dually equivalent to $(\mathbb{BL}_k)_{fin}$ is a full subcategory of the category of finite *weighted forests*, whose objects are finite forests such that each node is labeled with a positive natural dividing $k-1$, and whose morphisms are morphisms f of the underlying forests that *respect weights*, meaning that for each x in the domain there is $y \leq x$ such that $f(x) = f(y)$ and the weight of $f(y)$ divides the weight of y.

In [5] the structure of the dual objects to finitely generated free algebras in $(\mathbb{BL}_k)_{fin}$ is given through some recurrences. It is possible to apply our approach to these algebras in order to determine the structure and cardinality of their automorphism groups. We shall pursue this task in another paper.

VII. The subalgebra of automorphism invariant elements

Let φ be a formula over $\{x_1,\dots,x_n\}$. We say that the class of formulas $\varphi/\equiv$ logically equivalent with φ in the logic L is *automorphism invariant* if and only if, in L, $\models \varphi \leftrightarrow \sigma(\varphi)$, for each automorphism $\sigma \in \mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))$. Equivalently, $\{\sigma(\varphi) \mid \sigma \in \mathbf{Aut}(\mathbf{F}_n(\mathbb{L}))\} \subseteq \varphi/\equiv$, and then $(\varphi/\equiv) = (\sigma(\varphi)/\equiv)$.

With our standing assumption about the dual equivalences implemented by functors Spec and Sub:

Lemma 22: $\models \varphi \leftrightarrow \sigma(\varphi)$ if and only if $\mathrm{Spec}\,\varphi = \mathrm{Spec}\,\sigma(\varphi)$.

Proof: It follows at once from $(\varphi/\equiv) = (\sigma(\varphi)/\equiv)$. ■

Proposition 23: For each $n \geq 0$, the automorphism invariant elements of $\mathbf{F}_n(\mathbb{B})$ are exactly $\bot/\equiv$ and $\top/\equiv$.

Proof: Trivially, $\bot/\equiv$ and $\top/\equiv$ are automorphism invariant as $\operatorname{Spec}\bot = \emptyset$ and $\operatorname{Spec}\top = \mathcal{P}(n)$. Let φ be a formula over $\{x_1, \ldots, x_n\}$ not equivalent to $\bot$ or to $\top$. Then $\emptyset \subsetneq \operatorname{Spec}\varphi \subsetneq \mathcal{P}(n)$. Pick $p, q \in \mathcal{P}(n)$ such that $p \in \operatorname{Spec}\varphi$ while $q \notin \operatorname{Spec}\varphi$. Let $f\colon \mathcal{P}(n) \to \mathcal{P}(n)$ be any permutation of the points in $\mathcal{P}(n)$ exchanging p with q. Such a permutation trivially exists, and it is such that $\operatorname{Sub} f\colon \mathbf{F}_n(\mathbb{B}) \to \mathbf{F}_n(\mathbb{B})$ is an automorphism. To end the proof notice that $\operatorname{Spec}\varphi \neq \operatorname{Spec}((\operatorname{Sub} f)(\varphi))$, since $p \in \operatorname{Spec}\varphi$ and $p \notin \operatorname{Spec}((\operatorname{Sub} f)(\varphi))$. By Lemma 22, $\not\models \varphi \leftrightarrow (\operatorname{Sub} f)(\varphi)$. ■

Whence, the set of automorphism invariant elements of $\mathbf{F}_n(\mathbb{B})$ is the universe of the two-element subalgebra of $\mathbf{F}_n(\mathbb{B})$. Notice that those elements are characterised as the only elements which are independent from truth-value assignments, that is, their value under *some fixed* truth-value assignment coincides with the value under *any* truth-value assignment.

When we move from classical Boolean propositional logic to the many-valued logics considered in this paper, the situation gets more interesting. We shall deal with the case of Gödel propositional logic, and we shall see that the set of automorphism invariant elements of $\mathbf{F}_n(\mathbb{G})$ is again the universe of a subalgebra $\mathbf{AutInv}_n(\mathbb{G})$ of $\mathbf{F}_n(\mathbb{G})$. But $\mathbf{AutInv}_n(\mathbb{G})$ has a far more complex structure than the two-element Boolean algebra, and its elements are characterised in a more refined way.

Given an element x of a forest F, its *height* $H(x)$ is the length of $\downarrow x$. A subforest $F \in \operatorname{Spec}\mathbf{F}_n(\mathbb{G})$ is *symmetric* iff for all $x, y \in F$ with $H(x) = H(y)$ it holds that if $\uparrow x \cong \uparrow y$ as subposets of $\operatorname{Spec}\mathbf{F}_n(\mathbb{G})$, then $F \cap \uparrow x \cong F \cap \uparrow y$.

Lemma 24: $\varphi/\equiv$ is automorphism invariant if and only if $\operatorname{Spec}\varphi$ is symmetric.

Proof: Assume φ, over the variables $x_1, \ldots, x_n$, is not symmetric. Let $F = \operatorname{Spec}\varphi$. Then there are $x, y \in F$ with $H(x) = H(y)$ and $\uparrow x \cong \uparrow y$ such that $F \cap \uparrow x \ncong F \cap \uparrow y$. Clearly $\uparrow x$ can be mapped bijectively to $\uparrow y$ by an order-preserving permutation f of $\operatorname{Spec}\mathbf{F}(\mathbb{G})$. But obviously f does not map $F \cap \uparrow x$ bijectively onto $F \cap \uparrow y$. Whence $F \neq f(F)$ and φ is not automorphism invariant.

For the other way round, if φ is not automorphism invariant then there is an order-preserving permutation f of $\operatorname{Spec}\mathbf{F}(\mathbb{G})$ such that $F \neq f(F)$. Whence there must exist $x_0 \in \operatorname{Spec}\mathbf{F}(\mathbb{G})$ such that $x_0 \in F$ but $f(x_0) \notin F$. Clearly, if $f(x) = y$ then $H(x) = H(y)$ and $\uparrow x \cong \uparrow y$ and $f(\uparrow x) \cong \uparrow y$. But, $F \cap \uparrow x_0 \ncong F \cap \uparrow f(x_0)$. Whence, φ is not symmetric. ■

Let $AutInv_n$ be the set of all automorphism invariant classes of formulas over $\{x_1, \ldots, x_n\}$.

Lemma 25: $AutInv_n$ is a subuniverse of $\operatorname{Sub} G_n$.

Proof: Just notice that for any finite forest F, the operations of $\operatorname{Sub} F$ preserve symmetric elements. ■

Let $\mathbf{AutInv}_n(\mathbb{G})$ be the subalgebra of $\operatorname{Sub} G_m$ having $AutInv_n$ as universe.

An *ordered partition* of $x_1, \ldots, x_n$ is a partition $B_1, \ldots, B_m$ of $\{0, x_1, \ldots, x_n, 1\}$ with $m > 1$, equipped with the total order $B_i \preceq B_j$ iff $i \leq j$, such that $0 \in B_1$ and $1 \in B_m$. The set Ω_n of all ordered partitions of $x_1, \ldots, x_n$ is made into a poset stipulating that, for each pair $\pi_1, \pi_2 \in \Omega_n$, $\pi_1 \sqsubseteq \pi_2$ iff $\pi_1 = B_{1,1} \preceq \cdots \preceq B_{1,u}$, $\pi_2 = B_{2,1} \preceq \cdots \preceq B_{2,v}$, with $B_{1,j} = B_{2,j}$ for all $j < u$ and $B_{1,u} = \bigcup_{j=u}^{v} B_{2,j}$.

Lemma 26: The poset $(\Omega_n, \sqsubseteq)$ is isomorphic with G_n.

Proof: See, for instance [18]. ■

Notice that any ordered partition ρ of $x_1, \ldots, x_n$ can be displayed as

$$0 \trianglelefteq_0^\rho x_{\tau^\rho(1)} \trianglelefteq_1^\rho x_{\tau^\rho(2)} \trianglelefteq_2^\rho \cdots \trianglelefteq_{n-1}^\rho x_{\tau^\rho(n)} \trianglelefteq_n^\rho 1\,,$$

where $\tau^\rho\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation and $\trianglelefteq_i^\rho \in \{=, <\}$ for all $i = 0, 1, \ldots, n$. Then two ordered partitions ρ and ϱ are such that $\rho \sqsubseteq \varrho$ iff there is i such that $\trianglelefteq_j^\rho = \trianglelefteq_j^\varrho$ for all $0 \leq j \leq i$, while $\trianglelefteq_j^\rho$ is $=$ for all $i < j \leq n$. Whence, given an ordered partition ρ, letting i_ρ denote the smallest index such that $\trianglelefteq_j^\rho$ is $=$ for all $j > i_\rho$, we have that $\uparrow \rho$ is the set of all ordered partitions χ where $\trianglelefteq_j^\chi = \trianglelefteq_j^\rho$ and $\tau^\chi(j) = \tau^\rho(j)$ for all $0 \leq j \leq i_\rho$.

Lemma 27: In $(\Omega_n, \sqsubseteq)$ two ordered partitions ρ and ϱ are such that $H(\rho) = H(\varrho)$ and $\uparrow \rho' \cong \uparrow \varrho'$ as posets for all $\rho' \sqsubseteq \rho$ and $\varrho' \sqsubseteq \varrho$ with $H(\rho') = H(\varrho')$, iff $i_\rho = i_\varrho$ and $\trianglelefteq_j^\rho = \trianglelefteq_j^\varrho$ for all $0 \leq j \leq i_\rho$.

Proof: (Sketch) One direction is clear. For the other assume first that $i_\rho \neq i_\varrho$. Then clearly $|\uparrow \rho| \neq |\uparrow \varrho|$. Assume then $i_\rho = i_\varrho$ but there is $j \in \{0, \ldots, i_\rho\}$ such that $\trianglelefteq_j^\rho \neq \trianglelefteq_j^\varrho$. Then $H(\rho) \neq H(\varrho)$ or there is $\rho' \sqsubseteq \rho$ and $\varrho' \sqsubseteq \varrho$ with $H(\rho') = H(\varrho')$ such that $\uparrow \rho' \ncong \uparrow \varrho'$. ■

By Lemma 27, if ρ and ϱ are such that $H(\rho) = H(\varrho)$ and $\uparrow \rho' \cong \uparrow \varrho'$ for all $\rho' \sqsubseteq \rho$ and $\varrho' \sqsubseteq \varrho$ with $H(\rho') = H(\varrho')$, then, after suitable renaming of the variables (just take the permutations $r \circ (\tau^\rho)^{-1}$ and $r \circ (\tau^\varrho)^{-1}$, for $r\colon i \mapsto n+1-i$), both ρ and ϱ can be displayed as

$$0 \trianglelefteq_0^\rho x_n \trianglelefteq_1^\rho x_{n-1} \trianglelefteq_2^\rho \cdots \trianglelefteq_{n-1}^\rho x_1 \trianglelefteq_n^\rho 1\,.$$

A *standard* ordered partition is an ordered partition ρ with $\tau^\rho(i) = n + 1 - i$ for all $i = 1, \ldots, n$.

Theorem 28: $\mathbf{AutInv}_n(\mathbb{G})$ is isomorphic with $\mathbf{F}_n(\mathbb{G})/\Theta_n$, for Θ_n being the congruence generated by $\{(x_{i+1} \Rightarrow x_i, 1) \mid i \in \{1, \ldots, n-1\}\}$.

Proof: The elements of $\mathbf{AutInv}_n(\mathbb{G})$ are all the symmetric elements of $\operatorname{Sub} G_n$. Take two ordered partitions ρ and ϱ such that $H(\rho) = H(\varrho)$ and $\uparrow \rho' \cong \uparrow \varrho'$ for all $\rho' \sqsubseteq \rho$ and $\varrho' \sqsubseteq \varrho$ with $H(\rho') = H(\varrho')$, Then, $\rho \in F$ iff $\varrho \in F$, for any symmetric element $F \in \operatorname{Sub} G_n$. In particular, by Lemma 27, the ordered partition ρ

$$0 \trianglelefteq_0^\rho x_{\tau^\rho(1)} \trianglelefteq_1^\rho x_{\tau^\rho(2)} \trianglelefteq_2^\rho \cdots \trianglelefteq_{n-1}^\rho x_{\tau^\rho(n)} \trianglelefteq_n^\rho 1$$

belongs to F, iff the standard ordered partition $\chi(\rho)$ displayed as

$$0 \trianglelefteq_0^\rho x_n \trianglelefteq_1^\rho x_{n-1} \trianglelefteq_2^\rho \cdots \trianglelefteq_{n-1}^\rho x_1 \trianglelefteq_n^\rho 1$$

belongs to F, too.

Let S_n be the subposet of G_n formed by all standard ordered partitions. The map sending each ordered partition ρ to its standard ordered partition $\chi(\rho)$ is a surjection of G_m onto S_n. By duality, this corresponds to an embedding f of $\mathrm{Sub}\, S_n$ into $\mathrm{Sub}\, G_n \cong \mathbf{F}_n(\mathbb{G})$. The embedding f sends each subforest $S \in \mathrm{Sub}\, S_n$ to the subforest $S' \in \mathrm{Sub}\, G_n$ given by all ordered partitions ρ such that $\chi(\rho) \in S$. By the above application of Lemma 27 the image of f into $\mathrm{Sub}\, G_n$ is precisely $AutInv_n$.

Further, by duality, the embedding of S_n into G_n also corresponds to a surjection of $\mathbf{F}_n(\mathbb{G})$ over its homomorphic image $\mathbf{F}_n(\mathbb{G})/\Theta$. Clearly, Θ is the congruence generated by the relations holding in all standard ordered partitions, which amounts to stipulate the validity of $x_{i+1} \Rightarrow x_i$ for all $i = 1, \ldots, n-1$. Equivalently, Θ is generated by $\{(x_{i+1} \Rightarrow x_i, 1) \mid i \in \{1, \ldots, n-1\}\}$. ■

For each $i \geq 0$ let K_i be the forest inductively defined as follows:

$$K_0 = \mathbf{1}, \qquad K_{i+1} = K_i + (K_i)_\perp \, .$$

Theorem 29: $K_n \cong S_n$. Whence, $\mathbf{AutInv}_n(\mathbb{G})$ is isomorphic with $\mathrm{Sub}\, K_n$.

Proof: By induction on i. The base $i = 0$ is clear, as the only ordered partition to consider is displayed as $0 < 1$. Assume the statement true for i. Observe that to produce all standard ordered partitions of $x_1, \ldots, x_{i+1}$ we just have to insert in all possible ways x_{i+1} into all standard ordered partitions χ of $x_1, \ldots, x_i$. This amounts to inserting $= x_{i+1}$ or $< x_{i+1}$ just between 0 and $\trianglelefteq_0^\chi x_i$ for all χ. Ir is clear that inserting $= x_{i+1}$ produces K_i, as this action adds no block to any χ, while inserting $< x_{i+1}$ gives $(K_i)_\perp$, as this corresponds to adding a new common minimum block to all χ. ■

Corollary 30: The cardinality k_n of $\mathbf{AutInv}_n(\mathbb{G})$ is given by the following recurrence:

$$k_0 = 1, \qquad k_{i+1} = k_i^2 + k_i \, .$$

Corollary 31: The forest K_n contains exactly 2^n leaves.

VIII. Conclusion

Theorem 28 allows us to characterise interpretatively the automorphism invariant elements of $\mathbf{F}_n(\mathbb{G})$. The kind of *fuzzy* interpretation we provide here is suggestively compared with the well-known temporal interpretation of formulas of Gödel logic. While in classical Boolean propositional logic the truth status of a formula under an assignment only depends on the truth-value assigned to the variables, in Gödel propositional logic we have to keep track of the *order* in which a variable attains the truth-value *true* ($x \to y$ is true if y becomes true not later than x does). When we restrict our attention to automorphism invariant elements, we again have to keep track of the order to truth, but we *name* variables according to their position in the same order (x_i becomes true not later than x_j, if $i < j$). This could be useful to recall in designing applications based on this interpretation of Gödel logic: if the desired model allows us to freely name variables (observables) by the order in which they become true (occur), then we can replace evaluation in $\mathbf{F}_n(\mathbb{G})$ by evaluation in the much smaller subalgebra $\mathbf{AutInv}_n(\mathbb{G})$. However, by Corollary 31, the evaluation of a formula φ in $\mathbf{AutInv}_n(\mathbb{G})$ has the same complexity of the evaluation of φ in classical Boolean propositional logic.

An analogous analysis of the algebra of automorphism invariant elements can be conducted for all logics considered in this paper, each logic providing some hints on viable natural fuzzy interpretations of such elements.

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