On the properties of orderings of extensional fuzzy numbers

Martin Štěpnička
CE IT4I - IRAFM
University of Ostrava
Ostrava, Czech Republic
martin.stepnicka@osu.cz

Nicole Škorupová
CE IT4I - IRAFM
University of Ostrava
Ostrava, Czech Republic
nicole.skorupova@osu.cz

Michal Holčapek
CE IT4I - IRAFM
University of Ostrava
Ostrava, Czech Republic
michal.holcapek@osu.cz

***Abstract*—The article stems from distinct studies on arithmetics of fuzzy numbers, orderings of fuzzy numbers, and metrics on fuzzy numbers. Trying to capture the existing knowledge in the mentioned areas and putting them together, we motivate the construction of metric-like spaces on fuzzy numbers by desirable connection to their arithmetics. The desirable "metric" should be mapping pairs of fuzzy numbers again to fuzzy numbers and thus, reflecting the vagueness of operation on fuzzy numbers. This leads to developing all such areas under the joint umbrella and connecting such basic notions as orderings of fuzzy numbers to arithmetics and elementary "metrics" such as the absolute value of the difference of two fuzzy numbers. This article focuses mainly on the orderings and investigation of the preservation of their most natural properties. However, links to further studies going towards applications are also foreshadowed and referred to.**

***Index Terms*—MI-algebras, extensional fuzzy numbers, similarity relation, extensionality, orderings, approximate reasoning**

I. Introduction

Fuzzy numbers formally representing vague quantities are used in distinct areas of the theory (fuzzy approximation, fuzzy interpolation, approximate reasoning, ranking, fuzzy regression) as well as in practical applications (automated control, decision-making system, modeling economical behaviour of companies and stocks changes). Plenty of research efforts has been invested into this areas but still, the topic does not seem to be exhausted and the area is not fully known. The main motivation for us lies in the desirable connections between distinct theoretical problems related to fuzzy numbers, that are not interconnected yet.

A typical example of such two not fully connected problems is the plenty of technically deeply elaborated arithmetics of fuzzy numbers [1]–[4] on one side, and the problem of orderings of fuzzy numbers [5]–[7] on the other side. As mentioned above, both problems are rather deeply elaborated, however, up to our best knowledge, we are not aware of a scientific study that would directly connect problems areas into a compact and tightly connected theory. The reason in our opinion lies in the missing umbrella topic that essentially needs both problems to be closely connected, such as metric-like functions operating on fuzzy numbers.

What do we mean by the above mentioned theme that, at first sight, needs no attention as distinct metrics (e.g. Hausdorff) can be defined for any sets and thus, consequently α-cut wise extended to fuzzy sets too?

The above described approach uses a standard metric space and standard metric function to map a pair of fuzzy numbers to a real number. How intuitive is to say that the distance of two quantities, that are only vaguely given to us, is precisely some $x \in \mathbb{R}$? And moreover, does this crisp distance expressed in the real number x reflects our intuitive expectations?

Let us consider a simple example taken from [8]. Let a fuzzy number A models the vague quantity *"about 3"* and let a fuzzy number B models the vague quantity *"about 5"*, see Figure 1. The Hausdorff distance [9] between fuzzy numbers A and B equals to 4.18. Obviously, any human asked to her/his expectation on the distance between *"about 3"* and *"about 5"* would answer something like *"about 2"*, nobody would state 4.18. This result is mathematically fully correct yet not that intuitive.

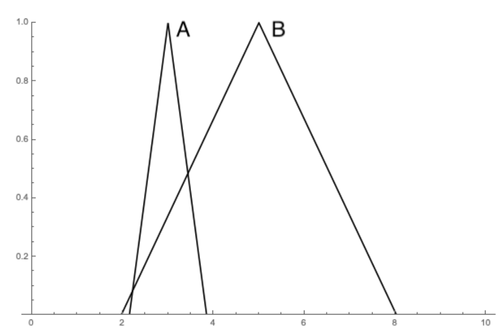

Fig. 1. Fuzzy numbers modeling the vague quantity *"about 3"* (A), and the vague quantity *"about 5"* (B). Their Hausdorff distance equals 4.18 .

However, even the intuitive result can be obtained by a mathematically fully correct way, in particular, by mimicking how the distances are constructed on real numbers, which means by the direct employment of the arithmetic of the reals. It is sufficient to construct an absolute value of the difference of the two fuzzy numbers A and B and to consider the

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result as a the "distance" between A and B. Formally, instead of building a classical metric operating on fuzzy numbers $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, we build a metric-like fuzzy function that maps pairs of fuzzy numbers again to fuzzy numbers $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

This intuitive generalization of the classical approach, however, needs lot of work on the formal level. For instance, the metric-like function defined fuzzy numbers should possess four axioms mimicking the same properties as the axioms of classical metric functions, including the triangle inequality. The triangle inequality, in the environment of fuzzy numbers, will necessarily employ the ordering of fuzzy numbers. Moreover, the ordering of fuzzy numbers is essential also for the definition of the absolute value, that is required for the construction of the most natural distance between fuzzy numbers. Thus, the topic related to the distances does not serves only as the motivation but possesses also the umbrella theme role joining orderings and arithmetics of fuzzy numbers.

Our investigation adopts the particular *arithmetic of extensional fuzzy numbers* [10], [11] that has been deeply elaborated from the algebraic point of view [11]–[14]. The first work initiating this direction towards metrics and orderings has been presented in [15] and uncovered the potential of this research as well as the next directions. Currently, we may refer readers to a deeper study on the metrics of fuzzy numbers in [8] and devote this article mainly to the orderings and their properties. Note, that the importance of the orderings is much higher than just in the metric-like spaces of extensional fuzzy numbers, that allow to proceed further in areas such as fuzzy approximation, mathematical analysis with vague quantities, construction of limits, convergences etc., but also in areas such as approximate reasoning. Indeed, the notion of a monotone fuzzy rule base has been studied already in [16]–[18] and for the implicative rule bases in [19], [20]. All studies considered monotonicity of the fuzzy rule base with respect interval orderings of α-cuts and the preservation of the monotonicity in the resulting function was naturally studied with respect to only crisp inputs and outputs. The fuzzy inputs were considered later on only from the interpolation point of view [21], [22]. Preservation of the monotonicity in the case of a monotone fuzzy rule base and order fuzzy input is a natural direct that can be approached only based on firm theoretical foundations related to orderings of fuzzy sets (e.g. fuzzy numbers).

II. Preliminaries

A. Arithmetics of extensional fuzzy numbers

Let us briefly recall the arithmetic of extensional fuzzy numbers. It is tightly connected to similarity relations and extensional hulls. Let us note that these are fundamental notions also in the investigations of orderings of fuzzy numbers presented by Bodenhofer [5], [6] and thus, it foreshadows that developing the arithmetic of extensional fuzzy numbers and their orderings jointly is promisingly based on the same foundations.

Let us recall, that a given t-norm $\otimes$, the *similarity relation* (more precisely $\otimes$-similarity, or also fuzzy equivalence relation) [23] is binary relation on a given universe that is *reflexive*, *symmetric*, and $\otimes$-*transitive*. As the purpose of similarity relations in this study is to determine fuzzy sets of "close values" to a given real number, we may freely restrict our focus on similarity relations defined on real numbers, i.e., to consider only $S : \mathbb{R} \times \mathbb{R} \to [0, 1]$ below.

Example 2.1: Binary fuzzy relations on $\mathbb{R}$ given by

$$S_p(x, y) = (1 - p|x - y|) \vee 0\ , \quad p > 0 \tag{1}$$

are the $\otimes$-similarity relations where $\otimes$ is the Łukasiewicz t-norm.

Binary fuzzy relations on $\mathbb{R}$ given by

$$S_p(x, y) = e^{-p|x-y|}\ , \quad p > 0 \tag{2}$$

are the $\otimes$-similarity relations where $\otimes$ is the product t-norm.

A fuzzy set A on the same universe, i.e., $A \in \mathcal{F}(\mathbb{R})$, is called *extensional* [24] w.r.t. the given similarity S if

$$A(x) \otimes S(x, y) \le A(y)$$

holds for arbitrary $x, y \in \mathbb{R}$.

The least fuzzy superset of a given fuzzy set $A \in \mathcal{F}(\mathbb{R})$ that is extensional with respect to the similarity S, is called *extensional hull* of A and will be denoted as $\mathrm{EXT}_S(A)$. It can be constructed as follows:

$$\mathrm{EXT}_S(A)(x) = \bigvee_{y \in \mathbb{R}} (A(y) \otimes S(x, y)) \tag{3}$$

and for more details, we refer readers to [25].

If we consider that each real number $x \in \mathbb{R}$ can be represented as singleton $\tilde{x} \in \mathcal{F}(\mathbb{R})$ ($\tilde{x}(x) = 1$, and $\tilde{x}(y) = 0$ for any $y \neq x$) and we neglect the difference between the real number x and its singleton representation $\tilde{x}$, we may construct the extensional hull of the real number. Such an object – an element of $\mathcal{F}(\mathbb{R})$ – is called fuzzy point [25]–[27] or *extensional fuzzy number* [10]–[12].

The construction of the extensional fuzzy number is straightforward and underlines their semantics – (*number x with closely neighboring numbers*) – where the closeness is given by the used similarity S.

Mathematically, the extensional fuzzy number with the semantics *"around x"*, denoted by $x_S \in \mathcal{F}(\mathbb{R})$, is given as follows:

$$x_S(y) = \mathrm{EXT}_S(x)(y)$$

and the particular calculation may be done as follows [11]:

$$x_S(y) = S(x, y), \quad y \in \mathbb{R}.$$

Remark 2.1: For the purpose of this investigation, we restrict our focus to such similarities that the fuzzy numbers x_S constructed as above are formed by α-cuts $(x_S)_\alpha$ that are closed intervals in $\mathbb{R}$.

In our opinion, there is not better representation of a vague quantity that would be reflecting its semantics more appropriately already in its construction. Moreover, the construction allows to develop specific mathematical model of the arithmetics, that was firstly proposed in [10], [12]. Algebraically, the arithmetics led structures with more identity-like elements, which gave them the name MI-algebras [11]. However, we will omit more general algebraic perspective and stay with the primary motivation for their rise – with arithmetics of extensional fuzzy numbers based on systems of similarity relations.

Let us consider a system of nested $\otimes$-similarity relations $\mathcal{C}$ with the bottom element $\perp_{\mathcal{C}}$ and the set of all fuzzy numbers that are extensional with respect to a similarity from the given system:

$$\mathcal{F}_{\mathcal{C}}(\mathbb{R}) = \{x_S \mid x \in \mathbb{R} \text{ and } S \in \mathcal{C}\}.$$

For the purpose of this paper, we restrict ourselves to such nested systems $\mathcal{C}$ that for any $x_S \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ the set $(x_S)_\alpha$ is a closed interval in $\mathbb{R}$ for any $\alpha \in (0,1]$.

Then the addition and the multiplication of two extensional fuzzy numbers from $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ defined with help of any $S, T \in \mathcal{C}$, are given as

$$x_S + y_T = (x+y)_{\max(S,T)}, \quad x_S \cdot y_T = (x \cdot y)_{\max(S,T)},$$

where the maximum of two similarities comes from the inclusion ordering on $\mathcal{C}$:

$$S \subseteq T \quad \text{if} \quad S(x,y) \leq T(x,y)\ , \quad x, y \in \mathbb{R}\ .$$

The use of the maximum of two similarities is mathematically correct due to the assumption that $\mathcal{C}$ is a system of nested similarities, which imposes the linearity assumption on the ordering $\subseteq$.

The proposed arithmetic brings several positive features. First of all, it mimic how humans calculated with vague quantities when, e.g., the summation of "around 50" and "around 40", leads to the result "around 90". Indeed, even humans sum up numbers 50 and 40 in the first step, and additionally, the human cognition realizes some tolerance to close values. Secondly, the calculus is computationally extremely cheap. Thirdly, the arithmetic operations do not widen the resulted fuzzy numbers as it happens when the α-cut based arithmetic obtained from the Zadeh's extensional principle is applied. Finally, we can preserve more of the desirable algebraic properties known from the classical arithmetic [11].

The bottom element $\perp_{\mathcal{C}}$ is the "narrowest " similarity from $\mathcal{C}$ and its role for the arithmetics lies in the so-called strong identity elements of both operations:

$$\mathbf{0} = 0_{\perp_{\mathcal{C}}} = \perp_{\mathcal{C}}(0,\cdot), \quad \mathbf{1} = 1_{\perp_{\mathcal{C}}} = \perp_{\mathcal{C}}(1,\cdot)$$

The inversions are obtained in the standard way, e.g., the inversion of the addition is defined as follows $-x_S = (-x)_S$. The (non-strong) identity elements, (*pseudoidentities*) are the elements of $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ obtained by applying the operations to elements and their inverses, for instance, the set of additive pseudoidentities is given as:

$$I^0_{\mathcal{C}} = \{x_S + (-x)_S \mid S \in \mathcal{C}\}.$$

Distinct algebraic structures, such as MI-prefield $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, \cdot, -, ^{-1})$ where many appropriate properties including the distributivity are preserved, are studied in [11]. In this study, the main focus will be on the additive MI-pregroups $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$.

Example 2.2: Consider an interval of positive parameters $p \in [\ell, r]$. Then $\mathcal{C} = \{S_p \mid p \in [\ell, r]\}$ where the S_p is given by (1) or (2) forms a system of nested Łukasiewicz (product) similarities with the bottom element $\perp_{\mathcal{C}} = S_r$.

The summation in the respective structure $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$ is calculated as follows:

$$x_{S_p} + y_{S_{p'}} = (x+y)_{S_{p''}}, \quad p'' = \min\{p, p'\}\ .$$

Note, that the crisp equality "=" is an equivalence relation and thus, it is also $\otimes$-similarity relation for any t-norm. Therefore, it can be an element in the system of nested similarities and in such case, it constitutes the bottom element.

Example 2.3: Consider an interval of positive parameters $p \in [\ell, +\infty]$. Then $\mathcal{C} = \{S_p \mid p \in [\ell, +\infty)\}$ where the S_p is given by (1) or (2) forms a system of nested Łukasiewicz (product) similarities. Let S_∞ be the crisp equality, i.e., $S_\infty(x,y) = 1$ if and only if $x = y$. Then $\mathcal{C}_\infty = \mathcal{C} \cup S_\infty$ forms the system of nested similarities with the bottom element $\perp_{\mathcal{C}_\infty} = S_\infty$.

Let us fix the denotation $\mathcal{C}_\infty$ for a system containing the crisp equality and let us call them *systems of nested similarities with the crisp bottom element*.

III. Orderings of extensional fuzzy numbers

As we stated above, one of the main motivations was the intended construction of the metric-like functions operating on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and mapping the values again to $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$. As the last axiom of metrics is the triangle inequality, that either holds or not, it required to construct orderings of fuzzy numbers in a sort of "binary" style, yet, the construction should not completely lose the vagueness information about the relationship between the compared fuzzy numbers. Moreover, in most cases, such binary orderings are not total anymore [28], which might bring some complications for the metric-like spaces. Typical example of such binary ordering that does not preserve any further information about the vagueness is the ordering of intervals

$$[a,b] \leq_i [c,d] \;\Leftrightarrow\; a \leq c \;\text{ and }\; b \leq d$$

applied to all α-cuts of the fuzzy numbers:

$$A \leq_i B \;\Leftrightarrow\; A_\alpha \leq_i B_\alpha \quad \forall \alpha \in (0,1]\ . \tag{4}$$

For some studies, it is fully sufficient and it can be very helpful, e.g., in the investigation of monotone fuzzy rule bases [18], [20], [29] and their interpolativity [21], [22]. However, it is not a total ordering and, for example, two fuzzy numbers depicted on Figure 1 are not comparable, although one would intuitively expect that A is smaller than B, and the arithmetic confirms that there is a significant difference between them.

Bodenhofer [5], [6] noticed this weakness and proposed a sort of widening of the fuzzy numbers by constructing their extensional hulls with respect to wider similarity relations. The resulted extensional hulls are often already comparable using the interval ordering. Still, we are on the level of crisp ordering. In order to capture the vagueness, Bodenhofer also developed an ordering preserving some degree up to which two fuzzy sets are ordered. This interesting concept is very intuitive and inspiring, however, for the metrics, preservation of some type of binarity is essential. Therefore, we stem from the Bodenhofer's extension to hulls and we "store" the vagueness in the truth and false values.

Mathematically, we construct "fuzzy truth values" and "fuzzy falses" as extensional hulls of 0 and 1 and the designed ordering will be a mapping from pairs of extensional fuzzy numbers to these extended truth/false values. Due to the bipolarity, that is, because we construct extensional hulls of 0 (truth) and 1 (false), we will refer to 0 and 1 as the "Boolean" truth values, and as their extensional hulls use similarities from the set $\mathcal{C}$, we will refer to them as the $\mathcal{C}$-Boolean valued truth values.

Definition 3.1: [15] Consider a system $\mathcal{C}$ of nested $\otimes$-similarities on $\mathbb{R}$ and $S \in \mathcal{C}$. The fuzzy sets $\mathrm{T}_S, \mathrm{F}_S \in \mathcal{F}([0,1])$:

$$\mathrm{T}_S = \mathrm{EXT}_S(1) \ , \tag{5}$$

$$\mathrm{F}_S = \mathrm{EXT}_S(0) \tag{6}$$

will be called *$\mathcal{C}$-Boolean valued truth*, and *$\mathcal{C}$-Boolean valued false*, respectively. Furthermore, we will denote the sets:

$$\mathrm{T}_\mathcal{C} = \{\mathrm{EXT}_S(1) \mid S \in \mathcal{C}\} \ , \tag{7}$$

$$\mathrm{F}_\mathcal{C} = \{\mathrm{EXT}_S(0) \mid S \in \mathcal{C}\} \ , \tag{8}$$

$$\mathrm{TF}_\mathcal{C} = \mathrm{T}_\mathcal{C} \cup \mathrm{F}_\mathcal{C} \ . \tag{9}$$

Now, we can define the ordering as a mapping to the set of fuzzy truth/false values.

Definition 3.2: [15] Let $(\mathcal{F}_\mathcal{C}(\mathbb{R}), +, -)$ be an MI-pregroup of extensional fuzzy numbers with respect to a system $\mathcal{C}$ of nested $\otimes$-similarities on $\mathbb{R}$. A mapping $\leq_\mathcal{C} \colon \mathcal{F}_\mathcal{C}(\mathbb{R}) \times \mathcal{F}_\mathcal{C}(\mathbb{R}) \to \mathrm{TF}_\mathcal{C}$ is called *$\mathcal{C}$-Boolean valued ordering* if:

(i) $(a_S \leq_\mathcal{C} a_T) \in \mathrm{T}_\mathcal{C}$, (reflexivity)

(ii) $(a_S \leq_\mathcal{C} b_T) \in \mathrm{T}_\mathcal{C} \ \& \ (b_T \leq_\mathcal{C} a_S) \in \mathrm{T}_\mathcal{C} \Rightarrow (a_S - b_T) \in I^0_\mathcal{C}$, (anti-symmetry)

(iii) $(a_S \leq_\mathcal{C} b_T) \in \mathrm{T}_\mathcal{C} \ \& \ (b_T \leq_\mathcal{C} c_R) \in \mathrm{T}_\mathcal{C} \Rightarrow (a_S \leq_\mathcal{C} c_R) \in \mathrm{T}_\mathcal{C}$, (transitivity).

Convention 3.1: If there will be no need to consider the particular width of the truth or false, we will use the following shorter notation: proposed in [15]:

$$a_S \leq_\mathcal{C} b_T \text{ denotes } (a_S \leq_\mathcal{C} b_T) \in \mathrm{T}_\mathcal{C} \ ,$$
$$a_S \not\leq_\mathcal{C} b_T \text{ denotes } (a_S \leq_\mathcal{C} b_T) \in \mathrm{F}_\mathcal{C} \ .$$

Let us recall some examples from [8] foreshadowed already in [15].

Let $(\mathcal{F}_\mathcal{C}(\mathbb{R}), +, -)$ be an MI-pregroup. Then

$$a_S \leq_{\max} b_T = \begin{cases} \mathrm{T}_{\max(S,T)}, & \text{if } a \leq b, \\ \mathrm{F}_{\max(S,T)}, & \text{otherwise.} \end{cases}$$

is a $\mathcal{C}$-Boolean valued ordering.

Mapping $\leq_{\max}$ demonstrates a $\mathcal{C}$-Boolean valued ordering reflecting the arithmetic operations: the width of the result of the arithmetic operations determines also the width of the fuzzy truth of the order. If we mirror this idea into particular arithmetics from Example 2.2, we get the following:

$$a_{S_p} \leq_{\max} b_{S_{p'}} = \begin{cases} \mathrm{T}_{S_{p''}}, \ p'' = \min\{p, p'\}, & \text{if } a \leq b \\ \mathrm{F}_{S_{p''}}, \ p'' = \min\{p, p'\}, & \text{otherwise,} \end{cases}$$

which is, for the system of Łukasiewicz similarities, demonstrated on Figure 2.

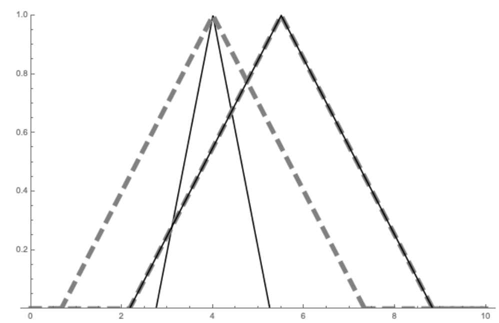

Fig. 2. Fuzzy numbers $x_{S_{p'}}$ and $y_{S_{p''}}$ determined by the Łukasiewicz similarities from Example 2.2 based on values $x = 4$ and $p' = 0.8$ ("left" solid fuzzy set), $y = 5.5$ and $p'' = 0.3$ ("right" solid fuzzy set). Fuzzy sets $x_{S_{p'}}$ and $y_{S_{p''}}$ cannot be ordered by $\leq_i$ of α-cuts however, if we use $\leq_{\max}$, we obtain the dashed fuzzy sets $\mathrm{EXT}_{S_{p''}}(x_{S_{p'}})$ and $\mathrm{EXT}_{S_{p''}}(y_{S_{p''}})$, respectively, and the subsequent conclusion $x_{S_{p'}} \leq_{\max} y_{S_{p''}} = \mathrm{T}_{S_{p'''}}$.

Another example of a $\mathcal{C}$-Boolean valued orderings requires the assumption that $\mathcal{C}$ has the greatest element $\top_\mathcal{C}$ (i.e. $R \subseteq \top_\mathcal{C}$ for all $R \in \mathcal{C}$). Then $\leq_{\top_\mathcal{C}} \colon \mathcal{F}_\mathcal{C}(\mathbb{R}) \times \mathcal{F}_\mathcal{C}(\mathbb{R}) \to \mathrm{TF}_\mathcal{C}$ may be constructed as follows:

$$a_S \leq_{\top_\mathcal{C}} b_T = \begin{cases} \mathrm{T}_{\top_\mathcal{C}}, & \text{if } \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T), \\ \mathrm{F}_{\top_\mathcal{C}}, & \text{otherwise.} \end{cases}$$

is a $\mathcal{C}$-Boolean valued ordering too.

Let us consider, e.g., the product t-norm and

$$\mathcal{C} = \{S_p \mid p \in [1,5] \text{ and } S_p(x,y) = e^{-p|x-y|}\} \ .$$

The ordering $\leq_{\top_\mathcal{C}}$ can be visually demonstrated by Figure 3 where one can see two extensional fuzzy numbers $4_{S_{2.5}}$ and $5.5_{S_{1.2}}$ (displayed by solid lines) and their extensional hulls $\mathrm{EXT}_{S_5}(4_{S_{2.5}}) = 4_{S_5}$ (left dashed fuzzy set) and $\mathrm{EXT}_{S_5}(5.5_{S_{1.2}}) = 5.5_{S_5}$ (right dashed fuzzy set) that due to their interval ordering $4_{S_5} \leq_i 5.5_{S_5}$ allow to order the original fuzzy numbers $4_{S_{2.5}} \leq_{\top_\mathcal{C}} 5.5_{S_{1.2}}$.

Assume that $\mathcal{C}$ has all infima, i.e., that $\inf\{S \in \mathcal{C} \mid S \in D\} \in \mathcal{C}$ exists for any $D \subseteq \mathcal{C}$. Then, we can define the ordering that seeks for the "narrowest" similarity relation that is sufficient in order to get interval-ordered extensional hulls of the given fuzzy numbers. In particular, such a $\mathcal{C}$-Boolean

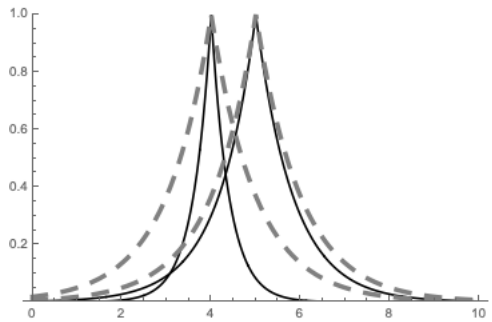

Fig. 3. Demonstration of the $\mathcal{C}$-Boolean valued ordering $\leq_{\top_{\mathcal{C}}}$.

valued ordering, denoted by $\leq_{\inf}$, will seek for the intersection of all such similarities:

$$a_S \leq_{\inf} b_T = \begin{cases} \mathrm{T}_E, \ \exists R \in \mathcal{C} : \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T), \\ \mathrm{F}_{\perp_{\mathcal{C}}}, \ \text{otherwise,} \end{cases}$$

where $E = \inf\{R \in \mathcal{C} \mid \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T)\}$. For the illustration of the ordering, see Figure 4.

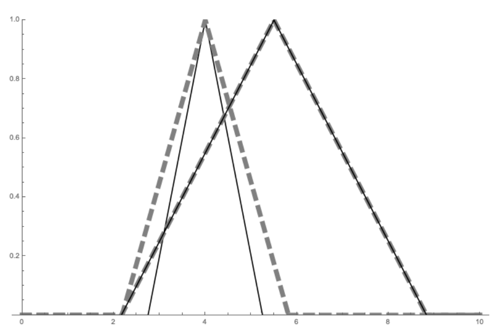

Fig. 4. Example of $\leq_{\inf}$ applied to fuzzy sets from Figure 2.

In [15], the authors introduced the so-called pre-order compatibility of a $\mathcal{C}$-Boolean valued ordering as one of the crucial properties. Let us recall the modified definition.

Definition 3.3: Let $\leq_{\mathcal{C}}$ be $\mathcal{C}$-Boolean valued ordering on an MI-pregroup $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$. The ordering $\leq_{\mathcal{C}}$ is called *pre-order compatible* if, for any $a_S, b_T \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and $R \in \mathcal{C}$,

$$(a_S \leq_{\mathcal{C}} b_T) = \mathrm{T}_R \ \Rightarrow \ \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T). \tag{10}$$

The pre-order compatibility is a property ensuring that the ordering is well-behaving in a sense elaborated by Bodenhofer [5], [6]. By that we mean: reflecting the necessary extension in the $\mathcal{C}$-Boolean valued truth respecting the order on $\mathbb{R}$.

The above recalled examples of $\mathcal{C}$-Boolean valued orderings were pre-order compatible. Let us show two cases of orderings that are not pre-order compatible, the first one harms the reflection of the necessary extension in the fuzzy truth:

$$a_S \leq_{\perp_{\mathcal{C}}} b_T = \begin{cases} \mathrm{T}_{\perp_{\mathcal{C}}}, & \text{if } a \leq b, \\ \mathrm{F}_{\perp_{\mathcal{C}}}, & \text{otherwise,} \end{cases}$$

the other one harms the expected respecting of the linear order on $\mathbb{R}$ because it reverses the order:

$$a_S \leq_{\max}^{rev} b_T = \begin{cases} \mathrm{T}_{\max(S,T)}, & \text{if } b \leq a \\ \mathrm{F}_{\max(S,T)}, & \text{otherwise.} \end{cases}$$

The ordering $\leq_{\perp_{\mathcal{C}}}$ is only a slight modification of $\leq_{\top_{\mathcal{C}}}$ (i.e., $\top_{\mathcal{C}}$ is replaced by $\perp_{\mathcal{C}}$) and it means that all the axioms of $\mathcal{C}$-Boolean valued ordering given by Definition 3.2 are clearly satisfied. But the problem is that $\mathrm{T}_{\perp_{\mathcal{C}}}$ does not reflect at all how much we had to extend the compared fuzzy numbers in order to obtain their extensional hulls that would be ordered according to their intervals. Thus, the fuzzy truth value is meaningless, it does not store any information. The other example, namely $\leq_{\max}^{rev}$, stores the necessary extension in the truth values but it ranks "about 4" below "about 2".

Though the pre-order compatibility is a very important property, later on, it turned out that for some proofs it is not sufficient. Much stronger would be to require the equivalence instead of the implication in (10). However, such a condition would be too restrictive and the only order that would in general meet such a requirement would be the $\leq_{\inf}$ which would reduce the investigation to a redundant game with classes of orderings instead of a single sample.

However, it is not necessary. As we will show below, the opposite implication need not reflect the particular similarity and it is fully sufficient to require that the interval order of extensional hulls of two extensional fuzzy numbers implies that these fuzzy numbers are (arbitrarily) ordered by $\leq_{\mathcal{C}}$.

Definition 3.4: Let $\leq_{\mathcal{C}}$ be $\mathcal{C}$-Boolean valued ordering on an MI-pregroup $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$. The ordering $\leq_{\mathcal{C}}$ is called *real-order compatible* if, for any $a_S, b_T \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and $R \in \mathcal{C}$,

$$\mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T) \Rightarrow (a_S \leq_{\mathcal{C}} b_T) \in \mathrm{T}_{\mathcal{C}}\,. \tag{11}$$

Moreover, if it is pre-order compatible and also real-order compatible, it is said to be *strongly compatible*.

Let us note that the above-shown cases $\leq_{\max}, \leq_{\top_{\mathcal{C}}}$, and $\leq_{\inf}$ are strongly compatible and thus, they constitute examples of well-behaving $\mathcal{C}$-Boolean valued orderings.

IV. Properties of $\mathcal{C}$-Boolean valued orderings

A. *Wang-Kerre properties of orderings of fuzzy numbers*

There are distinct studies developing orderings of fuzzy numbers. Many of them rely on the determination of indices that are compared instead of comparing the fuzzy numbers themselves [28]. In order to avoid partiality, some authors suggest the use of multiple indices. It is not our goal to consider and compare the dozens of existing methods for rankings of fuzzy numbers as each of them has its own motivation. The goal we have is to check whether our approach, that was motivated by developing metric-like functions on extensional fuzzy numbers and joining the topics of arithmetics of fuzzy numbers and their rankings under the common umbrella allowing to develop well-founded mathematical analysis on these objects, preserve the most natural properties. In our task, we stem from the work of Wang and Kerre [28], [30] who set up the most intuitive properties of orderings of fuzzy numbers and made a detailed analysis of existing methods to find out, whether they preserve them or not. Our approach for constructing the orderings cannot stay out of this test of the preservation.

Let us briefly recall the properties established by Wang and Kerre for any ordering $\preceq$ as the expected ones. Note, that the orderings based on indices often operate on a set of fuzzy numbers, say on a finite subset $\mathcal{G}$ of the set of all fuzzy numbers to which the ordering can be applied. It is often because the construction of distinct orderings (indices) compares the two ranked fuzzy numbers to a reference set.

Wang-Kerre properties: Let $\mathcal{F}(\mathbb{R})$ be the set of fuzzy number on $\mathbb{R}$ and $\mathcal{G}, \mathcal{G}_1, \mathcal{G}_2$ be arbitrary finite subsets of $\mathcal{F}(\mathbb{R})$. Consider $\mathrm{Supp}(A) = \{x \in \mathbb{R} \mid A(x) \geq 0\}$. Then

A1: $A \preceq A$, for $A \in \mathcal{G}$;
A2: $A \preceq B$ and $B \preceq A \Rightarrow A \sim B$, for $A, B \in \mathcal{G}$;
A3: $A \preceq B$ and $B \preceq C \Rightarrow A \preceq C$, for $A, B, C \in \mathcal{G}$;
A4: $\sup \mathrm{Supp}(A) \leq \inf \mathrm{Supp}(B) \Rightarrow A \preceq B$, for $A, B \in \mathcal{G}$;
A5: Let $A, B \in \mathcal{G}_1 \cap \mathcal{G}_2$. Then
$A \preceq B$ in $\mathcal{G}_1 \Leftrightarrow A \preceq B$ in $\mathcal{G}_2$;
A6: Let $A \preceq B$ in $\{A, B\}$. Then
$A + C \preceq B + C$ in $\{A + C, B + C\}$;
A7: Let $0 \leq h$ and $A, B, h \cdot A, h \cdot B \in \mathcal{G}$. Then
$A \preceq B$ in $\{A, B\}$ implies $h \cdot A \preceq h \cdot B$ in $\{h \cdot A, h \cdot B\}$.

Remark 4.1: Note, that properties A4 and A6 were published also in their strict variants A4' and A6' working with the strict ordering $\prec$ instead or $\preceq$.

For the moment, we may abstract from the precise definitions of the symbols $\preceq$ and $\sim$ as we need to accompany the meaning of the Wang-Kerre axioms into our formalism anyhow, and we believe, that the meaning of the axioms is very intuitive for the readers.

B. Preservation of the Wang-Kerre properties by $\mathcal{C}$-Boolean valued orderings

Let us consider the above recalled Wang-Kerre axioms rewritten into the formalism of extensional fuzzy numbers and their orderings $\leq_{\mathcal{C}}$, i.e., let us consider the following properties for arbitrary $a_S, b_T, c_R \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$.:

A1$_{\mathcal{C}}$: $a_S \leq_{\mathcal{C}} a_S$;
A2$_{\mathcal{C}}$: $a_S \leq_{\mathcal{C}} b_T$ and $b_T \leq_{\mathcal{C}} a_S \;\Rightarrow\; (a_S - b_T) \in I^0_{\mathcal{C}}$;
A3$_{\mathcal{C}}$: $a_S \leq_{\mathcal{C}} b_T$ and $b_T \leq_{\mathcal{C}} c_R \Rightarrow a_S \leq_{\mathcal{C}} c_R$;
A4$_{\mathcal{C}}$: $\sup \mathrm{Supp}(a_S) \leq \inf \mathrm{Supp}(b_T) \;\Rightarrow\; a_S \leq_{\mathcal{C}} b_T$;
A6$_{\mathcal{C}}$: $a_S \leq_{\mathcal{C}} b_T \;\Rightarrow\; a_S + c_R \leq_{\mathcal{C}} b_T + c_R$;
A7$_{\mathcal{C}}$: $0 \leq h$ and $a_S \leq_{\mathcal{C}} b_T \;\Rightarrow\; h \cdot a_S \leq_{\mathcal{C}} h \cdot b_T$;

Remark 4.2: Note that A5$_{\mathcal{C}}$ is not introduced as modification of A5 makes no sense. The Wang-Kerre properties were defined in such a way as many indices ranking fuzzy numbers were dependent on the chosen reference set. The suggested orderings $\leq_{\mathcal{C}}$ of extensional fuzzy numbers do not have any reference set and always operate on a non-reduced set of all fuzzy reals.

Now, we will focus on the preservation of A1$_{\mathcal{C}}$-A4$_{\mathcal{C}}$, A6$_{\mathcal{C}}$, and A7$_{\mathcal{C}}$. First of all, let us present an important lemma.

Lemma 4.1: Let $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup such that $\leq_{\mathcal{C}}$ is pre-order compatible. If $a_S \leq_{\mathcal{C}} b_T$ for certain $S, T \in \mathcal{C}$ then $a \leq b$.
Proof: Since $\leq_{\mathcal{C}}$ is pre-order compatible, then $a_S \leq_{\mathcal{C}} b_T$ implies the existence of $R \in \mathcal{C}$ such that

$$\mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T).$$

Consider $\alpha = 1$, and assume that

$$\mathrm{EXT}_R(a_S)(z) = \bigvee_{y \in \mathbb{R}} a_S(y) \otimes R(y, z) = 1$$

for a certain $z \in \mathbb{R}$. But

$$1 = \bigvee_{y \in \mathbb{R}} a_S(y) \otimes R(y, z) = \bigvee_{y \in \mathbb{R}} S(a, y) \otimes R(y, z) \leq$$
$$\bigvee_{y \in \mathbb{R}} \max(S, R)(a, y) \otimes \max(S, R)(y, z) \leq \max(S, R)(a, z),$$

where $\max(S, R) \in \mathcal{C}$ since $\mathcal{C}$ is a nested system. As a consequence of separability property, we find that $\max(S, R)(a, z) = 1$ if and only if $a = z$. Thus, we have $\mathrm{EXT}_R(a_S)_1 = \{a\}$ and similarly $\mathrm{EXT}_R(b_T)_1 = \{b\}$. From $\mathrm{EXT}_R(a_S)_1 \leq \mathrm{EXT}_R(b_T)_1$, we obtain $a \leq b$, which concludes the proof. □

Proposition 4.2: Let $\leq_{\mathcal{C}}$ be $\mathcal{C}$-Boolean valued ordering on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$. Then properties A1$_{\mathcal{C}}$, A2$_{\mathcal{C}}$, and A3$_{\mathcal{C}}$ are satisfied.
Sketch of the proof: Directly follows from Definition 3.2. □

Proposition 4.3: Let $\leq_{\mathcal{C}}$ be a real-order compatible $\mathcal{C}$-Boolean valued ordering. Then A4$_{\mathcal{C}}$ is satisfied.
Sketch of the proof: From $\sup \mathrm{Supp}(a_S) \leq_{\mathcal{C}} \inf \mathrm{Supp}(b_T)$ we derive $a_S \leq_i b_T$ and thus, there exists $R \in \mathcal{C}$ such that

$$\mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T).$$

Then the assumption of the real-order compatibility suffices to ensure that $a_S \leq_{\mathcal{C}} b_T$. □

Proposition 4.4: Let $\leq_{\mathcal{C}}$ be a strongly compatible $\mathcal{C}$-Boolean valued ordering on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$. Then A6$_{\mathcal{C}}$ is satisfied.
Sketch of the proof: From $a_S \leq_{\mathcal{C}} b_T$ and the pre-order compatibility, we get $a \leq b$ which implies $a + c \leq b + c$ and consequently with help of the real-order compatibility, we may prove $(a + c)_{\max(S,R)} \leq_{\mathcal{C}} (b + c)_{\max(T,R)}$ which completes the proof. □

Proposition 4.5: Let $\leq_{\mathcal{C}}$ be a strongly compatible $\mathcal{C}$-Boolean valued ordering on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ with the crisp bottom element. Then A7$_{\mathcal{C}}$ is satisfied.
Sketch of the proof: The crisp bottom element in $\mathcal{C}$ (crisp equality) allows to deal with scalar multiplication:

$$h \cdot a_S = h_{\perp_{\mathcal{C}}} \cdot a_S = (h \cdot a)_S$$

which together with the strong compatibility suffices to prove the property. □

V. Conclusions – Links to Related and Future Studies

The $\mathcal{C}$-Boolean valued orderings have been designed and their properties investigated. The importance of this step is many-fold. First of all, the above mentioned motivation to define metric-like functions operating on extensional fuzzy numbers and mapping again back to extensional fuzzy numbers. The intuitiveness of such an approach has been discussed above and mainly in details in [8]. Indeed, if the metrics will reflect human intuition in a sense that distance between "about a" and "about b" will be equal to the absolute value of their difference |"about a" - "about b"|, the approached based on such metric will reflect human expectations. So, the mathematical analysis that is intended to be developed on such foundations should be as intuitive and reliable as the standard mathematical analysis based on real numbers, however, with the ability to capture and process vagueness (imprecision, tolerance) through the whole calculations. Thus, the next steps in this direction should go towards limits, convergences, approximation methods, numerical algorithms, and related problems.

Apart form focusing on mathematical analysis and related subjects, already the ordering itself is an important notion that leads to crucial studies. Recall, e.g., the studies on monotonicity fuzzy rule based systems [17], [18]. Assuming some ordering of fuzzy sets, it is rather intuitive to define a monotonous fuzzy rule bases, i.e., a fuzzy rule bases capturing the monotonous dependency of consequents on antecedents. Then, the crucial thing is to investigate, whether this monotonicity is preserved also for the resulting function, i.e., whether the defuzzified outputs preserve the ordering of the crisp inputs [20], [29].

The above considered monotonicity has been studied based on the interval ordering of α-cuts. The natural question is whether the monotonicity can be studied also more generally for the $\mathcal{C}$-Boolean valued orderings, that tolerate also small harms of the interval orderings as long as the extensional hulls are correctly ordered. Moreover, such a general setting may also allow to study the preservation of the monotonicity in the case of fuzzy inputs and defuzzification employed at the end of the inference process.

These problems are also closely related to the interpolativity that also works with fuzzy inputs and investigates, for more details regarding the interpolativity of monotone fuzzy rule bases, we refer readers to [21], [22]. However, also the interpolativity investigations stemmed from the interval ordering of α-cuts that is too restrictive compared the proposed concept of $\mathcal{C}$-Boolean valued ordering.

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