A Parabolic Based Fuzzy Data Envelopment Analysis Model with an Application

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***Abstract*—A Fuzzy Data Envelopment Analysis (FDEA) is a popular technique to measure the relative efficiency of decision-making units (DMUs) with imprecise and vague data for multiple inputs and outputs. In real-life applications, there are two types of outputs: desirable outputs and undesirable outputs. In this paper, we have proposed a new version of FDEA model, named as Parabolic based Fuzzy Data Envelopment Analysis (PFDEA) model that computes parametric efficiency of a DMU in the presence of undesirable outputs. The inputs and outputs are represented in the form of asymmetric parabolic fuzzy numbers in the proposed model. A new technique is introduced to convert PFDEA model into a linear programming problem using $\alpha-$ cut approach with a novel section formula based method, named as Ratio Division Method. This method is used to perform the complete ranking of the DMUs in a numerical example using Cross-Efficiency Method to provide a complete ranking of the DMUs.**

***Index Terms*—Data envelopment analysis, Fuzzy data envelopment analysis, Undesirable outputs, Fuzzy arithmetic operations**

I. Introduction

Lotfi A. Zadeh generalized the classical set theory by introducing Fuzzy Set (FS) theory [40]. The fuzzy sets, which are defined on the set $\mathbb{R}$ of real numbers, are known as Fuzzy Numbers (FNs). A fuzzy number is the special types of fuzzy set which are normal, compact and convex. The most commonly used fuzzy numbers are triangular, trapezoidal, and parabolic of membership functions. Since the fuzzy numbers describe the physical world situations more realistically than the crisp numbers, therefore, they are utilized to solve various real-world problems such as in image processing [3], machine learning [11], data envelopment analysis (DEA) [5], and many others.

Data envelopment analysis (DEA) is a non-parametric linear programming technique to construct the experimental production frontiers, to evaluate the relative efficiency of the organizations under study or decision-making units (DMUs) with multiple inputs and outputs data. DEA, which was proposed by Charnes et al. [5] has been widely, implemented in several contexts, such as the efficiency of hospitals in providing their services [19] manufacturing efficiency [28] productivity of OECD countries [9]. The a recent theoretical survey in DEA has been discussed in [10].

Initially, in the traditional DEA, such as CCR and BCC models, all inputs and outputs values of DMUs are considered as specific numerical values or crisp numbers. However, in real-world applications, many complicated factors are involved, which results to imprecision and vagueness in the data. To deal with such uncertain information, the notion of fuzziness was introduced in the Data Envelopment Analysis model and is popularly known as Fuzzy Data Envelopment Analysis (FDEA) model. Several techniques have been developed in FDEA to deal with uncertainty in data. For instance, based on $\alpha-$ level approaches, Kao and Liu [18] formulated a pair of parametric programs to derive bounds on the membership functions of the efficiency score of the fuzzy BCC model and applied their work to the efficiency score of 24 university libraries in Taiwan. Saati et al. [30] suggested a fuzzy CCR model as a possibilistic programmings problems. Wang, Luo, and Liang [37] constructed the FDEA models from the perspective of fuzzy arithmetic operations and applied them to the manufacturing enterprise. Angiz et al. [2] used a discrete approach based on a fuzzy CCR model. Puri and Yadav [26] formalized a concept of fuzzy input mix-efficiency in FDEA and applied for their work in the banking sector. All the mentioned researcher utilized the triangular-fuzzy numbers (TFNs).

The assumption of standard DEA models, which proposed by Charnes et al. [5] are for desirable data. However, in real-life applications, the undesirable inputs and/or outputs might be present in the manufacturing process, which also needs to be counted. In order to deal with undesirable outputs in DEA, Scheel [32] coordinated direct and indirect approaches in DEA. Indirect approaches transforms the undesirable outputs by a monotone decreasing function to make it desirable; it can be seen in various studies of Scheel [32], Seiford [33] and Liu et al. [23]. On the other side, direct approaches avoid the transformation process and incorporate the undesirable outputs directly into the DEA and FDEA models. The literature of direct approaches on DEA and FDEA are discussed in [20] and [26] respectively. DEA models have been extended to various new version [13], [27], [31] to handle real-world applications of imprecise and undesirable data. In [20], the desirable and undesirable outputs are considered as a weighted sum in the optimization function of the DEA model. The positive weights

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are allotted to desirable outputs, whereas negative weights are given for undesirable outputs, to increase the efficiency of the DMUs. The model has been named as the DEA model with undesirable output (DEA-UO). Meanwhile, to deal with the impreciseness and vagueness of complex data, a fuzzy version of the DEA models have been introduced in the literature [28], [29] and the model has been named as FDEA-UO.

DEA as well as FDEA models are producing more than one efficient DMUs and no further discrimination between them. Therefore; the major draw back of DEA and FDEA models are the lack of discrimination power. To increase the discrimination power of DEA and FDEA models, there are various accessible approaches in the literature, (see Adler, Friedman, and Sinuany-Stern [1], Liang et al. [22], Jahanshahloo, Lofti, Khanmohammadi, Kazemimanesh, and Rezaie [15], Wu, Sun, Liang, and Zha [35] and Guo and Wu [13]), and Sexton et al. [34]. One among them is the cross-efficiency technique, which is characterized to evaluate the efficiency score of each DMUs, n times via using the optimal weights of n linear programming (LP). In another word the DMUs are self and peer evaluated.

In the above literature, various approaches have been proposed to deal with inefficient DMUs for the imprecise and vague dataset. The efficiency of a DMU is improved by maximizing its output value and by minimizing its input value to approach the efficient-frontier. Therefore, study on performance of the parameter α is done to find the least value of the input and maximum value of the output to improve the efficiency of the DMU. The α-cut approach converts the fuzzy number based data into interval data. Most of the researchers have used triangular fuzzy numbers to deal uncertain data. In the paper, parabolic fuzzy numbers are used to deal the uncertainty of the data. A PFN extends the input and output intervals, thereby providing the smallest lower bound for the input value and largest upper bound for the output value and hence making the DMU more efficient. Any inefficient DMU corresponding to PFNs can reach faster on the efficient-frontier rather than using TFNs. In addition, we have proposed a parametric FDEA model to deal undesirable output named as Parabolic based Fuzzy Data Envelopment Analysis with Undesirable Outputs (PFDEA-UO) model. A new technique named as Ratio division is also introduced to convert the input and output intervals for each α to obtain their corresponding crisp numbers for solving their corresponding linear programming model and perform the complete ranking of the DMUs.

The rest of the paper is organized into four sections. Section II outlines the essential related preliminaries of the proposed work. In section III, a parametric FDEA model has been proposed, named as Parabolic Fuzzy Data Envelopment Analysis with undesirable outputs (PFDEA-UO) model with a novel defuzzification technique called Ratio Division Method. Section IV provides the experiment of the proposed model with its analysis using cross efficiency method. Finally, section V states the conclusion of the paper.

II. Preliminaries

This section gives some basic definitions and some mathematical formulations of concepts used in the paper.

A. *Definition:* ***Triangular Fuzzy Number*** *[40]*

A triangular fuzzy number (TFN) denoted by $\tilde{A}_1$ is a triplet (a^L, a^M, a^U) defined by the membership function $\mu_{\tilde{A}_1} : X \to [0,1]$ given by:

$$\mu_{\tilde{A}_1}(x) = \begin{cases} \frac{x-a^L}{a^M-a^L} & , a^L < x \le a^M \\ \frac{x-a^U}{a^M-a^U} & , a^M \le x < a^U \\ 0 & , otherwise \end{cases} \qquad (1)$$

B. *Definition:* ***Parabolic Fuzzy Number*** *[40]*

A parabolic fuzzy number (PFN) denoted by $\tilde{A}_2$ is a (a^L, a^M, a^U) defined by the membership function $\mu_{\tilde{A}_2} : X \to [0,1]$ given by:

$$\mu_{\tilde{A}_2}(x) = \begin{cases} \frac{(x-a^L)(x-2a^M+a^L)}{(a^M-a^L)(a^L-a^M)} & , a^L < x \le a^M \\ \frac{(x-a^U)(x-2a^M+a^U)}{(a^M-a^U)(a^U-a^M)} & , a^M \le x < a^U \\ 0 & , otherwise \end{cases} \qquad (2)$$

The triangular and parabolic fuzzy number can be illustrated in the Fig 1 as:

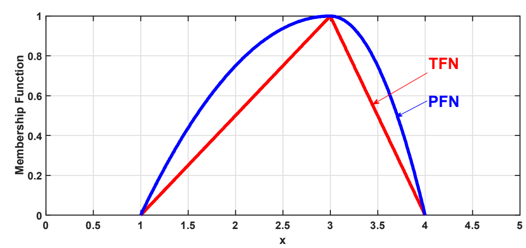

Fig. 1. Triangular and Parabolic Fuzzy number

C. *Definition:* $\alpha-$***cut Approach*** *[40]*

Let $\tilde{A}$ be a fuzzy set defined on universe discourse X. Let α be a parameter $\in [0,1]$, then the $\alpha-$ cut for $\tilde{A}$ can be defined as crisp sets as follows:

$$\tilde{A}_\alpha = \{x | \tilde{A}(x) \ge \alpha\} \qquad (3)$$

D. ***Arithmetic Operations on Fuzzy Numbers*** *[36]*

Suppose $\tilde{A} = (a^L, a^M, a^U)$ and $\tilde{B} = (b^L, b^M, b^U)$ be two positive triangular or parabolic fuzzy numbers. So, the fuzzy arithmetic operations are defined as:

$$\tilde{A} + \tilde{B} = (a^L + b^L, a^M + b^M, a^U + b^U) \qquad (4)$$

$$\tilde{A} - \tilde{B} = (a^L - b^U, a^M - b^M, a^U - b^L) \ , a^L > b^U \qquad (5)$$

$$\tilde{A} \cdot \tilde{B} = (a^L \cdot b^L, a^M \cdot b^M, a^U \cdot b^U) \qquad (6)$$

$$\tilde{A}/\tilde{B} = (\frac{a^L}{b^U}, \frac{a^M}{b^M}, \frac{a^U}{b^L}) \qquad (7)$$

E. *Fuzzy Data Envelopment Analysis (FDEA) Model [30]*

Suppose there are n decision making units (DMUs), with m different inputs to produce s different outputs. Let $X \in R^{m\times n}, Y \in R^{s\times n}$ be the input matrix and output matrix respectively. Consequently, for the $k^{th}, (k = 1, 2\cdots, n)$ DMU, the $\hat{x}_{ik}, (i = 1, 2, \cdots, m)$ and $\hat{y}_{rk}, (r = 1, 2, \cdots, s)$ are non-negative fuzzy inputs and fuzzy outputs respectively. Hence the mathematical formulation is as below:

$$
\begin{aligned}
\max \hat{E}_k =& \frac{\sum_{r=1}^{s} u_{rk}\hat{y}_{rk}}{\sum_{i=1}^{m} v_{ik}\hat{x}_{ik}} \\
s.t: & \frac{\sum_{r=1}^{s} u_{rj}\hat{y}_{rj}}{\sum_{i=1}^{m} v_{ij}\hat{x}_{ij}} \le \hat{1},\ \forall j = 1, 2, \cdots, n. \\
& u_{rk} \ge 0\ \forall r\ ,\ v_{ik} \ge 0 \forall i
\end{aligned} \tag{8}
$$

Where u_{rk} is the r^{th} weight of output and v_{ik} is the i^{th} weight of input.

F. *Cross-Efficiency Method [34]*

The cross-efficiency technique is a square matrix of $n \times n$, which is deduced from the calculation of the efficiency value of each DMU n times via using the optimal weights of n linear programming, and could be obtained through;

$$
\begin{aligned}
h_{kj} =& \frac{\sum_{r=1}^{s} u_{rk} y_{rj}}{\sum_{i=1}^{m} v_{ik} x_{ij}}, \forall k, j. \\
& k = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, n.
\end{aligned} \tag{9}
$$

Where u_{rk} is the r^{th} weight of output and v_{ik} is the i^{th} weight of input, similarly y_{rj} is the r^{th} output of j^{th} DMU and x_{ij} is the i^{th} input of j^{th} DMU.

III. Proposed Parabolic based Fuzzy Data Envelopment Analysis with Undesirable Outputs (PFDEA-UO) Model

In this section, an improved version of Fuzzy Data Envelopment Anaysis model, named as Parabolic based Fuzzy Data Envelopment Analysis with Undesirable Outputs (PFDEA-UO) model. Suppose there are n decision making units (DMUs) with m different inputs to produce s different outputs (s_1 desirable outputs and s_2 undesirable outputs), such that $s_1 + s_2 = s$. Let $X \in R^{m\times n}, Y^g \in R^{s_1\times n}, Y^b \in R^{s_2\times n}$ be the input matrix, desirable output matrix, and undesirable output matrix respectively. Consequently, for the $k^{th}, (k = 1, 2\cdots, n)$ DMU, the $x_{ik} = (x_{ik}^L, x_{ik}^M, x_{ik}^U), (i = 1, 2\cdots, m)$, $y_{rk}^g = (y_{rk}^{gL}, y_{rk}^{gM}, y_{rk}^{gU}), (r = 1, 2, \cdots, s_1)$, and $y_{pk}^b = (y_{pk}^{bL}, y_{pk}^{bM}, y_{pk}^{bU}), (p = 1, 2, \cdots, s_2)$ are non-negative parabolic fuzzy inputs, parabolic fuzzy desirable outputs and parabolic fuzzy undesirable outputs respectively. Hence the proposed PFDEA-UO model is defined as below:

Model 1: PFDEA-UO Model

$$
\begin{aligned}
& \max\ E_k = \\
& \frac{\sum_{r=1}^{s1} u_{rk}^g (y_{rk}^{gL}, y_{rk}^{gM}, y_{rk}^{gU}) - \sum_{p=1}^{s2} u_{pk}^b (y_{pk}^{bL}, y_{pk}^{bM}, y_{pk}^{bU})}{\sum_{i=1}^{m} v_{ik}(x_{ik}^L, x_{ik}^M, x_{ik}^U)} \\
& s.t: \\
& \frac{\sum_{r=1}^{s1} u_{rj}^g (y_{rj}^{gL}, y_{rj}^{gM}, y_{rj}^{gU}) - \sum_{p=1}^{s2} u_{pj}^b (y_{pj}^{bL}, y_{pj}^{bM}, y_{pj}^{bU})}{\sum_{i=1}^{m} v_{ij}(x_{ij}^L, x_{ij}^M, x_{ij}^U)} \\
& \le (1, 1, 1), (j = 1, 2, \cdots, n). \\
& u_{rk}^g \ge 0\ \forall r\ ,\ u_{pk}^b \ge 0\ \forall p\ ,\ v_{ik} \ge 0 \forall i
\end{aligned} \tag{10}
$$

where u_{rk}^g is the r^{th} weight of desirable output, u_{pk}^b is p^{th} weight of undesirable output and v_{ik} is i^{th} weight of input.

The analysis of the proposed model for each DMU is performed on the basis of its Efficiency, E_k computed on various levels as follows:

1. Firstly, the Efficiency, E_k for kth DMU is calculated on the bounds of Parabolic fuzzy numbers, known as lower bound Efficiency E_k^L, middle bound Efficiency E_k^M and upper bound Efficiency E_k^U.
2. Secondly, the Efficiency, E_k for kth DMU is also calculated between the bounds of Parabolic fuzzy numbers for each input and output value of kth DMU for the complete analysis of the performance of the DMUs.

Consequently, based upon fuzzy arithmetic operations [36] and linear transformation [5], the following three linear programming (LP) models are obtained. The first derived model is termed as Lower-Efficiency with Undesirable Outputs (LE-UO) Model. Its mathematical formulation is given as:

Model 2: LE-UO Model

$$
\begin{aligned}
& \max E_k^L = \sum_{r=1}^{s_1} u_{rk}^g y_{rk}^{gL} - \sum_{p=1}^{s_2} u_{pk}^b y_{pk}^{bU} \\
& s.t: \\
& \sum_{i=1}^{m} v_{ik} x_{ik}^U = 1 \\
& \sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gU} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{bL} - \sum_{i=1}^{m} v_{ij} x_{ij}^L \le 0\ ,\ \forall j. \\
& \sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gL} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{bU} > 0\ ,\ \forall j. \\
& u_{rk}^g \ge 0\ \forall r\ ,\ u_{pk}^b \ge 0\ \forall p\ ,\ v_{ik} \ge 0 \forall i
\end{aligned} \tag{11}
$$

The second derived model is termed as Middle-Efficiency with Undesirable Outputs (ME-UO) Model . Its mathematical formulation is given as:

Model 3: ME-OU Model

$$\max E_k^M = \sum_{r=1}^{s_1} u_{rk}^g y_{rk}^{gM} - \sum_{p=1}^{s_2} u_{pk}^b y_{pk}^{bM}$$

$s.t:$

$$\sum_{i=1}^{m} v_{ik} x_{ik}^M = 1$$

$$\sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gM} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{bM} - \sum_{i=1}^{m} v_{ij} x_{ij}^M \leq 0 \ , \ \forall j. \quad (12)$$

$$\sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gM} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{bM} > 0 \ , \ \forall j.$$

$$u_{rk}^g \geq 0 \ \forall r \ , \ u_{pk}^b \geq 0 \ \forall p \ , \ v_{ik} \geq 0 \forall i$$

The third derived model is termed as Upper-Efficiency with Undesirable Outputs (UpE-UO) Model. Its mathematical formulation is given as:

Model 4: UpE-UO Model

$$\max E_k^U = \sum_{r=1}^{s_1} u_{rk}^g y_{rk}^{gU} - \sum_{p=1}^{s_2} u_{pk}^b y_{pk}^{bL}$$

$s.t:$

$$\sum_{i=1}^{m} v_{ik} x_{ik}^L = 1$$

$$\sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gU} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{bL} - \sum_{i=1}^{m} v_{ij} x_{ij}^L \leq 0 \ , \ \forall j. \quad (13)$$

$$\sum_{r=1}^{s_1} u_{rj}^g y_{rj}^{gL} - \sum_{p=1}^{s_2} u_{pj}^b y_{pj}^{gU} > 0 \ , \ \forall j.$$

$$u_{rk}^g \geq 0 \ \forall r \ , \ u_{pk}^b \geq 0 \ \forall p \ , \ v_{ik} \geq 0 \forall i$$

The efficiency E_k for k^{th} DMU is to be found out for the values of imprecise data lying between the bounds of the PFNs as well. To do so, $\alpha-$ cut approach is implemented that provides intervals for each input and output value of the DMUs. So, parabolic fuzzy numbers the interval obtained using the α-cut approach is given as:

$$x \in [a^M(1-\sqrt{1-\alpha}) + a^L\sqrt{1-\alpha}, a^M(1-\sqrt{1-\alpha}) + a^U\sqrt{1-\alpha}] \quad (14)$$

To find the least and the most admissible bounds for the inputs and the outputs data, the nested interval in (14) are transformed to crisp numbers using proposed Ratio division method.

Ratio Division Method: Let $[A, B]$ be an interval, then for $0 \leq R \leq 1$, we get x_A and x_B using basic section formula.

$$x_A = \frac{A + R \cdot B}{1 + R} \quad (15)$$

$$x_B = \frac{A \cdot R + B}{1 + R} \quad (16)$$

x_A and x_B are the points close to A and B, respectively. The middle point of the interval is obtained when $R = 1$. The interval given in (14) gives $A = a^M(1-\sqrt{1-\alpha}) + a^L\sqrt{1-\alpha})$ and $B = a^M(1-\sqrt{1-\alpha}) + a^U\sqrt{1-\alpha}$.

Using the Ratio division method, we get the crisp values for each input interval and output interval using parametric bounds for each PFN input and PFN output value mentioned in (14).

$$\tilde{x}_{ik}^L = x_{ik}^M(1-\sqrt{1-\alpha}) + \frac{(x_{ik}^L + R \cdot x_{ik}^U)\sqrt{1-\alpha}}{1+R} \quad (17)$$

$$\tilde{y}_{rk}^{gL} = y_{rk}^{gM}(1-\sqrt{1-\alpha}) + \frac{(y_{rk}^{gL} + R \cdot y_{rk}^{gU})\sqrt{1-\alpha}}{1+R} \quad (18)$$

$$\tilde{y}_{pk}^{bL} = y_{pk}^{bM}(1-\sqrt{1-\alpha}) + \frac{(y_{pk}^{bL} + R \cdot y_{pk}^{bU})\sqrt{1-\alpha}}{1+R} \quad (19)$$

$$\tilde{x}_{ik}^U = x_{ik}^M(1-\sqrt{1-\alpha}) + \frac{(R \cdot x_{ik}^L + x_{ik}^U)\sqrt{1-\alpha}}{1+R} \quad (20)$$

$$\tilde{y}_{rk}^{gU} = y_{rk}^{gM}(1-\sqrt{1-\alpha}) + \frac{(R \cdot y_{rk}^{gL} + y_{rk}^{gU})\sqrt{1-\alpha}}{1+R} \quad (21)$$

$$\tilde{y}_{pk}^{bU} = y_{pk}^{bM}(1-\sqrt{1-\alpha}) + \frac{(R \cdot y_{pk}^{bL} + y_{pk}^{bU})\sqrt{1-\alpha}}{1+R} \quad (22)$$

Using Eqn. (17)-(22), we obtain two corresponding linear programming models named as Lower Parametric Efficiency with Undesirable Outputs (LPrE-UO) Model and Upper Parametric Efficiency with Undesirable Outputs (UpPrE-UO) Model as follows:

Model 5: LPrE-UO Model

$$\max \tilde{E}_k^L = \sum_{r=1}^{s_1} u_{rk}^g \tilde{y}_{rk}^{gL} - \sum_{p=1}^{s_2} u_{pk}^b \tilde{y}_{pk}^{gU}$$

$s.t$

$$\sum_{i=1}^{m} v_{ik} \tilde{x}_{ik}^U = 1$$

$$\sum_{r=1}^{s_1} u_{rj}^g \tilde{y}_{rj}^{gU} - \sum_{p=1}^{s_2} u_{pj}^b \tilde{y}_{pj}^{gL} - \sum_{i=1}^{m} v_{ij} \tilde{x}_{ij}^L \leq 0 \ , \ \forall j. \quad (23)$$

$$\sum_{r=1}^{s_1} u_{rj}^g \tilde{y}_{rj}^{gL} - \sum_{p=1}^{s_2} u_{pj}^b \tilde{y}_{pj}^{gU} > 0 \ , \ \forall j.$$

$$u_{rk}^g \geq 0 \ \forall r \ , \ u_{pk}^b \geq 0 \ \forall p \ , \ v_{ik} \geq 0 \forall i$$

Model 6: UpPrE-UO Model

$$\max \tilde{E}_k^U = \sum_{r=1}^{s_1} u_{rk}^g \tilde{y}_{rk}^{gU} - \sum_{p=1}^{s_2} u_{pk}^b \tilde{y}_{pk}^{gL}$$

$s.t$

$$\sum_{i=1}^{m} v_{ik} \tilde{x}_{ik}^L = 1$$

$$\sum_{r=1}^{s_1} u_{rj}^g \tilde{y}_{rj}^{gU} - \sum_{p=1}^{s_2} u_{pj}^b \tilde{y}_{pj}^{gL} - \sum_{i=1}^{m} v_{ij} \tilde{x}_{ij}^L \leq 0 \ , \ \forall j. \quad (24)$$

$$\sum_{r=1}^{s_1} u_{rj}^g \tilde{y}_{rj}^{gL} - \sum_{p=1}^{s_2} u_{pj}^b \tilde{y}_{pj}^{gU} > 0 \ , \ \forall j.$$

$$u_{rk}^g \geq 0 \ \forall r \ , \ u_{pk}^b \geq 0 \ \forall p \ , \ v_{ik} \geq 0 \forall i$$

The solution procedure of Model 5 and Model 6 are explained by the Algorithm 1.

Algorithm 1: Procedure to solve the Model 5 and Model 6

Input: Dataset (Parabolic Fuzzy inputs, desirable outputs, undesirable outputs) and α where $\alpha \in [0, 1]$.

Process:

1) Select a ratio $0 \leq R \leq 1$.
2) Select the parametric bounds of inputs with Eq. (17) and (20).
3) Select the parametric bounds of desirable outputs with Eq. (18) and (21).
4) Select the parametric bounds of undesirable outputs with Eq. (19) and (22).

Output:

1) Lower-efficiency score ($\tilde{E}_k^L$).
2) Upper-efficiency score ($\tilde{E}_k^U$).
3) Crisp efficiency or Middle efficiency score.

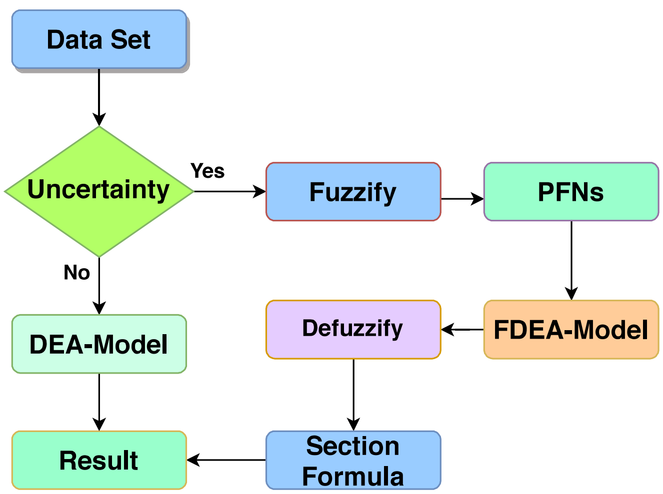

Fig. 2. Flowchart of Parametric FDEA Model

TABLE I
ARCHITECTURE OF THE PROPOSED MODELS

Model 1: Proposed PFDEA-UO Model

Three bounds of Parabolic Fuzzy Numbers splits the Model using Fuzzy Arithmetic Operations as below:

1) **Model 2**: Lower Efficiency with Undesirable Outputs Model
2) **Model 3**: Middle Efficiency with Undesirable Outputs Model
3) **Model 4**: Upper Efficiency with Undesirable Outputs Model

Using the $\alpha-$cut approach and the Ratio Division method the final models are obtained as:

1) **Model 5**: Lower Parametric Efficiency with Undesirable Outputs (LPrE-UO) Model
2) **Model 6**: Upper Parametric Efficiency with Undesirable Outputs (UpPrE-UO) Model

These models are producing more than one efficient DMUs and no further discrimination between them. Therefore; the major draw back of DEA models are the lack of discrimination power. To increase the discrimination power of DEA models, there are various accessible approaches in the literature. Cross-efficiency is one of the techniques, which is characterized to evaluate the efficiency score of each DMUs, n times via using the optimal weights of n linear programming (LP). Eventually, the output result is the cross-efficiency matrix of $n \times n$ for each stage of $\alpha \in [0, 1]$, and could be obtained through:

$$\tilde{h}_{kj} = \frac{\sum_{r=1}^{S_1} u_{rk}^g \tilde{y}_{rj}^g - \sum_{p=1}^{S_2} u_{pk}^b \tilde{y}_{pj}^b}{\sum_{i=1}^{m} v_{ik} \tilde{x}_{ij}}, \forall k, j. \tag{25}$$

In addition, all the entries of cross-efficiency matrix are between zero and one,i.e $0 < \tilde{h}_{kj} \leq 1$. The diagonal $\tilde{h}_{kk}$ demonstrates the standard DEA efficiency score, $\tilde{h}_{kk} = 1$, for efficient and $\tilde{h}_{kk} < 1$, for inefficient units. Furthermore; the average of these efficiencies are known as the average cross-efficiency(ACE). So, ACE is defined by:

$$\bar{h}_{kj} = \frac{1}{n} \sum_{k=1}^{n} \tilde{h}_{kj} \tag{26}$$

ACE is used to rank all the DMUs (efficient as well as inefficient). According to its decreasing values, the complete ranking produce by the algorithm 2.

Algorithm 2: Ranking of DMUs using Cross-Efficiency Method

Input: Dataset (Parabolic Fuzzy inputs, desirable outputs, undesirable outputs) and the values of $\alpha \in [0, 1]$.

Process:

1) Select a ratio $0 \leq R \leq 1$.
2) Select the parametric bounds of inputs with Eq. (17) and (20).
3) Select the parametric bounds of desirable outputs with Eq. (18) and (21).
4) Select the parametric bounds of undesirable outputs with Eq. (19) and (22).

Output:

1) Cross-efficiency matrices ($\tilde{h}_{kj}$).
2) Average cross-efficiency matrix ($\bar{h}_{kj}$).
3) The ranking result of DMUs with respect to decreasing values of $\bar{h}_{kj}$.

IV. EXPERIMENT AND ITS ANALYSIS

This section shows the empirical part of our theoretical work. In this section, the numerical example, along with tables and figures, are included. The numerical example consist of 5 DMUs with two inputs, one desirable output and one undesirable output given in [27]. The data is specified as Parabolic fuzzy numbers and arranged in TABLE II.

TABLE III presents the result of the Model 5 and Model 6 for different values of $\alpha-$cut approach. To increase the value of α where $\alpha \in [0, 1]$ the efficiency score of Model 5 is increasing while the efficiency score of Model 6 is decreasing, and for $\alpha = 1$ both models of Model 5 and Model 6 provide similar efficiency score, which is equivalent to the efficiency

TABLE II

INPUT-OUTPUT DATA

DMUs	Input 1	Input 2	Desirable Output	Undesirable Output
A	(2,4,5)	(3,3,3)	(1.5,1.5,1.5)	(0.5,0.7,0.8)
B	(5,7,8)	(3,3,3)	(3,3,3)	(0.2,0.3,0.35)
C	(7.5,8,8)	(1,1,1)	(4,4,4)	(0.7,0.8,0.9)
D	(3,4,6)	(2,2,2)	(2.5,2.5,2.5)	(0.3,0.35,0.45)
E	(1,2,3)	(4,4,4)	(1.2,1.2,1.2)	(0.13,0.15,0.15)

TABLE III

LOWER EFFICIENCY AND UPPER EFFICIENCY OF TABLE II

DMUs	Lower-efficiency score ($(E_k)^L_\alpha$)										
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
A	0.3779	0.3835	0.3896	0.3963	0.4061	0.4195	0.4351	0.4537	0.4772	0.5102	0.6000
B	0.6005	0.6095	0.6193	0.6300	0.6418	0.6550	0.6702	0.6881	0.7104	0.7411	0.8241
C	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
D	0.6464	0.6583	0.6713	0.6858	0.7021	0.7208	0.7426	0.7690	0.8030	0.8521	1.0000
E	0.4429	0.4583	0.4755	0.4948	0.5169	0.5427	0.5735	0.6119	0.6627	0.7397	1.0000
DMUs	**Upper-efficiency score ($(E_k)^U_\alpha$)**										
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
A	0.8100	0.7936	0.7772	0.7609	0.7445	0.7278	0.7107	0.6927	0.6731	0.6501	0.6000
B	0.8612	0.8590	0.8568	0.8544	0.8519	0.8492	0.8463	0.8431	0.8394	0.8347	0.8241
C	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
D	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

TABLE IV

AVERAGE CROSS-EFFICIENCY VALUE

DMUs	Average cross-efficiency (ACE) values										
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
A	0.6527	0.6271	0.6400	0.6141	0.6007	0.5868	0.5721	0.3597	0.3346	0.3027	0.2259
B	0.7594	0.7542	0.7568	0.7514	0.7486	0.7455	0.7422	0.7747	0.7703	0.7647	0.7525
C	0.8142	0.8208	0.8174	0.8244	0.8282	0.8324	0.8370	0.7992	0.8033	0.8085	0.8215
D	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E	0.7673	0.7696	0.7684	0.7709	0.7722	0.7737	0.7754	0.7774	0.7797	0.7829	0.7828

TABLE V

RANKING OF DMUS

DMUs	Ranking										
	$\alpha = 0.0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.3$	$\alpha = 0.4$	$\alpha = 0.5$	$\alpha = 0.6$	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
A	5	5	5	5	5	5	5	5	5	5	5
B	4	4	4	4	4	4	4	4	4	4	4
C	2	2	2	2	2	2	2	2	2	2	2
D	1	1	1	1	1	1	1	1	1	1	1
E	3	3	3	3	3	3	3	3	3	3	3

value of crisp data. Accordingly, three distinct efficiencies are defined as follows:

1) Fully efficient or E_k^{++}: All those DMUs, which are efficient concerning the lower efficiency model.
2) Efficient or E_k^{+}: All those DMUs, which are efficient concerning the middle-efficiency model or model of crisp data.
3) Inefficient or E_k^{-}: All those DMUs, which have an optimal value of less than one concerning the upper-efficiency model.

Therefore, TABLE III classifies the given DMUs in three categories on the basis of their efficiencies obtained from the Model 5 and Model 6 as follows:

1) Fully efficient or E_k^{++}: The DMU C is fully efficient because it has the optimal value of 1 concerning upper efficiency as well as the lower efficiency.
2) Efficient or E_k^{+}: The DMUs, C, D, and E are efficient because they have an optimal value of 1 with the upper-efficiency model.
3) Inefficient or E_k^{-}: The DMUs A and B are inefficient

because their optimal value is less than 1 for different α values with the upper-efficiency model.

The optimal objective function values, E_k using LPrE-UO Model and UpPrE-UO Model are the final efficiencies of kth DMUs for different α, where $\alpha \in [0, 1]$. In TABLE IV the values of average cross-efficiency (ACE) are indicated which assign the complete ranking method for the given DMUs at different stages of $\alpha-$cut approach, where $\alpha \in [0, 1]$. TABLE V shows the final ranking results, which is characterized with the decreasing values of average cross-efficiency (ACE).

Consequently, the Fig. 3 presents the efficiency result of 5 DMUs in crisp form while the Fig. 4 illustrates the result of upper efficiency score of fuzzy input-output data. Besides this, Fig. 4 depicts the impact of uncertainty on the efficiency score of each DMUs. Therefore, the efficiency result of fuzzy input-output data are more realistic and legible rather than the results produced using crisp data. Fig. 4 also determines the variation of $\alpha-$ level that are generating the variation of almost every DMUs. Thus, it implies the alterations in the $\alpha-$ level affect the efficiency results as well as the ranking of the DMUs.

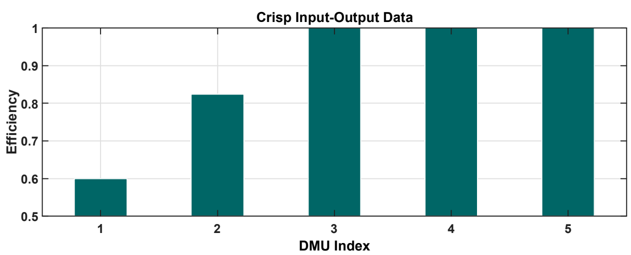

Fig. 3. The efficiency result of crisp input-output data

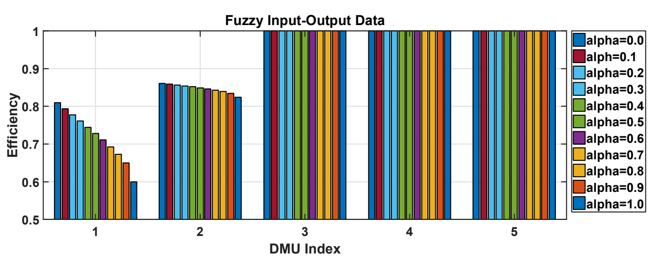

Fig. 4. The efficiency results of fuzzy Input-Output Data

V. Conclusion

In the paper, we have proposed an improved model of fuzzy data envelopment analysis for undesirable outputs, named as Parabolic Fuzzy Data Envelopment Analysis with Undesirable Outputs (PFDEA-UO). The impreciseness of the given data is well handled using parabolic fuzzy numbers in the proposed model. The three bounds of the parabolic fuzzy numbers are used to obtain three efficiency models with the help of fuzzy arithmetic operations, namely the Lower Efficiency with Undesirable Outputs (LE-UO) Model, the Middle-Efficiency with Undesirable Outputs (ME-UO) Model, and Upper-Efficiency with Undesirable Outputs (UpE-UO) Model. The asymmetrical geometry of the parabolic fuzzy numbers is exploited in the α-cut approach to obtain the interval-based Efficiency models. We also proposed a novel technique, named as Ratio Division to transform the interval efficiency models into two crisp parametric models named as Lower Parametric Efficiency with Undesirable Outputs (LPrE-UO) Model and the Upper Parametric Efficiency with Undesirable Outputs (UpPrE-UO) Model . Moreover, the Ratio Division method provides the crisp parametric data set for cross-efficiency technique to results in the complete ranking of DMUs at any α where $\alpha \in [0, 1]$ is the parameter. We used the numerical example to verify the effectiveness of the Ratio Division in the proposed PFDEA-UO Model.

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