From arithmetics of extensional fuzzy numbers to their distances

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***Abstract*—The notion of the metric space that allows to measure a distance between objects of the given space, has a crucial importance for distinct parts of mathematics, for instance, for the approximation theory, interpolation methods, data analysis, optimization etc. In fuzzy mathematics, the same areas of applications have an analogous importance and thus, not surprisingly measuring the distance between objects possesses a desirable importance. In many cases, e.g., in fuzzy clustering, the use of the standard metric spaces is absolutely sufficient. However, if we deal with vague quantities represented by fuzzy numbers, though the application of a standard metric to fuzzy numbers is mathematically correct, it may lead to counterintuitive and undesirable results. Our investigation constructs the "metric-like" spaces enabling to measure the distance between two fuzzy numbers in a way that is not disconnected from the used arithmetic of fuzzy numbers. Following the analogy from the classical math where the most natural distance between two numbers is the absolute value of their difference, in the case of fuzzy numbers and under the assumption that the distance is connected to the arithmetic, the most natural distance of two fuzzy numbers is the absolute values of their difference too. But then, naturally, the distance should map fuzzy numbers again to fuzzy numbers, not to crisp numbers. This article is a contribution to this area that guides readers from the fundamental notions to the final construction supported by some theoretical results.**

***Index Terms*—MI-algebras, extensional fuzzy numbers, similarity, extensionality, orderings, approximate reasoning**

I. Introduction

Arithmetics of fuzzy numbers are used for calculations with vague quantities in distinct fields such as fuzzy regression, fuzzy optimization, risk evaluation, or decision-making. Interestingly, this application impact of the calculus of fuzzy numbers is not that often followed by developing extended mathematical structures stemming from the arithmetics of fuzzy numbers. For instance, distances between fuzzy numbers are mappings to the real numbers. It is not unnatural as standard metrics (functions mapping to non-negative reals) are operating on distinct objects and such an approach is mathematically correct. On the other hand, in the case of distance on fuzzy numbers it may be viewed also a bit unintuitive. Indeed, fuzzy numbers are models of vague quantities and their distance is not a fuzzy number but a precise real number. But the most intuitive distance between two real numbers a and b is the absolute value of their difference $|a - b|$. If we replace the real values by fuzzy numbers, the most intuitive way how to measure their distance is to determine an absolute value of their difference which is again a fuzzy number. This naturally leads to the construction of metric-like spaces on fuzzy numbers where the "metrics" would be mappings to fuzzy numbers reflecting the chosen arithmetic. Note that ideas about considering a distance of two fuzzy numbers expressed again as a fuzzy number have not been deeply elaborated yet but it does not mean that they were not existing, we refer readers to an interesting construction published in [1].

This article is a contribution to this topic that directly stems from the *arithmetics of extensional fuzzy numbers* that form MI-prefields [2]–[4]. From the construction of the ordering of extensional fuzzy numbers [5] we get to the definition of specific metric-like spaces that are used to measure distanced between fuzzy numbers. We end up with intuitive tools that preserve most of the classical properties well-known from the classical analysis. Such advantages, including fast calculus, allow us to continue in constructing further notions from the mathematical analysis (limits, convergences, approximation spaces, interpolation) in the environment of extensional fuzzy numbers. Thus, the whole apparatus may be very helpful in developing efficient tools for tasks from approximate reasoning or data analysis, especially if the data is harmed by imprecision or vagueness – typically in social or economical sciences.

II. Preliminaries

A. Motivation

Definitions of fuzzy numbers differ at distinct publications [6], [7] but independently on the chosen definitions, a fuzzy number is a specific fuzzy set that serves as a model of a vague quantity [8], [9]. Although so-called *fuzzy metric spaces* have been defined [10] and although metric spaces can be generally on distinct domains that allow to measure distances between arbitrary objects, we are convinced that in order to capture intuitive expectations, the arithmetics of the fuzzy numbers have to be taken as the starting point. Let us consider a simple

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example. Let a fuzzy number A models the vague quantity *"about 3"* and let a fuzzy number B models the vague quantity *"about 5"*, see Figure 1.

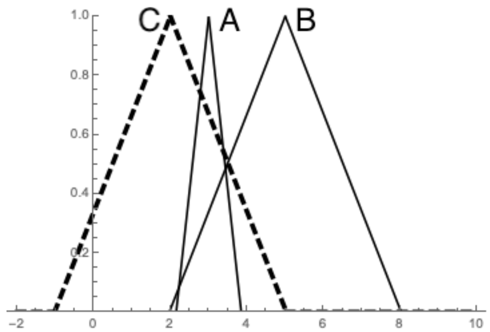

Fig. 1. Fuzzy numbers modeling the vague quantity *"about 3"* (A), and the vague quantity *"about 5"* (B). Their Hausdorff distance equals 4.18 . Their distance obtained as the absolute value of their difference is the vague quantity *"about 2"* (C).

As one can easily check, that, e.g., Hausdorff distance [11], that is often used for measuring the distance between fuzzy numbers [12] of the two fuzzy numbers A and B equals to 4.18. This result is mathematically fully correct yet not that intuitive. Indeed, the value 4.18 is obtained by applying the Hausdorff distance to all α-cuts and taking the supremum of such distances, which is in this case the Hausdorff distance for the supports of both fuzzy sets. This is really perfectly reasonable when talking about distances of sets or even fuzzy sets. But such an approach does not reflect the semantics of fuzzy numbers and one would intuitively expect that the distance between *"about 3"* and *"about 5"* is *"about 2"*.

Such an intuitive result would be obtained if the distance would stem from an arithmetic. Indeed, if we designed an absolute value of a fuzzy number, we should expect that the absolute value of the difference of two fuzzy numbers would be their most intuitive distance. The difference of two fuzzy numbers is a fuzzy number and the absolute value of a fuzzy number is again expected to be a fuzzy number. Therefore, instead of building a classical metric (distance) operating on fuzzy numbers $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathbb{R}$, we build a metric-like fuzzy function that maps pairs of fuzzy numbers again to fuzzy numbers $d : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$.

B. Arithmetics of extensional fuzzy numbers

Already the first step that influences the rest of the investigations in the area of arithmetics of fuzzy numbers is the definition of a fuzzy number. There are several approaches, most often, fuzzy number is considered to be a normal, convex and continuous or upper semi-continuous, sometimes also symmetry is expected. Such requirements are usually technically motivated, e.g., by preservation of the properties by distinct operations, which makes the family of fuzzy numbers closed with respect to those operations.

We prefer the restriction to *extensional fuzzy numbers* that is motivated by their semantics and origin. Extensional fuzzy numbers, sometimes also called fuzzy points [13]–[15], are constructed as extensional hulls of crisp numbers. Thus, their construction is simple and straightforward and their semantics (*a number with partial membership of its neighboring numbers*) is clear and unquestionable. Let us briefly recall the construction formally.

Definition 2.1: Let $\otimes$ be a left-continuous t-norm. A binary fuzzy relation on the set of real numbers $S : \mathbb{R}^2 \to [0,1]$ is called $\otimes$-*similarity on* $\mathbb{R}$ if the following axioms holds for all $x, y, z \in \mathbb{R}$

a) $S(x,x) = 1$, (reflexivity)
b) $S(x,y) = S(y,x)$, (symmetry)
c) $S(x,y) \otimes S(y,z) \leq S(x,z)$. ($\otimes$-transitive)

We say that a $\otimes$-similarity on $\mathbb{R}$ is *separated* if $S(x,y) = 1$ implies $x = y$ for any $x, y \in \mathbb{R}$ (see [4]).

Definition 2.2: [16] Let S be a $\otimes$-similarity relation on $\mathbb{R}$. A fuzzy set $A \in \mathcal{F}(\mathbb{R})$ is said to be *extensional* w.r.t. S if

$$A(x) \otimes S(x,y) \leq A(y)$$

holds for any $x, y \in \mathbb{R}$.

If a fuzzy set is not extensional, we may construct its extensional hull, that is its least fuzzy superset. Naturally, an extensional hull of an extensional fuzzy set is the fuzzy set itself. Mathematically, the extensional hull of a fuzzy set $A \in \mathcal{F}(\mathbb{R})$ may be constructed as follows [13]:

$$\mathrm{EXT}_S(A)(x) = \bigvee_{y \in \mathbb{R}} (A(y) \otimes S(x,y)) . \qquad (1)$$

We can also determine the extensional hull of a crisp number as each crisp number $x \in \mathbb{R}$ can be represented as singleton fuzzy set $\tilde{x}$ that is defined as $\tilde{x}(x) = 1$, and $\tilde{x}(y) = 0$ for any $y \in \mathbb{R}$, $y \neq x$. The singleton $\tilde{x}$ is already a fuzzy set to which we can apply formula (1). Following the previous studies, our denotation will omit the distinction between a crisp number x and its singleton $\tilde{x}$ and we will simply consider an extensional hull of x under the name *extensional fuzzy number* [2]–[4].

For a given similarity relation S, we can introduce the following denotation:

$$\mathrm{EXT}_S(x)(y) = x_S(y)$$

and one can easily check [4] that calculation is as simple as follows

$$x_S(y) = S(x,y), \quad y \in \mathbb{R}.$$

We argue that from the conceptual point of view, extensional fuzzy number provides us with a genuine representation of a vague quantity. Its origin precisely reflects a real number with its neighborhood, where the given similarity relation S models the particular way how we deal with the close values in the neighborhood.

A specific arithmetic has been developed for extensional fuzzy numbers [2] and it gave rise to algebraic structures with more identity-like elements, to the so-called MI-algebras[1] [4].

Let S, T be $\otimes$-similarities on $\mathbb{R}$. We say that S is *less than or equal to* T (symbolically $R \subseteq T$) if $S(x,y) \leq T(x,y)$ for any $x, y \in \mathbb{R}$.

[1]The abbreviation MI stands for many identities.

In this paper, we assume that $\mathcal{C}$ is a non-empty system of nested separated $\otimes$-similarities on $\mathbb{R}$ with respect to $\subseteq$ which has the least element $\bot_{\mathcal{C}}$ and the set of all extensional fuzzy numbers defined with respect to the similarities from the given system:

$$\mathcal{F}_{\mathcal{C}}(\mathbb{R}) = \{x_S \mid x \in \mathbb{R} \text{ and } S \in \mathcal{C}\}.$$

For the purpose of this paper, we restrict ourselves to such nested systems $\mathcal{C}$ that for any $x_S \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ the set $(x_S)_\alpha$ is a closed interval in $\mathbb{R}$ for any $\alpha \in (0, 1]$. The addition and the multiplication of two extensional fuzzy numbers from $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ are given as

$$x_S + y_T = (x + y)_{\max(S,T)}, \quad x_S \cdot y_T = (x \cdot y)_{\max(S,T)},$$

for any $S, T \in \mathcal{C}$, where the correctness of the definition follows from the assumption on the linearity ordering of the $\otimes$-similarities in the system $\mathcal{C}$ which ensures the existence of the maximum of two similarities (cf. [4]).

The "narrowest" $\otimes$-similarity from $\mathcal{C}$, i.e., the bottom element $\bot_{\mathcal{C}} = \inf\{S \mid S \in \mathcal{C}\}$ is used to determine the (strong) identity elements for both operations:

$$\mathbf{0} = 0_{\bot_{\mathcal{C}}} = \bot_{\mathcal{C}}(0, \cdot), \quad \mathbf{1} = 1_{\bot_{\mathcal{C}}} = \bot_{\mathcal{C}}(1, \cdot)$$

The inverse operations are defined naturally, e.g., for the additive operation as $-x_S = (-x)_S$. The non-strong neutral elements called *pseudoidentities* are determined with help of the inverse operations. For example, the set of pseudo-zeros is $I^0_{\mathcal{C}} = \{x_S + (-x)_S \mid S \in \mathcal{C}\}$.

The structures $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$ and $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), \cdot, ^{-1})$ form so-called MI-pregroups, and $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, \cdot, -, ^{-1})$ forms the MI-prefield where, for instance, the distributive law

$$(x_R + y_S) \cdot z_T = (x_R \cdot z_T) + (y_S \cdot z_T)$$

holds for any $R, S, T \in \mathcal{C}$. For details, we refer to [4].

It can be seen that the proposed arithmetic attempts to capture a human-intuitive calculus of vague quantities in a formal mathematical construction. If a person sums up quantities "around \$20" and "around \$30", the human cognition leads to the result "around \$50" obtained as a summation 20+30 and additionally, the tolerance values "around" the crisp result 50 is involved.

Example 2.1: [5] Let $\otimes$ be the Łukasiewicz t-norm and the closed system of embedded $\otimes$-similarities be $\mathcal{C}$:

$$\mathcal{C} = \{S_p \mid p \in [\ell, r] \text{ and } S_p(x, y) = (1 - p|x - y|) \vee 0\},$$

where $0 < \ell \le r$. Then $\max(S_p, S_{p'}) = S_{p''}$ for $p \le p'$ and thus

$$x_{S_p} + y_{S_{p'}} = (x + y)_{S_{p''}}, \quad p'' = \min\{p, p'\}$$

with the strong identity element $\mathbf{0} = 0_{S_r} = S_r(0, \cdot)$.

Analogously to Example 2.1, we could build other closed embedded systems of similarities for distinct t-norms. For instance, for the product t-norm, the parametric similarities could be constructed as follows $S_p(x, y) = e^{-p|x-y|}$.

At some cases, it is advantageous if the crisp equality "=" is contained in the closed system of embedded similarities. As the equality is an equivalence relation, it is a $\otimes$-similarity for any t-norm $\otimes$, and such a requirement is mathematically correct.

Example 2.2: Let $\otimes$ be the Łukasiewicz t-norm and the closed system of embedded $\otimes$-similarities be $\mathcal{C}$:

$$\mathcal{C} = \{S_p \mid p \in [\ell, +\infty) \text{ and } S_p(x, y) = (1 - p|x - y|) \vee 0\},$$

where $0 < \ell$. Let S_∞ be the crisp equality, i.e., $S_\infty(x, y) = 1$ if and only if $x = y$. Then the set $\mathcal{C}$ extended by the crisp equality forms a closed system of embedded $\otimes$-similarities $\mathcal{C}_\infty = \mathcal{C} \cup S_\infty$ and naturally, $\bot_{\mathcal{C}} = S_\infty$.

For the rest of the paper, let us fix the denotation $\mathcal{C}_\infty$ for a system containing the crisp equality.

III. "Metrics" on extensional fuzzy numbers

Now, we can start building the metric-like spaces of extensional fuzzy numbers based on their arithmetics and thus, capture the intuitive expectations and avoid the unwanted effects of standard metrics such as the one demonstrated on Figure 1.

Let us briefly recall, that $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a *metric* (distance) on $\mathbb{R}$ if for all $x, y, z \in \mathbb{R}$:

$$\begin{aligned} &d(x, y) \ge 0 \ , \\ &d(x, y) = 0 \Leftrightarrow x = y \ , \\ &d(x, y) = d(y, x) \ , \\ &d(x, z) \le d(x, y) + d(y, z) \ . \end{aligned}$$

The intended replacement of real values x, y, z by fuzzy numbers in the definition above seems straightforward however, the last axioms (triangle inequality) directly uses the ordering of fuzzy numbers, which is a notion that can be approached from different perspectives [17]–[19].

A. Orderings of extensional fuzzy numbers

The triangle inequality is a binary property – either it holds or not. So, also the ordering applied to extensional fuzzy numbers should give the true/false answer to be used in the triangle inequality axiom. Note, that it is not so automatic as some orderings work on distinct indexes, some allow two fuzzy sets to be order in a certain degree [20]. One might use, e.g., the interval ordering $\le_i$:

$$[a, b] \le_i [c, d] \ \Leftrightarrow \ a \le c \ \text{ and } \ b \le d$$

applied to all α-cuts of the fuzzy numbers:

$$A \le_i B \ \Leftrightarrow \ A_\alpha \le_i B_\alpha \quad \forall \alpha \in (0, 1]. \tag{2}$$

But the fuzzy numbers from Figure 1 would not be comparable according to such ordering. Bodenhofer [20], [21] proposed to overcome this weakness by "widening" the fuzzy numbers by constructing their extensional hulls with respect to wider similarity relations and the obtained hulls are already ordered according to the interval ordering of their α-cuts.

In [5], the authors have followed this idea however, they captured the necessarily used extensionality in the truth/false values of the ordering. Mathematically, "fuzzy truth values" and "fuzzy falses" are constructed and the ordering has been constructed as a mapping from pairs of extensional fuzzy numbers to these extended truth/false values. This again mimics the classical ordering that maps pairs of numbers (or other objects) to the set containing two elements $\{0, 1\}$ – the truth and the false. Let us briefly recall the construction mathematically.

The truth-values 0 and 1 are represented by their extensional hulls "around 0" and "around 1". In order to avoid possible confusion between extensional fuzzy numbers "around 0" or "around 1" used in the arithmetic calculations and the truth-values, we opt for the denotation T (true) and F (false) and their extensional hulls.

Definition 3.1: [5] Consider a system $\mathcal{C}$ of nested $\otimes$-similarities on $\mathbb{R}$ and $S \in \mathcal{C}$. The fuzzy sets $\mathrm{T}_S, \mathrm{F}_S \in \mathcal{F}([0,1])$:

$$\mathrm{T}_S = \mathrm{EXT}_S(1) \ , \tag{3}$$
$$\mathrm{F}_S = \mathrm{EXT}_S(0) \tag{4}$$

will be called *$\mathcal{C}$-Boolean valued truth*, and *$\mathcal{C}$-Boolean valued false*, respectively. Furthermore, we will denote the sets:

$$\mathrm{T}_{\mathcal{C}} = \{\mathrm{EXT}_S(1) \mid S \in \mathcal{C}\} \ , \tag{5}$$
$$\mathrm{F}_{\mathcal{C}} = \{\mathrm{EXT}_S(0) \mid S \in \mathcal{C}\} \ , \tag{6}$$
$$\mathrm{TF}_{\mathcal{C}} = \mathrm{T}_{\mathcal{C}} \cup \mathrm{F}_{\mathcal{C}} \ . \tag{7}$$

Definition 3.2: [5] Let $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$ be an MI-pregroup of extensional fuzzy numbers with respect to a system $\mathcal{C}$ of nested $\otimes$-similarities on $\mathbb{R}$. A mapping $\leq_{\mathcal{C}} : \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \to \mathrm{TF}_{\mathcal{C}}$ is called *$\mathcal{C}$-Boolean valued ordering* if:

(i) $(a_S \leq_{\mathcal{C}} a_T) \in \mathrm{T}_{\mathcal{C}}$, (reflexivity)
(ii) $(a_S \leq_{\mathcal{C}} b_T) \in \mathrm{T}_{\mathcal{C}} \ \& \ (b_T \leq_{\mathcal{C}} a_S) \in \mathrm{T}_{\mathcal{C}} \ \Rightarrow \ (a_S - b_T) \in I^0_{\mathcal{C}}$, (anti-symmetry)
(iii) $(a_S \leq_{\mathcal{C}} b_T) \in \mathrm{T}_{\mathcal{C}} \ \& \ (b_T \leq_{\mathcal{C}} c_R) \in \mathrm{T}_{\mathcal{C}} \ \Rightarrow \ (a_S \leq_{\mathcal{C}} c_R) \in \mathrm{T}_{\mathcal{C}}$, (transitivity).

Notation 3.1: In cases where it will not possibly cause any confusion and the particular "width" of the $\mathcal{C}$-Boolean valued truth will not be important, we will adopt the denotation from [5] and instead of $(a_S \leq_{\mathcal{C}} b_T) \in \mathrm{T}_{\mathcal{C}}$ or $(a_S \leq_{\mathcal{C}} b_T) \in \mathrm{F}_{\mathcal{C}}$, we will shortly write $a_S \leq_{\mathcal{C}} b_T$ or $a_S \not\leq_{\mathcal{C}} b_T$, respectively.

Example 3.1: Let $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$ be an MI-pregroup. Then

$$a_S \leq_{\max} b_T = \begin{cases} \mathrm{T}_{\max(S,T)}, & \text{if } a \leq b, \\ \mathrm{F}_{\max(S,T)}, & \text{otherwise,} \end{cases}$$

is a $\mathcal{C}$-Boolean valued ordering.

Example 3.1 demonstrates a $\mathcal{C}$-Boolean valued ordering reflecting transitive closure of the union of the similarities used in the arithmetic in the determination of the width of the obtained truth. Let us present two particular examples. First, if we consider the system of Łukasiewicz similarities $\mathcal{C}$ from Example 2.1, we would come up to the following ordering:

$$a_{S_p} \leq_{\max} b_{S_{p'}} = \begin{cases} \mathrm{T}_{S_{p''}}, \ p'' = \min\{p, p'\}, & \text{if } a \leq b, \\ \mathrm{F}_{S_{p''}}, \ p'' = \min\{p, p'\}, & \text{otherwise.} \end{cases}$$

which can be visually illustrated on Figure 2.

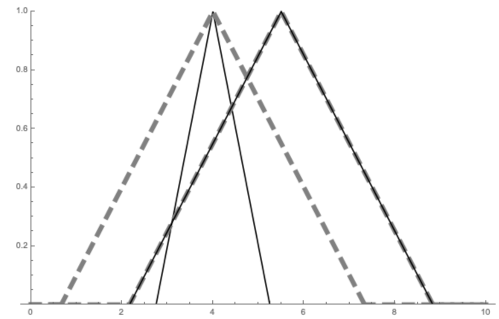

Fig. 2. Fuzzy numbers $x_{S_{p'}}$ and $y_{S_{p''}}$ from Example 2.1, one is determined by $x = 4$ and $p' = 0.8$ ("left" solid fuzzy set), the other one by $y = 5.5$ and $p'' = 0.3$ ("right" solid fuzzy set). Fuzzy sets $x_{S_{p'}}$ and $y_{S_{p''}}$ cannot be ordered by $\leq_i$ of α-cuts however, if we use $\leq_{\max}$, we obtain the dashed fuzzy sets $\mathrm{EXT}_{S_{p''}}(x_{S_{p'}})$ and $\mathrm{EXT}_{S_{p''}}(y_{S_{p''}})$, respectively, and the subsequent conclusion $x_{S_{p'}} \leq_{\max} y_{S_{p''}} = \mathrm{T}_{S_{p''}}$.

The way how $\leq_{\max}$ is defined is by far not the only way how to design a $\mathcal{C}$-Boolean valued orderings. Assuming that $\mathcal{C}$ possesses the greatest element $\top_{\mathcal{C}}$ (i.e. $R \subseteq \top_{\mathcal{C}}$ for all $R \in \mathcal{C}$), then for example, the mapping $\leq_{\top_{\mathcal{C}}} : \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \to \mathrm{TF}_{\mathcal{C}}$ given by:

$$a_S \leq_{\top_{\mathcal{C}}} b_T = \begin{cases} \mathrm{T}_{\top_{\mathcal{C}}}, & \text{if } \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T), \\ \mathrm{F}_{\top_{\mathcal{C}}}, & \text{otherwise,} \end{cases}$$

is a $\mathcal{C}$-Boolean valued ordering too. Note that the composition of $\otimes$-similarities $S \circ R = \max(S, R) \in \mathcal{C}$ for any $S, T \in \mathcal{C}$, which is a simple consequence of the fact that $\mathcal{C}$ is nested. Hence, we find that $\mathrm{EXT}_R(a_S) = a_{S \circ R} \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$, which ensures the correctness of the definition of $\leq_{\top_{\mathcal{C}}}$.

Let us consider, e.g., the product t-norm, and

$$\mathcal{C} = \{S_p \mid p \in [1, 5] \text{ and } S_p(x, y) = e^{-p|x-y|}\} \ .$$

The ordering $\leq_{\top_{\mathcal{C}}}$ can be visually demonstrated by Figure 3, where one can see two extensional fuzzy numbers $4_{S_{2.5}}$ and $5.5_{S_{1.2}}$ (displayed by solid lines) and their extensional hulls $\mathrm{EXT}_{S_5}(4_{S_{2.5}}) = 4_{S_5}$ (left dashed fuzzy set) and $\mathrm{EXT}_{S_5}(5.5_{S_{1.2}}) = 5.5_{S_5}$ (right dashed fuzzy set) that due to their interval ordering $4_{S_5} \leq_i 5.5_{S_5}$ allow to order the original fuzzy numbers $4_{S_{2.5}} \leq_{\top_{\mathcal{C}}} 5.5_{S_{1.2}}$.

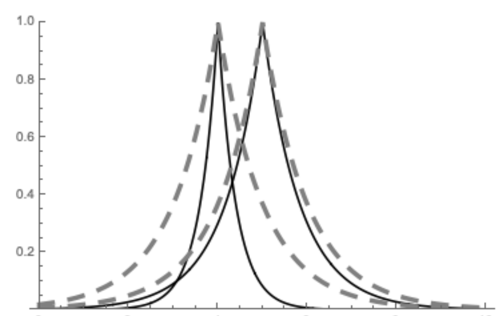

Fig. 3. Demonstration of the $\mathcal{C}$-Boolean valued ordering $\leq_{\top_{\mathcal{C}}}$.

Assume that $\mathcal{C}$ has all infima, i.e., for any subset $D \subseteq \mathcal{C}$, we have $\inf\{S \in \mathcal{C} \mid S \in D\} \in \mathcal{C}$. Then, another ordering applies the rule of the "minimal size" of the similarity relation that is sufficient in order to get extensional hulls of the given fuzzy numbers, that are already ordered with respect to $\leq_i$. Technically, this $\mathcal{C}$-Boolean valued ordering, denoted by $\leq_{\inf}$, will seek for the infimum of all such similarities:

$$a_S \leq_{\inf} b_T = \begin{cases} \mathrm{T}_E, \ \exists R \in \mathcal{C} : \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T), \\ \mathrm{F}_{\perp_{\mathcal{C}}}, \ \text{otherwise}, \end{cases}$$

where $E = \inf\{R \in \mathcal{C} \mid \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T)\}$. The demonstration of such ordering can be illustrated in Figure 4.

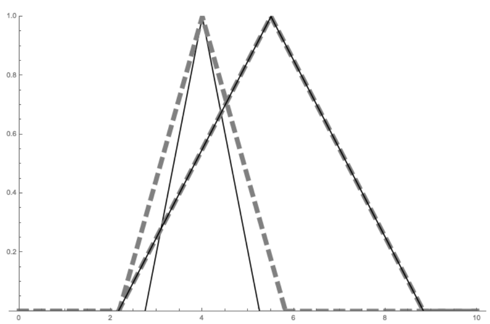

Fig. 4. Example of $\leq_{\inf}$ applied to fuzzy sets from Figure 2.

All the $\mathcal{C}$-Boolean valued orderings mentioned above are so-called *strongly compatible* which means well-behaving with respect to the ordering of fuzzy sets based on fuzzy pre-order [20], [21] and thus, (1) reflecting the necessary extension (of the two fuzzy numbers being ordered) in the $\mathcal{C}$-Boolean valued truth; (2) respecting the order of $\mathbb{R}$. Formally, these properties are defined as follows

Definition 3.3: Let $\leq_{\mathcal{C}}$ be $\mathcal{C}$-Boolean valued ordering on an MI-pregroup $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -)$. The ordering $\leq_{\mathcal{C}}$ is called *pre-order compatible* if, for any $a_S, b_T \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and any $R \in \mathcal{C}$:

$$(a_S \leq_{\mathcal{C}} b_T) = \mathrm{T}_R \ \Rightarrow \ \mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T) \ . \quad (8)$$

The ordering $\leq_{\mathcal{C}}$ is called *real-order compatible* if, for any $a_S, b_T \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and $R \in \mathcal{C}$,

$$\mathrm{EXT}_R(a_S) \leq_i \mathrm{EXT}_R(b_T) \Rightarrow (a_S \leq_{\mathcal{C}} b_T) \in \mathrm{T}_{\mathcal{C}} \, . \quad (9)$$

Moreover, if an ordering $\leq_{\mathcal{C}}$ is pre-order compatible and also real-order compatible, it is said to be *strongly compatible*.

We will demonstrate the properties on examples that do not meet them. For example, consider the following modification of $\leq_{\top_{\mathcal{C}}}$ (only $\top_{\mathcal{C}}$ is replaced by $\perp_{\mathcal{C}}$):

$$a_S \leq_{\perp_{\mathcal{C}}} b_T = \begin{cases} \mathrm{T}_{\perp_{\mathcal{C}}}, & \text{if } a \leq b, \\ \mathrm{F}_{\perp_{\mathcal{C}}}, & \text{otherwise}, \end{cases}$$

that is real-order compatible but it is not pre-order compatible as the use of the narrowest similarity $\perp_{\mathcal{C}}$ in $\mathrm{T}_{\perp_{\mathcal{C}}}$ does not reflect the necessary extensions of the two fuzzy sets being compared, for more details, see [22].

The following example of $\leq_{\mathcal{C}}$ that is neither pre-order compatible nor real-order compatible [22] adopts the principles of $\leq_{\max}$ but it reverses its order and so, it would rank "about 7" below "about 5":

$$a_S \leq_{\max}^{rev} b_T = \begin{cases} \mathrm{T}_{\max(S,T)}, & \text{if } b \leq a, \\ \mathrm{F}_{\max(S,T)}, & \text{otherwise.} \end{cases}$$

When studying ordered structures that serve for the arithmetics, both qualities of the algebraic structure should be connected. If we impose the standard property connecting the additive operation and the ordering that is given for any $a_T, b_S, c_R \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ as follows:

$$a_S \leq_{\mathcal{C}} b_T \Leftrightarrow a_S + c_R \leq_{\mathcal{C}} b_T + c_R,$$

we obtain the so-called *$\mathcal{C}$-ordered MI-pregroup* $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$. A sufficient condition on the ordering $\leq_{\mathcal{C}}$ under which an MI-pregroup becomes the $\mathcal{C}$-ordered MI-pregroup, is provided in [22].

B. $\mathcal{C}$-valued metrics on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$

Now, we may define "metric-like functions" operating on extensional fuzzy numbers. From the mathematical point of view, it is a necessary step for the development of further of mathematical analysis with extensional fuzzy numbers. From the application point of view the construction may have a huge potential for distinct applications calculating with imprecise or vague quantities, which typically falls into social sciences including economy [19], especially in time series modelling where we need to aggregate such quantities (e.g. due to the imprecise measurement or the lack of information) to one or more quantities representing their important attributes and measure their dispersion to optimize model parameters.

Recall that we assume a system $\mathcal{C}$ of separated nested $\otimes$-similarities on $\mathbb{R}$. The motivation for the separated $\otimes$-similarities comes from the metric axiom saying that $d(x, y) = 0$ if and only if $x = y$. Therefore, to design the metric-like functions for extensional fuzzy numbers in the spirit of the metric functions, the separateness of $\otimes$-similarities is a natural requirement. Define $x_S - y_T = x_S + (-y_T)$ for any $x_S, y_T \in \mathcal{F}_{\mathcal{C}}(\mathbb{R})$.

Definition 3.4: Let $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup. A mapping $d_{\mathcal{C}} : \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ preserving the following axioms for all $R, S, T \in \mathcal{C}$:

$$\begin{aligned} &d_{\mathcal{C}}(x_S, y_T) \geq_{\mathcal{C}} 0_R \ , \\ &d_{\mathcal{C}}(x_S, y_T) = 0_R \Leftrightarrow x_S - y_T = 0_R \ , \\ &d_{\mathcal{C}}(x_S, y_T) = d_{\mathcal{C}}(y_T, x_S) \ , \\ &d_{\mathcal{C}}(x_S, z_R) \leq_{\mathcal{C}} d_{\mathcal{C}}(x_S, y_T) + d_{\mathcal{C}}(y_T, z_R) \end{aligned}$$

will be called *$\mathcal{C}$-valued metric function on $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$*.
On can easily see that from the separateness of the $\otimes$-similarities we obtain that if $d_{\mathcal{C}}(x_S, y_T) = 0_R$ then $x = y$. Indeed, if $d_{\mathcal{C}}(x_S, y_T) = 0_R$, then $x_S - y_T = (x - y)_U = 0_R$ for a certain $U = \max(S, T) \in \mathcal{C}$. Hence, we obtain $U(x - y, 0) = R(0, 0) = 1$. Since U is separated, we find that $x - y = 0$, i.e., $x = y$.

Let us start from examples. Let us consider $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\max})$ and we will obtain the following definition of the absolute value

$$|x_S| = \begin{cases} x_S, & 0_E \leq_{\max} x_S \quad \forall E \in \mathcal{C} \\ -x_S, & \text{otherwise.} \end{cases}$$

We can check that right-hand side condition $0_E \leq_{\max} x_S$ is equivalent to the condition $0 \leq x$ and thus, the definition may be written as

$$|x_S| = \begin{cases} x_S, & 0 \leq x \\ -x_S, & \text{otherwise} \end{cases}$$

which is consequently in a concise form:

$$|x_S| = |x|_S \ . \tag{10}$$

If we fix the structure $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\top_{\mathcal{C}}})$ and mimic the same definition of the absolute value

$$|x_S| = \begin{cases} x_S, & 0_E \leq_{\top_{\mathcal{C}}} x_S \quad \forall E \in \mathcal{C} \\ -x_S, & \text{otherwise} \end{cases}$$

in which, we will expand the right-hand side $0_E \leq_{\top_{\mathcal{C}}} x_S$ as $\mathrm{EXT}_{\top_{\mathcal{C}}}(0_E) \leq_i \mathrm{EXT}_{\top_{\mathcal{C}}}(x_S)$, we will end up with $0_{\top_{\mathcal{C}}} \leq_i x_R$ which leads to $0 \leq x$, and consequently to (10).

If we consider the structure $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\inf})$ and:

$$|x_S| = \begin{cases} x_S, & 0_E \leq_{\inf} x_S \quad \forall E \in \mathcal{C} \\ -x_S, & \text{otherwise} \end{cases}$$

we may expand the right-hand side $0_E \leq_{\inf} x_S$ as an existence of $R \in \mathcal{C}$ such that $\mathrm{EXT}_R(0_E) \leq_i \mathrm{EXT}_R(x_S)$ which is equivalent to $0_{\max(R,E)} \leq_i x_{\max(R,S)}$. These two fuzzy numbers may preserve interval ordering for all α-cuts (including 1-cut) only if $0 \leq x$ and thus, (10) holds.

As we can observe, in all the three examples above led to formula (10). This, at first sight tiny observation, is extremely important. Indeed, the absolute value is a function and thus, stems from the given arithmetic. And observe, that no matter which $\mathcal{C}$-Boolean valued ordering we adopt, we will land to the same absolute value. Furthermore, it also follows the technical part of the arithmetic in which, the operations on extensional fuzzy numbers were calculated as operations on crisp numbers with additionally applied tolerance to close values by the calculus on similarity relations. The absolute values acts according to (10) in the same way: instead of the absolute value of an extensional fuzzy number, we determine the absolute value of the crisp number, and in the second step, we build its extensional hull.

The natural question that rises on can be formulated as follows: *"do all absolute values of extensional fuzzy numbers lead to formula (10)?"*

The answer to the question above is, on a general level, negative. It can be easily checked by a counterexample. Consider the $\mathcal{C}$-Boolean valued ordering $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\max})$ that reverses the order of the real line. Then

$$|x_S| = \begin{cases} x_S, & 0_E \leq_{\max}^{rev} x_S, \quad \forall E \in \mathcal{C}, \\ -x_S, & \text{otherwise} \end{cases}$$

and the right hand side $0_E \leq_{\max}^{rev} x_S$ occurs when $x \leq 0$ and thus $|x_S| = -|x|_S$.

We may have noticed, that while the three positive cases when formula (10) held, occurred in the case of pre-order compatibility, the counterexample was built on a $\mathcal{C}$-Boolean valued ordering that is not pre-order compatible. As we show below, this observation can be generalized to all such cases proved formally.

Lemma 3.1: [22] Let $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup such that $\leq_{\mathcal{C}}$ is pre-order compatible. If $a_S \leq_{\mathcal{C}} b_T$ for certain $S, T \in \mathcal{C}$ then $a \leq b$.

Theorem 3.2: Consider a $\mathcal{C}$-ordered MI-pregroup $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ with $\leq_{\mathcal{C}}$ being strongly compatible. Let $S \in \mathcal{C}$, and let the absolute value $|\cdot| \colon \mathcal{F}_{\mathcal{C}}(\mathbb{R}) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ be defined as:

$$|x_S| = \begin{cases} x_S, & 0_E \leq_{\mathcal{C}} x_S, \quad \forall E \in \mathcal{C} \\ -x_S, & \text{otherwise.} \end{cases}$$

Then the following holds $|x_S| = |x|_S$.

Sketch of the proof: If $0_E \leq_{\mathcal{C}} x_S$ for all $E \in \mathcal{C}$ then by Lemma 3.1 we get $0 \leq x$ and hence, $|x_S| = x_S = |x|_S$. We have to ensure that the case of $0_E \leq_{\mathcal{C}} x_S$ and simultaneously $0_F \not\leq_{\mathcal{C}} x_S$ for some $E, F \in \mathcal{C}$ dos not occur. From $0_E \leq_{\mathcal{C}} x_S$ by lemma 3.1, we get $0 \leq x$. Let us take $R = \max(S, F)$. Then we get

$$\mathrm{EXT}_R(0_F) = 0_R \leq_i x_R = \mathrm{EXT}_R(x_S)$$

which, with the help of the real-order compatibility leads to $0_F \leq_{\mathcal{C}} x_S$. Finally, by similar arguments, one can show that if $0_E \not\leq_{\mathcal{C}} x_S$ for all $E \in \mathcal{C}$, then $x < 0$. Hence, we simply get $|x_S| = -x_S = |x|_S$. ☐

As pointed out above, the most natural metric or distance of two fuzzy numbers would be the absolute value of the difference of these two fuzzy numbers. In our case, restricted to the set of extensional fuzzy numbers, the calculations should not be so complicated and thus, under the restriction to the structures with pre-order compatible $\mathcal{C}$-Boolean valued orderings, we will check if such a construction meets the axioms of Definition 3.4.

Proposition 3.3: Consider a $\mathcal{C}$-ordered MI-pregroup $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ with strongly compatible $\leq_{\mathcal{C}}$. Then $d_{\mathcal{C}}(x_S, y_T) = |x_S - y_T|$ is a $\mathcal{C}$-valued metric function.

Sketch of the proof: The non-negativity $|x_S - y_T| \geq_{\mathcal{C}} 0_E$ is ensured by the use of strongly compatible order $\leq_{\mathcal{C}}$ and the use of Theorem 3.2.

The second axiom follows from the fact that

$$|x_S - y_T| = |(x-y)_{\max(S,T)}| = |x - y|_{\max(S,T)}$$

and this can be equal to 0_E if and only if $|x - y| = 0$ and $\max(S, T) = E$.

The symmetry is proved as follows:

$$|x_S - y_T| = |(x-y)_{\max(S,T)}| = |x - y|_{\max(S,T)}$$

which due to $|x-y| = |y-x|$ equals to

$$|y-x|_{\max(S,T)} = |(y-x)_{\max(S,T)}| = |y_T - x_S|.$$

The triangle inequality is proved again using the strong compatibility. □

C. Generalization to higher dimensions

We have shown that if we approach the metric-like function on fuzzy numbers from their arithmetic as the starting point, we may get very natural results. Indeed, the distance of two fuzzy numbers is naturally expected to be equal to the absolute value of their difference which is again a fuzzy number. However, this resultn, if being left for the single dimension only, is interesting only from the motivation point of view. But in order to employ the machinery in real applications, e.g., in approximation and interpolation of vague data, automated approximate reasoning, calculations with vague quantities in social sciences and (economics and finance), we necessarily have to step to higher dimensions.

As the construction of extensional fuzzy numbers from crisp reals is well founded, the natural approach to increase the dimension is based on dealing with n-dimensional vectors of extensional fuzzy numbers. So, the set $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$ will serve as the domain of our metric-like functions, whenever we will need to apply some operations to vectors $\bar{x}_S = (x_{1,S}, \dots, x_{n,S})$, $\bar{y}_T = (y_{1,T}, \dots, y_{n,T})$ from $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$, they will be applied component-wise:

$$\bar{x}_S - \bar{y}_T = (x_{1S} - y_{1T}, \dots, x_{nS} - y_{nS}).$$

Definition 3.5: Let $(\mathcal{F}^n_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup. A mapping $d_{\mathcal{C}} : \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ preserving the following axioms for all $R, S, T \in \mathcal{C}$:

$$\begin{aligned}
&d_{\mathcal{C}}(\bar{x}_S, \bar{y}_T) \geq_{\mathcal{C}} 0_R \ , \\
&d_{\mathcal{C}}(\bar{x}_S, \bar{y}_T) = 0_R \Leftrightarrow \bar{x}_S - \bar{y}_T = \bar{0}_R \ , \\
&d_{\mathcal{C}}(\bar{x}_S, \bar{y}_T) = d_{\mathcal{C}}(\bar{y}_T, \bar{x}_S) \ , \\
&d_{\mathcal{C}}(\bar{x}_S, \bar{z}_R) \leq_{\mathcal{C}} d_{\mathcal{C}}(\bar{x}_S, \bar{y}_T) + d_{\mathcal{C}}(\bar{y}_T, \bar{z}_R)
\end{aligned}$$

will be called *$\mathcal{C}$-valued metric function on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$*.

Now, we may define distinct metric-like functions on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$ that mimic the classical metric functions on vectors of reals and show that they meet the axioms of Definition 3.5.

Proposition 3.4: Let $(\mathcal{F}^n_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup with strongly compatible $\leq_{\mathcal{C}}$. The mapping $d_M : \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ called Manhattan distance on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$ and defined by

$$d_M(\bar{x}_S, \bar{y}_T) = \sum_{i=1}^{n} |x_{iS} - y_{iT}|$$

is a $\mathcal{C}$-valued metric function on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$.

Sketch of the proof: Satisfaction of the first axiom $d_M(\bar{x}_S, \bar{y}_T) \geq_{\mathcal{C}} 0_R$ follows from the non-negative nature of the absolute value assured by Theorem 3.2.

The proof of the satisfaction of the second axiom is also based on the fact that a summation of non-negative fuzzy numbers may be equal to 0_R if and only if all the summands are equal to 0_E for some $E \in \mathcal{C}$. Then each summand $|x_{iS} - y_{iT}|$ is equal to $0_{\max(S,T)}$ and the summation $\sum_i^n 0_{\max(S,T)}$ is equal to 0_R if and only if $\max(S,T) = R$ which leads $\bar{x}_S - \bar{y}_T = \bar{0}_R$.

Satisfaction of the third axiom is easily proved using the fact:

$$\sum_{i=1}^{n} |x_{iS} - y_{iT}| = \sum_{i=1}^{n} |y_{iT} - x_{iS}| \ .$$

The triangle inequality is derived from the triangle inequality of the absolute value of the difference of two extensional fuzzy numbers, see Proposition 3.3. □

The Manhattan metric is one of the "classical" distances used on the $\mathbb{R}^n$. Another classical alternative is the maximum metric. We can again mimics its construction for the vectors of extensional fuzzy numbers. However, first of all, we have to define the maximum function on $(\mathcal{F}_{\mathcal{C}}(\mathbb{R}), +, - \leq_{\mathcal{C}})$ which can be naturally done as follows:

$$x_S \vee y_T = \begin{cases} x_S, & x_S \leq_{\mathcal{C}} y_T \\ y_T, & y_T \leq_{\mathcal{C}} x_S \ . \end{cases} \tag{11}$$

The above definition of the maximum requires *totality* of the used $\mathcal{C}$-Boolean valued ordering, for details, we refer to [5]. The definition of the maximum given by (11) can be again applied to $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$ component-wise and thus, enable to define the maximum metric.

Proposition 3.5: Let $(\mathcal{F}^n_{\mathcal{C}}(\mathbb{R}), +, -, \leq_{\mathcal{C}})$ be a $\mathcal{C}$-ordered MI-pregroup with strongly compatible $\leq_{\mathcal{C}}$. The mapping $d_\vee : \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \times \mathcal{F}^n_{\mathcal{C}}(\mathbb{R}) \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ called maximum distance on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$ and defined by

$$d_\vee(\bar{x}_S, \bar{y}_T) = \bigvee_{i=1}^{n} |x_{iS} - y_{iT}|$$

is a $\mathcal{C}$-valued metric function on $\mathcal{F}^n_{\mathcal{C}}(\mathbb{R})$.

Sketch of the proof: The first axiom $d_\vee(\bar{x}_S, \bar{y}_T) \geq_{\mathcal{C}} 0_R$ again follows from the non-negative nature of the absolute value assured by Theorem 3.2.

The maximum metric $d_\vee(\bar{x}_S, \bar{y}_T) = |x_{jS} - y_{jT}|$ for some particular j, and this absolute value $|x_{jS} - y_{jT}|$ can be equal to 0_R (where $R = \max(S,T)$) if and only if $x_j - y_j = 0$. But then also all other components $|x_{iS} - y_{iT}|$ for $i \neq j$ are equal to 0_R and thus, $\bar{x}_S - \bar{y}_T = \bar{0}_R$.

The third axiom is proved using the fact (Proposition 3.3):

$$|x_{jS} - y_{jT}| = |y_{jT} - x_{jS}| \ .$$

The triangle inequality of the maximum metric is derived from the triangle inequality of the absolute value of the difference of two extensional fuzzy numbers, see Proposition 3.3. □

Let us note, that in a very similar way, we could construct also other $\mathcal{C}$-valued metric functions, including the most common one – the Euclidean distance. Of course, we have to keep in mind that such a construction requires technical steps, e.g., to construct power and mainly squared root and prove their properties (non-negativity etc.). Due to the limited extent of this article we do not include these technical steps and the whole construction into this article and we only argue that though technically different, in principle the whole construction is analogous.

IV. Conclusions and future work

This article is devoted to the concept of metric (distance) for extensional fuzzy numbers where the choice of the extensional fuzzy numbers was made because of their simple arithmetics [4]. In contrast to the standard approaches to the metric spaces of fuzzy numbers, we proposed metric-like functions that assign an extensional fuzzy number to each pair of extensional fuzzy numbers and moreover, we admitted that the axioms for our metric-like functions can be satisfied only in certain $\mathcal{C}$-Boolean valued truth degrees which are special extensional fuzzy numbers in the unit interval. More precisely, the proposed definition of $\mathcal{C}$-valued metric functions is based on the concept of $\mathcal{C}$-valued ordering relation defined on a set of extensional fuzzy numbers using which an extensional fuzzy number is less than or equal to another extensional fuzzy number in a certain $\mathcal{C}$-Boolean valued truth degree [22]. Further, we were seeking for a computationally reasonable, and mainly natural, definition of the absolute value for extensional fuzzy numbers that should be considered in the definition of the most natural metric (at least for real numbers) given as the absolute value of the difference between extensional fuzzy numbers, where the difference is obtained from the arithmetic with extensional fuzzy numbers. We showed that under the assumption that a $\mathcal{C}$-valued ordering is strong, i.e., it is connected in an appropriate way with the interval ordering of α-cuts of extensional hulls, we can introduce very simple definition of the absolute value such that $|x_S| = |x|_S$, where x_S is an extensional fuzzy number, and moreover, the absolute value of the difference of extensional fuzzy numbers is a $\mathcal{C}$-valued metric function. This result demonstrates a nice link between the particular metric for the real numbers and the extensional fuzzy numbers, namely, there is no difference between them in principle. Moreover, this analogous behaviour of the absolute value for the real numbers and the extensional fuzzy numbers gives us a possibility to construct other metrics based on the absolute value. This benefit is demonstrated in several natural constructions of $\mathcal{C}$-valued metric functions for vectors of extensional fuzzy numbers.

Although, the presented results are only preliminary ones, a basic theory similar to mathematical analysis for real numbers can be developed now for the extensional fuzzy numbers. The introduction of the concepts like limits, derivatives or integrals in a space of extensional fuzzy numbers is very challenging for us and it is the main topic for our further research.

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