On k-Lipschitzian (T, N)-Implications

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***Abstract*—Implications and k-Lipschitz condition have been recurrent topics of studies in fuzzy theory. Additionally, they play a singular role concerning the robustness of rule-based fuzzy reasoning. Therefore, in this paper, we investigate the properties required for fuzzy negations N, t-norms T and $(\mathbf{T}, \mathbf{N})$-implications satisfying the k-Lipschitz condition for a specific, and arbitrary, k. Besides, we generate new Lipschitzian implications from $(\mathbf{T}, \mathbf{N})$-implications using automorphisms.**

***Index Terms*—Fuzzy operators, k-lipschitz condition, $(\mathbf{T}, \mathbf{N})$-implications.**

I. Introduction

Implication functions play a significant role in a logical system and, in a fuzzy context, they are used to represent imprecise knowledge as well as to perform inferences in a vague space of truth. So, fuzzy systems may have distinct behaviors depending on their fuzzy implication definitions. Therefore, since Zadeh's seminal work [1], fuzzy operators, specially the fuzzy implications, have been the subject of much research over the past decades [2]–[9].

In this sense, fuzzy implication classes have been defined and investigated, amongst other things, in order to: 1) find out some properties they satisfy and under which conditions they satisfy them; 2) generate and map fuzzy implications to each fuzzy implication class and to the intersections between those classes; and 3) with the theoretical knowledge of the previous items, to adapt fuzzy system's behavior (e.g. in a fuzzy control or in an image processing) through the election of an adequate fuzzy implication.

There are various ways and motivations to define a fuzzy implication class. Generally, they are generated by a specific generator function, or obtained by other fuzzy operators. These ones are based on t-norms, uninorms, aggregations, quasi-copulas, etc. In this paper, particularly, we introduce the best comprehension about the relation between (T, N)-implications and **k-Lipschitz** condition. The former is a N-dual implication of T [10] and can also be obtained from a (S, N)-implication, if (T, S, N) satisfies De Morgan's Law [4]. The Lipschitz condition of aggregation functions has good advantages in a practical point of view: it is reasonable to model dynamic process, found on fuzzy neural networks [11], and for rule-based fuzzy reasoning [12]. That condition has been well studied with respect to triangular norms, but not so much related to fuzzy implications. Moreover, there is no previous work relating the Lipschitz condition and the (T, N)-implications.

Therefore, in the following section, we briefly mention basic notions about fuzzy negations, t-norms, fuzzy implications, automorphisms and **k-Lipschitz** condition. Then, we demonstrate results relating T, N and I_T^N – a (T, N)-implication – to the **k-Lipschitz** condition, as well as we demonstrate under which conditions Φ-conjugate functions of T, N and I_T^N maintain their properties, including the **k-Lipschitz** one. Finally, in section IV we summarize the main contributions of this paper.

II. Preliminaries

In this section we recall some definitions and results already known in the literature.

Definition 2.1: [6] A function $T\colon [0,1]^2 \to [0,1]$ is a **triangular norm**, **t-norm** for short, if the following properties are satisfied:

(T1) Symmetry: $T(x, y) = T(y, x)$, for each $x, y \in [0, 1]$;
(T2) Associativity: $T(x, T(y, z)) = T(T(x, y), z)$, for each $x, y, z \in [0, 1]$;
(T3) Monotonicity: If $x \leq y$ then $T(x, z) \leq T(y, z)$, for each $x, y, z \in [0, 1]$;
(T4) Boundary condition: $T(x, 1) = x$, for each $x \in [0, 1]$.

Follow examples of some basic t-norms:

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- Gödel or minimum: $T_M(x, y) = \min(x, y)$;
- Product: $T_P(x, y) = x \cdot y$;
- Łukasiewicz: $T_L(x, y) = \max(0, x + y - 1)$.

Proposition 2.1: [6] Let T be a t-norm. Then $T(0, y) = 0$ for each $y \in [0, 1]$.

Definition 2.2: A function $N\colon [0,1] \to [0,1]$ is a **fuzzy negation** if

(N1) N is antitonic, i.e., $N(x) \le N(y)$ whenever $y \le x$;
(N2) $N(0) = 1$ and $N(1) = 0$;

In addition, a fuzzy negation N is

1) **strict** if (N3) N is continuous and (N4) $N(x) < N(y)$ whenever $y < x$;
2) **strong** if (N5) $N(N(x)) = x$, for each $x \in [0, 1]$;

All strong fuzzy negation is strict but not all strict fuzzy negation is strong. For example, $N(x) = 1 - x^2$ is strict but not strong whereas the standard fuzzy negation $N_S(x) = 1 - x$ is strong (and therefore strict) [3], [13].

Definition 2.3: Let $k \in [0, \infty)$. A function $f : [0,1]^n \to [0,1]$ is **k-Lipschitzian** if for all $x_i, y_i \in [0, 1]$,

$$\left|f(x_1, \cdots, x_n) - f(y_1, \cdots, y_n)\right| \le k \left(\sum_{i=1}^{n} |x_i - y_i| \right). \quad (1)$$

Evidently, if f is k-Lipschitzian, then is also p-Lipschitzian for any $p > k$. The smallest constant k satisfying (1) is called **Lipschitz constant**.

Theorem 2.1: [14] Let N be a 1-Lipschitzian negation. Then $N = N_S$.

Proposition 2.2: [14] Let N be a k-Lipschitzian fuzzy negation. Then $k \ge 1$.

That is, there is an unique negation whose Lipschitz constant is less than or equal to 1, namely $N(x) = 1 - x$.

We denote by Φ the family of all increasing bijections $\varphi\colon [0,1] \to [0,1]$, called automorphisms. We say that functions $f, g\colon [0,1]^n \to [0,1]$ are Φ-conjugate (see Kuczma [15], p. 156), if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) = \varphi^{-1}(f(\varphi(x_1), \ldots, \varphi(x_n))),$$

for all $x_1, \ldots, x_n \in [0, 1]$.

Proposition 2.3: [11] Let φ be an automorphism. Then φ^{-1} also is an automorphism.

Proposition 2.4: [3, Proposition 1.4.8] If $\varphi \in \Phi$ and N is a fuzzy (strict, strong) negation, then N_φ is also a fuzzy (strict, strong) negation.

Proposition 2.5: [6, Proposition 2.28] If $\varphi \in \Phi$ and T is a t-norm, then T_φ is also a t-norm.

Proposition 2.6: [11] Let $\varphi : [0,1] \to [0,1]$ be an increasing function. Then φ is bijective if and only if $\varphi(1) = 1, \varphi(0) = 0$ and φ is continuous and strictly increasing.

Definition 2.4: A function $f : [0,1]^2 \to [0,1]$ is called **convex** if:

$$f(tx + (1 - t)y) \le tf(x) + (1 - t)f(y),$$

for all $x, y \in [0,1]^2$ and $t \in [0, 1]$.

Definition 2.5: [3], [5], [7] A function $I : [0,1]^2 \to [0,1]$ is a **fuzzy implication** if, for all $x, y, z \in [0, 1]$, the following properties are satisfied:

(I1) If $x \le z$ then $I(x, y) \ge I(z, y)$;
(I2) If $y \le z$ then $I(x, y) \le I(x, z)$;
(I3) $I(0, y) = 1$;
(I4) $I(x, 1) = 1$;
(I5) $I(1, 0) = 0$.

We denote by $\mathcal{FI}$ the set of all fuzzy implications.

Among the various classes of fuzzy implications, such as (S, N), R- and QL-implication, we highlight a new class of fuzzy implication named (T, N)-implications. This implication was firstly presented by Bedregal in [10], and the idea was to provide an implication by the composition of a fuzzy negation and a t-norm. The (T, N)-implications were later investigated in [16]–[19]. The (T, N)-implication is defined as follows.

Definition 2.6: Let T be a t-norm and let N be a fuzzy negation. The function I_T^N defined by

$$I_T^N(x, y) = N(T(x, N(y)))$$

is called $(\mathbf{T}, \mathbf{N})$**-implication**.

III. k LIPSCHITZIAN (T, N)-IMPLICATIONS

In [12], Chen et. al. deeply investigated the implications from the perspective of Lipschitz condition and copula characteristic. In addition, some k-Lipschitzian implications were found. Another work was developed by Kolesárová and Mesiar in [20], in which they showed that there is a duality between the classes of all quasi-copulas and 1-Lipschitzian fuzzy implications.

In this sense, we will now study the Lipschitz condition for the (T, N)-implications from the t-norm and fuzzy negation that define it.

Proposition 3.1: Let I_T^N be a (T, N)-implication. Then, I_T^N satisfy 1-Lipschitz condition if and only if N and T satisfy the 1-Lipschitz condition.

PROOF: Indeed,
$(\Rightarrow)$ Since I_T^N satisfy 1-Lipschitz condition, then

$$\left|N(T(x, N(z))) - N(T(y, N(w)))\right| \le |x - y| + |z - w|, \quad (2)$$

for all $x, y, z, w \in [0, 1]$. In particular, for $z = w = 0$

$$\left|N(T(x, 1)) - N(T(y, 1))\right| \le |x - y|,$$

by (T4),

$$\left|N(x) - N(y)\right| \le |x - y|.$$

Therefore, N satisfy 1-Lipschitz condition. So, by the Theorem 2.1 and (2),

$$\left|T(y, N(w)) - T(x, N(z))\right| \le |x - y| + |z - w|,$$

i.e.,

$$\left|T(x, N(z)) - T(y, N(w))\right| \le |x - y| + |z - w|,$$

for all $x, y, z, w \in [0,1]$. Again, by the Theorem 2.1,

$$\begin{aligned}\big|T(x,z) - T(y,w)\big| &= \big|T(x, N(1-z)) - T(y, N(1-w))\big| \\ &\leq |x-y| + |(1-z) - (1-w)| \\ &= |x-y| + |z-w|,\end{aligned}$$

for all $x, y, z, w \in [0,1]$. Therefore, T satisfy 1-Lipschitz condition.

($\Leftarrow$) Since N is 1-Lipschitzian, we have by the Theorem 2.1 that $N(x) = 1 - x$. So, for all $x, y, z, w \in [0,1]$

$$\begin{aligned}\big|I_T^N(x,z) - I_T^N(y,w)\big| &= \big|N(T(x, N(z))) - N(T(y, N(w)))\big| \\ &= \big|T(y, N(w)) - T(x, N(z))\big|,\end{aligned}$$

now, since T is 1-Lipschitzian,

$$\begin{aligned}\big|I_T^N(x,z) - I_T^N(y,w)\big| &= \big|T(y, N(w)) - T(x, N(z))\big| \\ &\overset{(1)}{\leq} |y-x| + |N(w) - N(z)| \\ &= |x-y| + |z-w|.\end{aligned}$$

Therefore, I_T^N is 1-Lipschitzian. ■

Corollary 3.1: Let I_T^N be a (T,N)-implication. If I_T^N satisfy 1-Lipschitz condition then $N = N_S$.

PROOF: It follows straight from Proposition 3.1 and Theorem 2.1. ■

Remark 3.1: From Corollary 3.1 we can see that a necessary condition for I_T^N to be 1-Lipschitzian is that N is the standard negation, i.e., $N = N_S$.

The following theorem guarantees another necessary condition for I_T^N to be 1-Lipschitzian, where the condition depends on the t-norm and fuzzy negation.

Theorem 3.1: Let I_T^N be a (T,N)-implication. If I_T^N is 1-Lipschitzian then:

(i) $T(x,y) \geq T_L(x,y)$ for all $x, y \in [0,1]$, where T_L is the Łukasiewicz t-norm;

(ii) $I_T^N(x,y) \geq N_S(x)$, for all $x, y \in [0,1]$.

PROOF: Indeed,

(i) as I_T^N is 1-Lipschitzian, then by the Proposition 3.1, N is 1-Lipschitzian and, in particular,

$$\big|I_T^N(x,z) - I_T^N(y,z)\big| \leq |x-y|,$$

for all $x, y, z \in [0,1]$. So by making $x = 1$ we get

$$\big|N(N(z)) - I_T^N(y,z)\big| \leq |1-y|.$$

Since N is 1-Lipschitzian, by the Theorem 2.1, $N(x) = 1 - x$ for all $x \in [0,1]$. So, $N(N(z)) = z$ and, by the Remark 1.5 in [6], $T(y, 1-z) \leq 1-z$, so $1 - T(y, 1-z) \geq z$, i.e., $I_T^N(y,z) \geq z$, hence for all $y, z \in [0,1]$

$$\begin{aligned}\big|z - I_T^N(y,z)\big| \leq 1 - y &\Rightarrow I_T^N(y,z) - z \leq 1 - y \\ &\Rightarrow 1 - T(y, 1-z) - z \leq 1 - y \\ &\Rightarrow T(y, 1-z) \geq y - z.\end{aligned}$$

For all $x, y \in [0,1]$,

$$T(x,y) = T(x, N(N(y))) \geq x - N(y) = x + y - 1.$$

Now, since $T(x,y) \geq 0$ for all $x, y \in [0,1]$, so $T(x,y) \geq \max(0, x+y-1)$, i.e., $T(x,y) \geq T_L(x,y)$.

(ii) Since I_T^N satisfy 1-Lipschitz condition, in particular,

$$\big|N(T(x, N(y))) - N(T(z, N(y)))\big| \leq |x-z|,$$

for all $x, y, z \in [0,1]$. So, for $z = 0$, we have, by the Proposition 2.1,

$$\big|I_T^N(x,y) - 1\big| \leq x,$$

for all $x, y \in [0,1]$. Hence, $I_T^N(x,y) \geq N_S(x)$ for all $x, y \in [0,1]$. ■

Proposition 3.2: Let I_T^N be a (T,N)-implication, where N is 1-Lipschitzian. Then I_T^N is k-Lipschitzian if and only if

$$\big|I_T^N(x,z) - I_T^N(y,z)\big| \leq k|x-y|, \tag{3}$$

for all $x, y, z \in [0,1]$.

PROOF: Indeed, if I_T^N is k-Lipschitzian then (3) follows straight. Now, considering that the inequality (3) is satisfied,

$$\begin{aligned}\big|I_T^N(x,z) - I_T^N(y,w)\big| &\leq \big|I_T^N(x,z) - I_T^N(y,z)\big| \\ &\quad + \big|I_T^N(y,z) - I_T^N(y,w)\big| \\ &\overset{(3)}{\leq} k|x-y| + \big|I_T^N(y,z) - I_T^N(y,w)\big|\end{aligned}$$

for all $x, y, z, w \in [0,1]$. Now, since N is 1-Lipschitzian, by the Theorem 2.1, we have that $N(x) = 1 - x$, so

$$\begin{aligned}\big|I_T^N(y,z) - I_T^N(y,w)\big| &= \big|T(y, N(w)) - T(y, N(z))\big| \\ &\overset{(T1)}{=} \big|T(N(w), y) - T(N(z), y)\big| \\ &= \big|I_T^N(N(z), N(y)) - I_T^N(N(w), N(y))\big| \\ &\overset{(3)}{\leq} k|N(z) - N(w)| = k|z-w|.\end{aligned}$$

So, for all $x, y, z, w \in [0,1]$,

$$\begin{aligned}\big|I_T^N(x,z) - I_T^N(y,w)\big| &\leq k|x-y| + k\big|z-w\big| \\ &\leq k(|x-y| + |z-w|).\end{aligned}$$

Therefore, I_T^N is k-Lipschitzian. ■

Proposition 3.3: Let I_T^N be a (T,N)-implication, where N satisfy 1-Lipschitz condition. Then T is k-Lipschitzian if and only if I_T^N is k-Lipschitzian.

PROOF: Indeed, by the Theorem 2.1, $N(x) = 1 - x$ for all $x \in [0,1]$, so

($\Rightarrow$) for all $x, y, z \in [0,1]$

$$\begin{aligned}\big|I_T^N(x,z) - I_T^N(y,z)\big| &= \big|T(y, N(z)) - T(x, N(z))\big| \\ &\leq k(|x-y|),\end{aligned}$$

since T is k-Lipschitzian. Therefore, by the Proposition 3.2, I_T^N is k-Lipschitzian.

($\Leftarrow$) Now, since I_T^N is k-Lipschitzian, then for all $x, y, z, w \in [0,1]$

$$\big|I_T^N(x,z) - I_T^N(y,w)\big| \leq k(|x-y| + |z-w|),$$

and so, again by the Theorem 2.1,

$$\big|T(x, N(z)) - T(y, N(w))\big| \leq k(|x-y| + |z-w|).$$

Thus, for all $x, y, z, w \in [0, 1]$

$$\begin{aligned}\big|T(x,z)-T(y,w)\big| &= \big|T(x,N(1-z))-T(y,N(1-w))\big| \\ &\leq k(|x-y|+|z-w|).\end{aligned}$$

Therefore, T is k-Lipschitzian. ■

Proposition 3.4: Let I_T^N be a (T, N)-implication. If I_T^N is k-Lipschitzian, then N is k-Lipschitzian. In addition if N is a strict concave negation then T also is k-Lipschitzian.

PROOF: Indeed, since I_T^N is k-Lipschitzian, then for all $x, y, z, w \in [0, 1]$

$$\big|N(T(x,N(z)))-N(T(y,N(w)))\big| \leq k(|x-y|+|z-w|),$$

for all $x, y, z, w \in [0, 1]$. In particular, for $z = w = 0$,

$$\big|N(T(x,1))-N(T(y,1))\big| \leq k|x-y|,$$

by (T4),

$$\big|N(x)-N(y)\big| \leq k|x-y|.$$

Therefore, N is k-Lipschitzian. On the other hand, if N is a concave strict negation then N^{-1} is a convex strict negation and therefore

$$|N(x)-N(y)| \geq |x-y| \tag{4}$$

and

$$|N^{-1}(x)-N^{-1}(y)| \leq |x-y| \tag{5}$$

for each $x, y \in [0, 1]$. Then, since N is k-Lipschitzian,

$$\begin{aligned}\big|T(x,z)-T(y,w)\big| &= \big|T(x,N(N^{-1}(z)))-T(y,N(N^{-1}(w)))\big| \\ &\overset{(4)}{\leq} \big|N(T(x,N(N^{-1}(z))) - N(T(y,N(N^{-1}(w))))\big| \\ &\leq k(|x-y|+|N^{-1}(z)-N^{-1}(w)|) \\ &\overset{(5)}{\leq} k(|x-y|+|z-w|),\end{aligned}$$

for all $x, y, z, w \in [0, 1]$. Therefore, T is k-Lipschitzian. ■

Corollary 3.2: Let I_T^N be a (T, N)-implication. If I_T^N is k-Lipschitzian, then $k \geq 1$.

PROOF: It follows straight from Propositions 3.4 and 2.2. ■

Corollary 3.3: Let I_T^N be a (T, N)-implication. If k_1 is the Lipschitz constant of I_T^N and k_2 is the Lipschitz constant of N, then $k_2 \leq k_1$.

PROOF: Suppose $k_2 > k_1$. Since k_1 is the Lipschitz constant of I_T^N, then

$$\big|I_T^N(x,z)-I_T^N(y,w)\big| \leq k_1(|x-y|+|z-w|).$$

In particular, for $z = w = 0$,

$$\big|N(x)-N(y)\big| \leq k_1|x-y|,$$

for all $x, y \in [0, 1]$. Contradiction, since k_2 is the Lipschitz constant of N. ■

Proposition 3.5: Let I_T^N be a (T, N)-implication. If I_T^N is k-Lipschitzian, then $N(x) \geq 1 - kx$ for all $x \in [0, 1]$.

PROOF: Since I_T^N is k-Lipschitzian, for all $x, y, z \in [0, 1]$

$$\big|I_T^N(x,z)-I_T^N(y,w)\big| \leq k(|x-y|+|z-w|).$$

Doing $x = z = w = 0$, by the Proposition 2.1, we get:

$$\big|1-I_T^N(y,0)\big| \leq ky.$$

So, $1 - N(T(y, 1)) \leq ky$, therefore $N(y) \geq 1 - ky$, for all $y \in [0, 1]$. ■

Proposition 3.6: Let I_T^N be a (T, N)-implication. If N is k_1-Lipschitzian and T is k_2-Lipschitzian, then I_T^N is $\tilde{k}$-Lipschitzian, where $\tilde{k} = k_1^2 \cdot k_2$.

PROOF: Indeed, since N is k_1-Lipschitzian and T is k_2-Lipschitzian, then, for all $x, y, z, w \in [0, 1]$

$$\begin{aligned}\big|I_T^N(x,z)-I_T^N(y,w)\big| &\leq k_1|T(x,N(z))-T(y,N(w))| \\ &\leq k_1 \cdot k_2(|x-y|+|N(z)-N(w)|) \\ &\leq k_1 \cdot k_2(|x-y|+k_1|z-w|).\end{aligned}$$

Now, by the Proposition 2.2 we have that $k_1 \geq 1$, so

$$\big|I_T^N(x,z)-I_T^N(y,w)\big| \leq k_1^2 \cdot k_2(|x-y|+|z-w|),$$

for all $x, y, z, w \in [0, 1]$. ■

Corollary 3.4: Let I_T^N be a (T, N)-implication, where T satisfy 1-Lipschitzian condition. If N is k-Lipschitzian then I_T^N is k^2-Lipschitzian.

PROOF: Let $k_1 = k$ and $k_2 = 1$ in the previous proposition. ■

In the next result, we show some relationships between (T, N)-implications and their conjugates.

Proposition 3.7: Let I_T^N be a (T, N)-implication and φ be an automorphism. Then,

$$(I_T^N)_\varphi(x,y) = I_{T_\varphi}^{N_\varphi}(x,y),$$

for all $x, y \in [0, 1]$.

PROOF: Given the automorphisms φ and an (T, N)-implication I_T^N, we have by the Proposition 2.4 that N_φ is a fuzzy negation and by the Proposition 2.5 that T_φ is a t-norm. So, $I_{T_\varphi}^{N_\varphi}$ is an (T, N)-implication. Moreover, for $x, y \in [0, 1]$,

$$\begin{aligned}(I_T^N)_\varphi(x,y) &= \varphi^{-1}(I_T^N(\varphi(x),\varphi(y))) \\ &= \varphi^{-1}(N(T(\varphi(x),N(\varphi(y))))) \\ &= \varphi^{-1}(N(\varphi(\varphi^{-1}(T(\varphi(x),\varphi(\varphi^{-1}(N(\varphi(y))))))))) \\ &= N_\varphi(\varphi^{-1}(T(\varphi(x),\varphi(N_\varphi(y))))) \\ &= N_\varphi(T_\varphi(x,N_\varphi(y))) \\ &= I_{T_\varphi}^{N_\varphi}(x,y).\end{aligned}$$

■

Proposition 3.8: Let I_T^N be a (T, N)-implication and $(I_T^N)_\varphi$ its Φ-conjugate. If φ is k_φ-Lipschitzian, φ^{-1} is $k_{\varphi^{-1}}$-Lipschitzian and I_T^N is k-Lipschitzian, then the (T, N)-implication $(I_T^N)_\varphi$ is $\widetilde{k}$-Lipschitzian, where $\widetilde{k} = k \cdot k_\varphi \cdot k_{\varphi^{-1}}$.

PROOF: Indeed, since I_T^N is k-Lipschitzian, φ is k_φ-Lipschitzian and φ^{-1} is $k_{\varphi^{-1}}$-Lipschitzian, then, for all $x, y, z, w \in [0,1]$

$$\begin{aligned}
&\left|(I_T^N)_\varphi(x,z) - (I_T^N)_\varphi(y,w)\right| = \\
&\quad = \left|\varphi^{-1}(I_T^N(\varphi(x),\varphi(z))) - \varphi^{-1}(I_T^N(\varphi(y),\varphi(w)))\right| \\
&\quad \leq k_{\varphi^{-1}}\left|I_T^N(\varphi(x),\varphi(z)) - I_T^N(\varphi(y),\varphi(w))\right| \\
&\quad \leq k_{\varphi^{-1}} \cdot k(|\varphi(x)-\varphi(y)| + |\varphi(z)-\varphi(w)|) \\
&\quad \leq k_{\varphi^{-1}} \cdot k \cdot k_\varphi(|x-y| + |z-w|).
\end{aligned}$$

So, $(I_T^N)_\varphi$ is $k \cdot k_\varphi \cdot k_{\varphi^{-1}}$-Lipschitzian. ■

Corollary 3.5: Let I_T^N be a k-Lipschitzian (T,N)-implication and $(I_T^N)_\varphi$ your Φ-conjugate. If φ is k_φ-Lipschitzian and φ^{-1} is $k_{\varphi^{-1}}$-Lipschitzian, then N_φ is $k \cdot k_\varphi \cdot k_{\varphi^{-1}}$-Lipschitzian.

PROOF: It follows straight from Propositions 3.8 and 3.4. ■

Corollary 3.6: Let I_T^N be a k-Lipschitzian (T,N)-implication and $(I_T^N)_\varphi$ your Φ-conjugate. If φ is k_φ-Lipschitzian, φ^{-1} is $k_{\varphi^{-1}}$-Lipschitzian and N is 1-Lipschitzian, then T_φ is $k \cdot k_\varphi \cdot k_{\varphi^{-1}}$-Lipschitzian.

PROOF: Since N is 1-Lipschitzian, by the Proposition 3.3, T is k-Lipschitzian. So, for all $x, y, z, w \in [0,1]$

$$\begin{aligned}
\left|T_\varphi(x,z)\right. &- \left.T_\varphi(y,w)\right| \leq \\
&\leq k_{\varphi^{-1}}\left|T(\varphi(x),\varphi(z)) - T(\varphi(y),\varphi(w))\right| \\
&\leq k_{\varphi^{-1}} \cdot k(|\varphi(x)-\varphi(y)| + |\varphi(z)-\varphi(w)|) \\
&\leq k_{\varphi^{-1}} \cdot k \cdot k_\varphi(|x-y| + |z-w|).
\end{aligned}$$

■

Definition 3.1: Let $f : [0,1]^n \to [0,1]$ be a function and φ be an automorphism. We said that φ **preserve** f, if for each $(x_1, \cdots, x_n) \in [0,1]^n$, $\varphi(f(x_1, \cdots, x_n)) = f(\varphi(x_1), \cdots, \varphi(x_n))$.

Proposition 3.9: Let I_T^N be a (T,N)-implication and φ be an automorphism. If φ preserve the t-norm T and the fuzzy negation N, then φ preserve I_T^N.

PROOF: Indeed, since φ preserve T and N, then $\varphi(T(x,y)) = T(\varphi(x),\varphi(y))$ and $\varphi(N(x)) = N(\varphi(x))$ for all $x, y \in [0,1]$. So,

$$\begin{aligned}
\varphi(I_T^N(x,y)) &= \varphi(N(T(x,N(y)))) = N(\varphi(T(x,N(y)))) \\
&= N(T(\varphi(x),\varphi(N(y)))) = N(T(\varphi(x),N(\varphi(y)))) \\
&= I_T^N(\varphi(x),\varphi(y)).
\end{aligned}$$

■

Proposition 3.10: Let I_T^N be a (T,N)-implication and φ be an automorphism. If φ preserve I_T^N then

(i) φ preserve the fuzzy negation N;
(ii) If N is strict, then φ preserve the t-norm T.

PROOF: Indeed, since φ preserve I_T^N, we have that

$$\varphi(N(T(x,N(y)))) = N(T(\varphi(x),N(\varphi(y)))) \tag{6}$$

for all $x, y \in [0,1]$.

(i) In particular for $y = 0$, by the Proposition 2.6,

$$\varphi(N(T(x,N(0)))) = N(T(\varphi(x),N(0))).$$

So, by (N2) and (T4),

$$\varphi(N(x)) = N(\varphi(x)).$$

Therefore, φ preserve the N.

(ii) By previous item, φ preserve N, then by (6),

$$N(\varphi(T(x,N(y)))) = N(T(\varphi(x),\varphi(N(y)))).$$

So, since N is strict,

$$\varphi(T(x,N(y))) = T(\varphi(x),\varphi(N(y)))$$

and for each $y' \in [0,1]$ there exists $y \in [0,1]$ such that $y' = N(y)$, so

$$\begin{aligned}
\varphi(T(x,y')) &= \varphi(T(x,N(y))) = T(\varphi(x),\varphi(N(y))) \\
&= T(\varphi(x),\varphi(y'))
\end{aligned}$$

foa all $x, y' \in [0,1]$. Therefore, φ preserve T.

■

Corollary 3.7: Let I_T^N be a (T,N)-implication and φ be an automorphism. If N is strict, then φ preserve I_T^N if and only if φ preserve T and N.

PROOF: It follows straight from the Propositions 3.9 and 3.10. ■

IV. Final Remarks

In function of the relevance of implications and k-Lipschitz condition, this work aimed to develop the first study on k-Lipschitz condition for the class of (T,N)-implications which can be the start point for more wide research on k-Lipschitzian implications considering theoretic and practical aspects. Therefore, this paper brings a seminal investigation of under which properties a (T,N)- implication satisfy the k-Lipschitz condition for a specific and generic k. Moreover, in order to promote the generation of new Lipschitzian implications from (T,N)-implications, we also demonstrate under which conditions it could be done using automorphisms.

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