Integral transforms on spaces of complete residuated lattice valued functions

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***Abstract*—The aim of this paper is to introduce two types of integral transforms that naturally generalize the lower and upper fuzzy transforms for the residuated lattice valued functions. For the construction of such integral transforms we consider a multiplication based fuzzy (qualitative) integral and an integral kernel in the form of a special binary fuzzy relation. We analyze the basic properties of the proposed integral transforms.**

***Index Terms*—Integral transform, Fuzzy transform, Residuated lattice, Kernel, Generalized Sugeno integral.**

I. Introduction

Integral transforms are mathematical operators that produce a new function $g(y)$ by integrating the product of an existing function $f(x)$ and an integral kernel function $K(x,y)$ between suitable limits. An integral kernel function forms a link between the domains of functions $f(x)$ and $g(y)$. The Fourier and Laplace transforms belong among the most popular integral transforms and are applied for real or complex valued functions. Integral transforms are very useful in solving practical problems from different areas of science and engineering as solving (partial) differential equations, signal and image processing, spectral analysis of stochastic processes (see, e.g., [1]–[3]).

In fuzzy set theory we usually deal with functions whose function values belong to an appropriate algebra of truth values as a residuated lattice and its special variants as the BL-algebra, MV-algebra, IMTL-algebra (see, e.g., [4]–[6]). Also for this type of (residuated) lattice valued functions we can recognize a type of integral transforms which are hidden under the name lattice valued upper and lower fuzzy transforms. These fuzzy transforms were proposed by Perfilieva in [7] and further developed in several papers [8]–[14]. To show that these fuzzy transforms are particular cases of integral transforms, let us briefly recall their definitions. We assume that L is a complete residuated lattice, and let $\mathcal{P}(X)$ and $\mathcal{F}(X)$ denote the power set of X and the set of all fuzzy sets $f : X \to L$, respectively.[1] Let X, Y be non-empty sets, and let $\mathbf{A} = \{A_y : X \to L \mid y \in Y\}$ be a fuzzy partition of X, i.e., $\bigcup_{y\in Y} A_y = X$ and $\mathrm{Core}(A_y) \cap \mathrm{Core}(A_{y'}) = \emptyset$ for $y \neq y'$, where $\mathrm{Core}(A_y) = \{x \in X \mid A_y(x) = \top\}$. The *upper fuzzy transform* is a map $F^{\uparrow}_{\mathbf{A}} : \mathcal{F}(X) \to \mathcal{F}(Y)$ given by

$$F^{\uparrow}_{\mathbf{A}}(f)(y) = \bigvee_{x\in X} f(x) \otimes A_y(x)$$

for any $f \in \mathcal{F}(X)$ and $y \in Y$. The *lower fuzzy transform* is a map $F^{\downarrow}_{\mathbf{A}} : \mathcal{F}(X) \to \mathcal{F}(Y)$ given by

$$F^{\downarrow}_{\mathbf{A}}(f)(y) = \bigwedge_{x\in X} A_y(x) \to f(x)$$

for any $f \in \mathcal{F}(X)$ and $y \in Y$. To interpret these fuzzy transforms as integral transforms, we consider a Sugeno like fuzzy integral for residuated lattice valued functions (see [15]–[17]) which is defined as

$$\int^{\otimes} f \; d\mu = \bigvee_{A\in\mathcal{F}} \Big(\bigwedge_{x\in A} f(x)\Big) \otimes \mu(A)$$

for any fuzzy measure space $(X, \mathcal{F}, \mu)$. First, if we consider $\mathcal{F} = \mathcal{P}(X)$ and $\mu(A) = \top$ for any $A \in \mathcal{F}$ such that $A \neq \emptyset$, then $\int^{\otimes} f \; d\mu = \bigvee_{x\in X} f(x)$. Defining from a fuzzy partition $\mathbf{A}$ an integral kernel function $K : X \times Y \to L$ by $K(x,y) = A_y(x)$ for any $x \in X$ and $y \in Y$, we simply find that

$$F^{\uparrow}_{\mathbf{A}}(f)(y) = \int^{\otimes} f(x) \otimes K(x,y)\, d\mu,$$

which can be seen as a particular case of a multiplication based (Sugeno) integral transform. Moreover, if we consider again $\mathcal{F} = \mathcal{P}(X)$, but $\mu(A) = \bot$ for any $A \in \mathcal{F}$ such that $A \neq X$, then $\int^{\otimes} f \; d\mu = \bigwedge_{x\in X} f(x)$. Considering the same integral kernel function K as before, we simply find that

$$F^{\downarrow}_{\mathbf{A}}(f)(y) = \int^{\otimes} K(x,y) \to f(x)\, d\mu,$$

which again can be recognized as a particular case of a residuum based (Sugeno) integral transform, where only the multiplication $\otimes$ is replaced by the residuum $\to$. It is well known that the lower and upper fuzzy transforms can approximate the original function [7], which is one of the valuable properties that hold for the classical integral transforms. In

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[1]For further notation and definitions of concepts used below, we refer to Sections II-A and II-B.

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contrast to the classical integral transforms the fuzzy transforms can fully reconstruct only extraordinary functions like constant functions, but the residuated lattice valued functions possess only very weak properties comparing them with real or complex-valued functions; therefore, nobody can expect that all properties of classical integral transforms can be simply preserved by the fuzzy transforms. Now, a natural question appears: what kinds of properties are satisfied by the previously introduced (Sugeno) integral transforms. This paper aims to introduce the theory of (Sugeno) integral transform and show some preliminary results that should stimulate further research focusing on the more advanced properties and the need to seek appropriate applications in areas like image or signal processing.

The paper is structured as follows. The next section is Preliminary, where we recall the basic definitions of complete residuated lattices, topological and fuzzy measure spaces and present several new results on the measurability of residuated lattice valued functions. The third section is devoted to the Sugeno like fuzzy integral for residuated lattice valued functions. The integral transforms are introduced in the fourth section, where we provide several basic properties for them. The last section is a conclusion.

II. Preliminaries

A. Algebra of truth values

We assume that the structure of truth values is a *complete residuated lattice*, i.e., an algebra $L = \langle L, \wedge, \vee, \otimes, \rightarrow, \bot, \top \rangle$ with four binary operations and two constants such that $\langle L, \wedge, \vee, \bot, \top \rangle$ is a complete lattice, where $\bot$ is the least element and $\top$ is the greatest element of L, $\langle L, \otimes, \top \rangle$ is a commutative monoid (i.e., $\otimes$ is associative, commutative and the identity $a \otimes \top = a$ holds for any $a \in L$) and the adjointness property is satisfied, i.e.,

$$a \leq b \rightarrow c \quad \text{iff} \quad a \otimes b \leq c \tag{1}$$

holds for each $a, b, c \in L$, where $\leq$ denotes the corresponding lattice ordering. A residuated lattice is *divisible*, if $a \otimes (a \rightarrow b) = a \wedge b$ holds for arbitrary $a, b \in L$. The operation of *negation* is defined as $\neg a = a \rightarrow \bot$. Then a residuated lattice satisfies the *law of double negation* if $\neg\neg a = a$ holds for any $a \in L$. A divisible residuated lattice satisfying the law of double negation is called an *MV-algebra*. For other information about residuated lattices, we refer to [4], [5].

Example 2.1: It is well-known (see, e.g., [18]) that the algebra

$$L_T = \langle [0,1], \min, \max, T, \rightarrow_T, 0, 1 \rangle,$$

where T is a left continuous t-norm [19] and $a \rightarrow_T b = \bigvee \{c \in [0,1] \mid T(a,c) \leq b\}$, defines the residuum, is a complete residuated lattice.

Example 2.2: Let $a, b \in [0, \infty]$ be such that $a < b$. One checks easily that $L_{[a,b]} = \langle [a,b], \min, \max, \min, \rightarrow, a, b \rangle$, where

$$c \rightarrow d = \begin{cases} b, & \text{if } c \leq d, \\ d, & \text{otherwise,} \end{cases} \tag{2}$$

is a complete residuated lattice. Note that $L_{[a,b]}$ is a special example of a more general residuated lattice called a *Heyting algebra*.

Recall that a topological space is a pair (X, τ), where X is a non-empty set and $\tau \subseteq \mathcal{P}(X)$ satisfies the following axioms:

(i) $\emptyset, X \in \tau$,
(ii) $\bigcup_{i \in I} U_i \in \tau$ for any $\{U_i\}_{i \in I} \subseteq \tau$,
(iii) $U \cap V \in \tau$ for any $U, V \in \tau$.

The elements of τ are called *open sets*. The following example shows a construction of topology on a complete residiuated lattice.

Example 2.3: Let $u : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ be defined as

$$u(X) = \{x \in L \mid \exists a \in X, \, a \leq x\} \tag{3}$$

for any $X \in \mathcal{P}(L)$. Obviously, $X \subseteq u(X)$. A set $X \in \mathcal{P}(L)$, for which $u(X) = X$ holds, is called the *upper set* or *upset*. We use $\mathcal{U}(L)$ to denote the set of all upsets in L, i.e., $\mathcal{U}(L) = \{u(X) \mid X \in \mathcal{P}(L)\}$. Trivially, we have $\emptyset, X \in \mathcal{U}(L)$. Moreover, one can simply prove that the intersection and the union of a non-empty family of upsets is an upset. Let us show that this is true for the intersection of a non-empty family of upsets. Let $C^{\downarrow} : \mathcal{U}(L) \rightarrow \mathcal{U}(L)$ be defined as

$$C^{\downarrow}(X) = \{x \mid \neg\neg x \in X\} \tag{4}$$

for any $X \in \mathcal{U}(L)$. Since X is an upset in L and $x \leq \neg\neg x$, we obtain $X \subseteq C^{\downarrow}(X)$. It is easy to show that

(i) $C^{\downarrow}(C^{\downarrow}(X)) = C^{\downarrow}(X)$ for any $X \in \mathcal{U}(L)$,
(ii) $\bigcup_{i \in I} C^{\downarrow}(X_i) = C^{\downarrow}(\bigcup_{i \in I} X_i)$ for any family $\{X_i \mid i \in I\} \subseteq \mathcal{U}(L)$,
(iii) $\bigcap_{i \in I} C^{\downarrow}(X_i) = C^{\downarrow}(\bigcap_{i \in I} X_i)$ for any family $\{X_i \mid i \in I\} \subseteq \mathcal{U}(L)$.

Hence, we find that $C^{\downarrow}$ is a closure operator on $\mathcal{U}(L)$. Note that if the negation $\neg$ is involutive, i.e. $\neg\neg x = x$ for any $x \in L$, we have $C^{\downarrow}(X) = X$ for any $X \in \mathcal{U}(L)$. Now, we can introduce a topology $\tau_{C^{\downarrow}}$ on L as the system of closed upsets in L with respect to $C^{\downarrow}$. Particularly, we define

$$\tau_{C^{\downarrow}} = \{C^{\downarrow}(X) \mid X \in \mathcal{U}(L)\}. \tag{5}$$

An extension of (4) can be done for an arbitrary $a \in L$ by the formula

$$C_a^{\downarrow}(X) = \{x \in L \mid \exists n \in \mathbb{N} : \, (\neg\neg a)^n \rightarrow \neg\neg x \in X\}.$$

In [20], it was proved that the map $C_a^{\downarrow}$ satisfies the properties (i)-(iii), where (iii) holds only for a finite family of upsets, and (5) defines a topology on L. Note that $C_{\top}^{\downarrow} = C^{\downarrow}$.

B. Fuzzy sets

Let L be a complete residuated lattice, and let X be a non-empty universe of discourse. A map $A : X \rightarrow L$ is called a *fuzzy set on* X. A value $A(x)$ is called a *membership degree of* x *in the fuzzy set* A. The set of all fuzzy sets on X is denoted by $\mathcal{F}(X)$. A fuzzy set A on X is called *crisp* if $A = \top_Z$ for a certain $Z \subseteq X$, where $\top_Z$ denotes the characteristic function of Z. Particularly, $\emptyset$ denotes the empty fuzzy set on

X, i.e., $\emptyset(x) = \perp$ for any $x \in X$. For the sake of better readability of the text, we do not distinguish between X and $\top_X$ and similarly for Y, Z etc. The set of all crisp fuzzy sets (i.e., subsets) on X is denoted in the same way as the power set of X, i.e., $\mathcal{P}(X)$. A fuzzy set A is said to be *constant* if there is $a \in L$ such that $A = a \otimes \top_X$. A constant fuzzy set on X is denoted by a_X, i.e., $a_X(x) = a$ for any $x \in X$. We use $\mathrm{Supp}(A) = \{x \mid x \in X \ \&\ A(x) > \perp\}$ and $\mathrm{Core}(A) = \{x \mid x \in X \ \&\ A(x) = \top\}$ to denote the *support* and the *core* of a fuzzy set A, respectively.

Let A, B be fuzzy sets on X. An extension of the operations $\wedge$, $\vee$, $\otimes$ and $\rightarrow$ on L to the operations on $\mathcal{F}(X)$ is given by

$$(A \wedge B)(x) = A(x) \wedge B(x), \tag{6}$$

$$(A \vee B)(x) = A(x) \vee B(x), \tag{7}$$

$$(A \otimes B)(x) = A(x) \otimes B(x), \tag{8}$$

$$(A \rightarrow B)(x) = A(x) \rightarrow B(x) \tag{9}$$

for any $x \in X$. Obviously, $A \wedge B$ and $A \vee B$ are the classical definitions of the *intersection* and *union* of fuzzy sets A and B, respectively. Further, we say that a fuzzy set A is a *fuzzy subset of* a fuzzy set B and denote it by $A \subseteq B$ whenever $A(x) \leq B(x)$ holds for any $x \in X$. Moreover, a fuzzy set A is *equal* to a fuzzy set B and denote it by $A = B$ whenever $A \subseteq B$ and $B \subseteq A$.

Let $X_1, \dots, X_n$ be non-empty universes for $n > 0$. A fuzzy set $K : X_1 \times \cdots \times X_n \to L$ is called an n-ary fuzzy relation. For the sake od simplicity, K is called simply a fuzzy relation if $n = 2$. An n-ary fuzzy relation K is said to be *normal*, whenever $\mathrm{Core}(K) \neq \emptyset$, and *normal in the i-th coordinate*, whenever $\mathrm{Core}(K_{x_i}) \neq \emptyset$ for any $x_i \in X_i$ where $K_{x_i} : X_1 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_n \to L$ is defined as

$$\begin{aligned} K_{x_i}(x_1, \dots, &x_{i-1}, x_{i+1}, \dots, x_n) = \\ &K(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n), \end{aligned} \tag{10}$$

for any $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_1 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_n$. An n-ary fuzzy relation K is said to be *complete normal* whenever K is normal in the i-th coordinate for any $i = 1, \dots, n$. A relaxation of the normality of n-fuzzy relation is a *semi-normal* n-ary fuzzy relation defined as $K \neq \emptyset$, i.e., there exists $(x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ such that $K(x_1, \dots, x_n) > \perp$. Similarly one can define *semi-normal in the i-th coordinate* and *complete semi-normal* fuzzy relation.

C. Fuzzy measure spaces

a) Measurable spaces and functions: Let us consider algebras of sets as follows.

Definition 2.1: Let X be a non-empty set. A subset $\mathcal{F}$ of $\mathcal{P}(X)$ is an *algebra of sets on X* provided that

(A1) $\emptyset, X \in \mathcal{F}$,
(A2) if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$,
(A3) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

Definition 2.2: An algebra $\mathcal{F}$ of sets on X is a *σ-algebra of sets* if

(A4) if $A_i \in \mathcal{F}$, $i = 1, 2, \dots$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

A pair $(X, \mathcal{F})$ is called a *measurable space* (on X) if $\mathcal{F}$ is an algebra (σ-algebra) of sets on X. Let $(X, \mathcal{F})$ be a measurable space and $A \in \mathcal{F}(X)$. We say that A is *$\mathcal{F}$-measurable* if $A \in \mathcal{F}$.

We now present some examples of algebras and σ-algebras of sets on a non-empty fuzzy set on X.

Example 2.4: The sets $\{\emptyset, X\}$ and $\mathcal{P}(X)$ are σ-algebras of fuzzy sets on X.

A very useful tool how to define an algebra (σ-algebra) of sets on X is to generate it from a family of sets.

Definition 2.3: Let $\mathcal{G} \subseteq \mathcal{P}(X)$ be a non-empty family of sets. The smallest algebra (σ-algebra) on X containing $\mathcal{G}$ is denoted by $\mathrm{alg}(\mathcal{G})$ ($\sigma(\mathcal{G})$) and is called the *generated algebra (σ-algebra) by family $\mathcal{G}$*.

The following well-known theorem shows how an algebra is determined from a family of sets.

Theorem 2.1: Let $\mathcal{G} \subseteq \mathcal{P}(X)$ be a non-empty family which contains X (or $\emptyset$). Then $\mathrm{alg}(\mathcal{G})$ is the set consisting of all complements and finite unions over $\mathcal{G}$.

Note that the elements of $\sigma(\mathcal{G})$ cannot be simply determined and a transfinite construction has to be considered for this purpose. Using the previous theorem, we can determine various examples of algebras of sets.

Example 2.5: Let τ_X be a topology on X. Let $\mathcal{B} = \tau_X$. The set consisting of all complements and finite unions over $\mathcal{B}$ is an algebra of sets on X. Obviously, this generated algebra is the smallest algebra of sets containing all sets of $\mathcal{B}$.

Example 2.6: Let L be a residuated lattice and $\mathcal{U}(L)$ denote the set of all upsets in L introduced in Example 2.3. The set of all complements and finite unions over $\mathcal{U}(L)$ is an algebra of sets on the residuated lattice L.

Example 2.7: Let L be a residuated lattice endowed by the topological space $\tau_{C_a^{\downarrow}}$ for $a \in L$ introduced in Example 2.3. Similarly to Example 2.5, the set of all complements and finite unions over $\tau_{C_a^{\downarrow}}$ is an algebra of sets on the residuated lattice L.

Example 2.8: Let L be a residuated lattice and $\mathcal{U}_1(L)$ denote the set of all upsets determined by one element of L, i.e., $\{x \in L \mid x \geq a\}$ for any $a \in L$. Similarly to Example 2.5, the set of all complements and finite unions over $\mathcal{U}_1(L)$ is an algebra of sets on the residuated lattice L. Obviously, $\mathcal{U}_1(L) \subseteq \mathcal{U}(L)$.

Remark 2.9: It is easy to see that if $\mathcal{F}$ is an algebra (σ-algebra) of sets, then the intersection of finite (countable) number of sets belongs to $\mathcal{F}$.

Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, and let $f : X \to Y$ be a function. We say that f is $\mathcal{F}$-$\mathcal{G}$-measurable if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$.

Lemma 2.2: Let $\mathcal{G} \subseteq \mathcal{P}(Y)$ be a subset such that $Y \in \mathcal{G}$, and let $(X, \mathcal{F})$ be a measurable space. A function $f : X \to Y$ is $\mathcal{F}$-$\mathrm{alg}(\mathcal{G})$-measurable if and only if $f^{-1}(Z) \in \mathcal{F}$ for any $Z \in \mathcal{G}$.

Sketch of the proof: The implication ($\Rightarrow$) is a simple consequence of $\mathcal{G} \subseteq \mathrm{alg}(\mathcal{G})$. The opposite implication ($\Leftarrow$) can be proved expressing the elements of $\mathrm{alg}(\mathcal{G})$ with the help of the following equalities:

$$f^{-1}(Y \setminus Z) = X \setminus f^{-1}(Z),$$
$$f^{-1}(\bigcup_{i=1}^{n} Z_i) = \bigcup_{i=1}^{n} f^{-1}(Z_i),$$
$$f^{-1}(\bigcap_{i=1}^{n} Z_i) = \bigcap_{i=1}^{n} f^{-1}(Z_i),$$

which hold for any $Z \in \mathcal{P}(Y)$, $\{Z_i \mid i = 1, \ldots, n\} \subseteq \mathcal{P}(Y)$ and a natural number n. □

Remark 2.10: Note that an analogous statement can be formulated for σ-algebras.

If f and g are $\mathcal{F}$-$\mathcal{G}$-measurable, the question arises when $f \star g$, where $\star \in \{\wedge, \vee, \otimes, \rightarrow\}$, is $\mathcal{F}$-$\mathcal{G}$-measurable. In the following part, we show results partially answering the previous question. For the purpose of this paper, we restrict ourselves to $\mathcal{G} = \mathcal{U}(L)$ and $\mathcal{G} = \tau_{C^{\downarrow}}$.

Theorem 2.3: Let L be linearly ordered, let $(X, \mathcal{F})$ be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-alg$(\mathcal{U}(L))$-measurable fuzzy sets. Then

$$f \wedge g, f \vee g \in \mathcal{B}, \quad f, g \in \mathcal{B}.$$

Proof: Since the proofs for both operations are analogous, we verify here only the case of $\wedge$. By Remark 2.10 and Lemma 2.2, we have to prove that for any $f, g \in \mathcal{B}$ and $Y \in \mathcal{U}(L)$, we obtain $(f \wedge g)^{-1}(Y) \in \mathcal{F}$. Put $h = f \wedge g$. We show that $h^{-1}(Y) = f^{-1}(Y) \cap g^{-1}(Y)$. Let $x \in h^{-1}(Y)$. Then $h(x) \in Y$. Since $f(x) \geq h(x)$ and $g(x) \geq h(x)$ and $h(x) \in Y$, we find that $f(x), g(x) \in Y$. Hence, we obtain $x \in f^{-1}(Y)$ and simultaneously $x \in g^{-1}(Y)$; therefore, $x \in f^{-1}(Y) \cap g^{-1}(Y)$, and thus $h^{-1}(Y) \subseteq f^{-1}(Y) \cap g^{-1}(Y)$. Now, let $x \in f^{-1}(Y) \cap g^{-1}(Y)$. Then $f(x) \in Y$ and $g(x) \in Y$. Since L is linearly ordered, we find that $h(x) = f(x)$ or $h(x) = g(x)$; therefore, $h(x) \in Y$. Hence, we obtain $f^{-1}(Y) \cap g^{-1}(Y) \subseteq h^{-1}(Y)$, and the equality is proved. Since $f^{-1}(Y), g^{-1}(Y) \in \mathcal{F}$, we find that $h^{-1}(Y) = f^{-1}(Y) \cap g^{-1}(Y) \in \mathcal{F}$. □

To extend the previous result for the non-linear residuated lattices, we need an additional assumption. The proof is omitted because of the lack of space.

Theorem 2.4: Let $(X, \mathcal{F})$ be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-alg$(\mathcal{U}(L))$-measurable fuzzy sets. If $\mathcal{F}$ is closed over arbitrary unions, then

$$f \otimes g, f \wedge g, f \vee g, \quad f, g \in \mathcal{B}.$$

The next statement shows a different condition under which the measurability of the multiplication of measurable functions is ensured. Similarly to the previous theorem, we omit the proof.

Theorem 2.5: Let $(X, \mathcal{F})$ be a σ-algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-alg$(\tau_{C^{\downarrow}})$-measurable fuzzy sets, where $\tau_{C^{\downarrow}}$ is the topology on L determined by closed upsets in L with respect to $C^{\downarrow}$. If L is a second-countable space (complete separable), then

$$f \otimes g \in \mathcal{B}, \quad f, g \in \mathcal{B}.$$

The last theorem of this series of statements is devoted to the measurability of the residuum operation which is principally different from the previous operations, because its monotonically decreasing in the first argument and the monotonically increasing in the second argument. Again the proof has to be omitted because of the space limitation.

Theorem 2.6: Let L be linearly ordered and dense. Let $(X, \mathcal{F})$ be an algebra, and let $\mathcal{B} \subseteq \mathcal{F}(X)$ be a set of all $\mathcal{F}$-alg$(\mathcal{U}(L))$-measurable fuzzy sets. If $\mathcal{F}$ is closed over arbitrary unions, then

$$f \rightarrow g \in \mathcal{B}, \quad f, g \in \mathcal{B}.$$

b) Fuzzy measures: Let us introduce the concept of fuzzy measure as follows. The definition is a modification of the definition of the normed measure with respect to truth values (e.g., [21], [22]).

Definition 2.4: Let $(X, \mathcal{F})$ be a measurable space. A map $\mu : \mathcal{F} \rightarrow L$ is called a *fuzzy measure* on $(X, \mathcal{F})$ if

(i) $\mu(\emptyset) = \bot$ and $\mu(X) = \top$,
(ii) if $A, B \in \mathcal{F}$ such that $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

A triplet $(X, \mathcal{F}, \mu)$ is called a *fuzzy measure space* whenever $(X, \mathcal{F})$ is a measurable space and μ is a fuzzy measure on $(X, \mathcal{F})$.

Definition 2.5: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space. We say that the fuzzy measure μ is

1) *lower-continuous* if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ such that $A_1 \subset A_2 \subset \cdots$ and $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, then
$$\bigvee_{n \to \infty} \mu(A_n) = \mu(A).$$
2) *upper-continuous* if $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ such that $A_1 \supset A_2 \supset \cdots$ and $A = \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$, then
$$\bigwedge_{n \to \infty} \mu(A_n) = \mu(A).$$

Example 2.11: Let L_{T} be an algebra from Example 2.1, where T is a continuous t-norm. Let X be a finite non-empty set, and let $\mathcal{F}$ be an arbitrary algebra. Then, we can define fuzzy measure space $(X, \mathcal{F}, \mu^r)$, where

$$\mu^r(A) = \frac{|A|}{|X|}$$

for all $A \in \mathcal{F}$, where $|A|$ and $|X|$ denote the cardinality of A and $|X|$, respectively.

III. Generalized Integrals for Functions Valued in Complete Residuated Lattices

This section is devoted to the multiplication based fuzzy integrals introduced in [15] (see also [16]) which, in some sense, generalizes the well known Sugeno integral [23].

The integrated functions are fuzzy sets on X. To keep the notation of integrals the same as in the classical measure theory, we prefer, in this section, to use denotations f, g for the integrated functions instead of A, B. Nevertheless, we deal with them as with fuzzy sets. For example, $f \cap g$ denotes the intersection of fuzzy sets. The multiplication based fuzzy integral is defined over the multiplication $\otimes$ of a complete residuated lattice as follows.

Definition 3.1: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and let $f : X \to L$. The *$\otimes$-fuzzy integral of f on X* is given by

$$\int^{\otimes} f \; d\mu = \bigvee_{A \in \mathcal{F}^-} \bigwedge_{x \in A} (f(x) \otimes \mu(A)) \tag{11}$$

Note that $\mathcal{F}$ has to be restrict to $\mathcal{F}^-$ in (11), otherwise, the value of the $\otimes$-fuzzy integral is trivially equal to $\top$ which is a consequence of $\bigwedge \emptyset = \top$. If the residuate lattice is an MV-algebra, we obtain an equivalent defintion of the $\otimes$-fuzzy integral.

Theorem 3.1 ([16]): Let L be a complete MV-algebra. Then

$$\int^{\otimes} f \; d\mu = \bigvee_{A \in \mathcal{F}} (\bigwedge_{x \in A} f(x)) \otimes \mu(A). \tag{12}$$

Note that Dubois, Prade and Rico in [17] defines their multiplitacion based fuzzy (qualitative) integral by formula (12). Let us emphasis that both definitions are not identical in general. In what follows, we restrict our consideration to the $\otimes$-fuzzy integral defined by formula (12).

One can see and might be surprised that we do not assume an $\mathcal{F}$-$\mathcal{G}$-measurability (e.g., $\mathcal{G} = \mathrm{alg}(\mathcal{U}(L))$) of the function f in the previous definitions of $\otimes$-fuzzy integral. If we consider $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L))$-measurability of the function f we obtain a very useful formula for the computation of the $\otimes$-fuzzy integral.

Theorem 3.2: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and let $f : X \to L$ be $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L))$-measurable. Then

$$\int^{\otimes} f \; d\mu = \bigvee_{a \in L} \left(a \otimes \mu(\{x \in X \mid f(x) \geq a\}) \right). \tag{13}$$

Proof: Let $a \in L$ and denote $L_a = \{x \in L \mid x \geq a\}$. Note that $u(\{a\}) = L_a$, where u is defined in Example 2.3. By the assumption on the $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L))$-measurability of f, we have $f^{-1}(L_a) \in \mathcal{F}$, where $f^{-1}(L_a) = \{x \in X \mid f(x) \geq a\}$. Put $I = \bigvee_{A \in \mathcal{F}} \left(\mu(A) \otimes \bigwedge_{x \in A} f(x)\right)$ and $J = \bigvee_{a \in L} \left(a \otimes \mu(f^{-1}(L_a))\right)$. First, we show that $I \leq J$. Let $\lambda_f : \mathcal{F} \to L$ be a map given by $\lambda_f(A) = \bigwedge_{x \in A} f(x)$. Obviously, $A \subseteq f^{-1}(L_{\lambda_f(A)})$, and thus $\mu(A) \leq \mu(f^{-1}(L_{\lambda_f(A)}))$, where we used the fact that f is $\mathcal{F}$-measurable. Since $\lambda_f(\mathcal{F}) \subseteq L$, we obtain

$$I \leq \bigvee_{A \in \mathcal{F}} \lambda_f(A) \otimes \mu(f^{-1}(L_{\lambda_f(A)})) \leq J.$$

Further, let $\varrho_f : L \to \mathcal{F}$ be given by $\varrho_f(a) = f^{-1}(L_a)$. From the $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L))$-measurability of f, the map ϱ_f is well defined. Obviously, $\bigwedge_{x \in \varrho_f(a)} f(x) \geq a$ for any $a \in L$ and $\varrho_f(L) \subseteq \mathcal{F}$. Then, we obtain

$$J \leq \bigvee_{a \in L} \left(\mu(\varrho_f(a)) \otimes \bigwedge_{x \in \varrho_f(a)} f(x) \right) \leq I.$$

Hence, we obtain $I = J$ which concludes the proof. □

We say that $f, g \in \mathcal{F}(X)$ are *comonotonic* if and only if there is no pair $x_1, x_2 \in X$ such that $f(x_1) > f(x_2)$ and simultaneously $g(x_1) < g(x_2)$.

Lemma 3.3: Let L be linearly ordered, and let $f, g \in \mathcal{F}(X)$. Denote $C_f = \{C_f(a) \mid a \in L\}$, where $C_f(a) = \{x \in X \mid f(x) \geq a\}$. Then C_f is a chain with respect to $\subseteq$, and if f and g are comonotonic, then $C_{f \star g}(a) = C_f(a)$ or $C_{f \star g}(a) = C_g(a)$ for any $a \in L$, where $\star \in \{\wedge, \vee\}$.

Proof: The first statement is trivial. To proof the second statement, we restrict ourselves to the case $\otimes = \wedge$. The second case can be verified analogously.

First, let us show that $C_f(a) \cap C_g(a) = C_{f \wedge g}(a)$ holds for any $a \in L$. Let $x \in C_f(a) \cap C_g(a)$. Then $f(x) \geq a$ and $g(x) \geq a$. Hence, $f(x) \wedge g(x) \geq a$, which implies $x \in C_{f \wedge g}(a)$. Now, let $x \in C_{f \wedge g}(a)$. Since $f(x) \wedge g(x) \geq a$, we immediately get $x \in C_f(a)$ and $x \in C_g(a)$. Hence, $x \in C_f(a) \cap C_g(a)$. Further, we show that $C_{f \wedge g}(a) = C_f(a)$ or $C_{f \wedge g}(a) = C_g(a)$ for any $a \in L$, whenever f and g are comonotonic. Assume that $C_f(a) \not\subset C_g(a)$ and simultaneously $C_g(a) \not\subset C_f(a)$ for some $a \in L$. From $C_f(a) \not\subset C_g(a)$ there exists $x \in C_f(a)$ and $x \notin C_g(a)$, which implies $g(x) < a \leq f(x)$, and similarly, from $C_g(a) \not\subset C_f(a)$ there exists $y \in X$ such that $y \in C_g(a)$ and $y \notin C_f(a)$, which implies $f(y) < a \leq g(y)$, where we used the linearity of L. But this is a contradiction with the comonotonicity of f and g, since there exist $x, y \in X$ with $f(x) < f(y)$ and simultaneously $g(y) < g(x)$. □

The following theorem shows that $\wedge$-fuzzy integral is comonotonically minitive and comonotonically maxitive (cf. [24, Theorem 4.44]).

Theorem 3.4: Let L be linearly ordered complete Heyting algebra, and let $f, g \in \mathcal{F}(X)$ be comonotonic maps that are $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L)$-measurable. Then

$$\int^{\wedge} (f \star g) \; d\mu = \int^{\wedge} f \; d\mu \star \int^{\wedge} g \; d\mu$$

for $\star \in \{\wedge, \vee\}$.

Proof: We restrict ourselves to the proof of the case $\otimes = \wedge$, the second case can be proved analogously. According to Theorem 2.4, the map $f \wedge g$ is $\mathcal{F}$-$\mathrm{alg}(\mathcal{U}(L)$-measurable. Hence, we can use formula (13) to compute the $\wedge$-fuzzy integral, i.e.,

$$\begin{aligned}\int^{\wedge} (f \wedge g) \; d\mu &= \bigvee_{a \in L} a \wedge \mu(\{x \in X \mid f(x) \wedge g(x) \geq a\}) \\ &= \bigvee_{a \in L} a \wedge \mu(C_{f \wedge g}(a)),\end{aligned}$$

where we used the notation from Lemma 3.3. Since $C_{f \wedge g}(a) = C_f(a) \subseteq C_g(a)$ or $C_{f \wedge g}(a) = C_g(a) \subseteq C_f(a)$ for any $a \in L$, we obtain

$$\mu(C_{f \wedge g}(a)) = \mu(C_f(a)) \wedge \mu(C_g(a)),$$

where we used the monotonicity of μ. Hence, we obtain

$$\begin{aligned}\int^{\wedge}(f\wedge g)\ d\mu &= \bigvee_{a\in L} a\wedge\mu(C_{f\wedge g}(a))\\ &= \bigvee_{a\in L} a\wedge(\mu(C_f(a))\wedge\mu(C_g(a)))\\ &\leq \bigvee_{a\in L} a\wedge\mu(C_f(a))\wedge\bigvee_{b\in L} b\wedge\mu(C_g(b))\\ &= \int^{\wedge} f\ d\mu\wedge\int^{\wedge} g\ d\mu.\end{aligned}$$

On the opposite side, we have

$$\begin{aligned}&\int^{\wedge} f\ d\mu\wedge\int^{\wedge} g\ d\mu =\\ &= \bigvee_{a\in L} a\wedge\mu(C_f(a))\wedge\bigvee_{b\in L} b\wedge\mu(C_g(b))\\ &= \bigvee_{a\in L}\bigvee_{b\in L}(a\wedge\mu(C_f(a)))\wedge(b\wedge\mu(C_g(b))\\ &\leq \bigvee_{a\in L}\bigvee_{b\in L}(a\wedge b)\wedge(\mu(C_f(a\wedge b))\wedge\mu(C_g(a\wedge b)))\\ &= \bigvee_{a\in L}\bigvee_{b\in L}(a\wedge b)\wedge\mu(C_{f\wedge g}(a\wedge b)) =\\ &= \bigvee_{a\in L} a\wedge\mu(C_{f\wedge g}(a) = \int^{\wedge}(f\wedge g)\ d\mu,\end{aligned}$$

where we used the distributivity of $\wedge$ over $\bigvee$, which holds in each Heyting algebra, and the fact that $C_f(a)\leq C_f(b)$ for any $a,b\in L$ with $b\leq a$. □

Remark 3.1: One can see that we used very strong assumption on the complete residuated lattice. The generalization of the previous theorem to non-linear lattices or more general multiplication requires likely a generalization of the definition of the concept of comonotonicity. This issue is a subject of our future research.

IV. Integral transforms for lattice valued functions

In this section we propose two types of integral transforms for functions whose function values are evaluated in a complete residuated lattice. For its definition, we use multiplication based fuzzy integral introduced in Subsection III. The integral transforms transforms fuzzy sets from $\mathcal{F}(X)$ to fuzzy sets from $\mathcal{F}(Y)$.

a) $(K,\mu,\otimes)$-integral transform: We start with the definition of the integral transform which is motivate by the $F^{\uparrow}$-transform.

Definition 4.1: Let $(X,\mathcal{F},\mu)$ be a fuzzy measure space, and let $K: X\times Y\to L$ be a semi-normal in the second component fuzzy relation. A map $F^{\otimes}_{(K,\mu)}:\mathcal{F}(X)\to\mathcal{F}(Y)$ defined by

$$F^{\otimes}_{(K,\mu)}(f)(y)=\int^{\otimes} K(x,y)\otimes f(x)\,d\mu. \tag{14}$$

is called a $(K,\mu,\otimes)$-*integral transform.*

One can see that the definition of $(K,\mu,\otimes)$-integral transform is dependent on a measure μ and a semi-normal in the second component fuzzy relation K. The fuzzy relation K will be called the *integral kernel* which corresponds to the standard notation in the theory of integral transforms. We should note that we assume the semi-normality in the second component for the integral kernels from a natural reason where it seems to be strange that an element $y\in Y$ has no relationship to any element from X, i.e., $K(x,y)=\perp$ for any $x\in X$. In this case, the $(K,\mu,\otimes)$-integral transform of any function f from $\mathcal{F}(X)$ at the point $y\in Y$ is trivially equal to $\perp$.

In what follows, let us assume that $(X,\mathcal{F},\mu)$ is a fuzzy measure space and $F^{\otimes}_{(K,\mu)}:\mathcal{F}(X)\to\mathcal{F}(Y)$ is a $(K,\mu,\otimes)$-integral transform. The next theorem shows basic properties of $(K,\mu,\otimes)$-integral transform.

Theorem 4.1: For any $f,g\in\mathcal{F}(X)$ and $a\in L$, we have
- (i) $F^{\otimes}_{(K,\mu)}(f)\leq F^{\otimes}_{(K,\mu)}(g)$ if $f\leq g$;
- (ii) $F^{\otimes}_{(K,\mu)}(f\cap g)\leq F^{\otimes}_{(K,\mu)}(f)\wedge F^{\otimes}_{(K,\mu)}(g)$;
- (iii) $F^{\otimes}_{(K,\mu)}(f)\vee F^{\otimes}_{(K,\mu)}(g)\leq F^{\otimes}_{(K,\mu)}(f\cup g)$;
- (iv) $a\otimes F^{\otimes}_{(K,\mu)}(f)\leq F^{\otimes}_{(K,\mu)}(a\otimes f)$;
- (v) $F^{\otimes}_{(K,\mu)}(a\to f)\leq a\to F^{\otimes}_{(K,\mu)}(f)$;

Proof: (i)-(iii) They are consequences of the monotonicity of the $\otimes$-integral.

(iv) We have

$$\begin{aligned}&F^{\otimes}_{(K,\mu)}(a\otimes f)(y)=\int^{\otimes} K(x,y)\otimes(a\otimes f(x))\,d\mu\\ &= \bigvee_{A\in\mathcal{F}}(\mu(A)\otimes\bigwedge_{x\in A}(K(x,y)\otimes(a\otimes f(x)))\\ &\geq \bigvee_{A\in\mathcal{F}}(\mu(A)\otimes a\otimes\bigwedge_{x\in A}(K(x,y)\otimes f(x))\\ &= a\otimes\bigvee_{A\in\mathcal{F}}(\mu(A)\otimes\bigwedge_{x\in A}(K(x,y)\otimes f(x)))\\ &= a\otimes\int^{\otimes} K(x,y)\otimes f(x)\,d\mu = a\otimes F^{\otimes}_{(K,\mu)}(f)(y),\end{aligned}$$

where we used $a\otimes\bigwedge_{i\in I} b_i\leq\bigwedge_{i\in I}(a\otimes b_i)$ and $a\otimes\bigvee_{i\in I} b_i\leq\bigvee_{i\in I}(a\otimes b_i)$.

(v) We have

$$\begin{aligned}&F^{\otimes}_{(K,\mu)}(a\to f)(y)=\int^{\otimes} K(x,y)\otimes(a\to f(x))\,d\mu\\ &= \bigvee_{A\in\mathcal{F}}(\mu(A)\otimes\bigwedge_{x\in A}(K(x,y)\otimes(a\to f(x)))\\ &\leq \bigvee_{A\in\mathcal{F}}(\mu(A)\otimes\bigwedge_{x\in A}(a\to(K(x,y)\otimes f(x)))\\ &= \bigvee_{A\in\mathcal{F}}(\mu(A)\otimes(a\to\bigwedge_{x\in A}(K(x,y)\otimes f(x)))\\ &\leq \bigvee_{A\in\mathcal{F}}(a\to(\mu(A)\otimes\bigwedge_{x\in A}(K(x,y)\otimes f(x)))\\ &\leq (a\to\bigvee_{A\in\mathcal{F}}(\mu(A)\otimes\bigwedge_{x\in A}(K(x,y)\otimes f(x)))\\ &= a\to\int^{\otimes} K(x,y)\otimes f(x)\,d\mu = a\to F^{\otimes}_{(K,\mu)}(f)(y),\end{aligned}$$

where we used $a \otimes (b \to c) = b \to (a \otimes c)$, $a \to \bigwedge_{i\in I} b_i = \bigwedge_{i\in I}(a \to b_i)$ and $a \to \bigvee_{i\in I} b_i \geq \bigvee_{i\in I}(a \to b_i)$. $\square$

Theorem 4.2: If L is an MV-algebra, then $a \otimes F^{\otimes}_{(K,\mu)}(f) = F^{\otimes}_{(K,\mu)}(a \otimes f)$ for any $f \in \mathcal{F}(X)$ and $a \in L$.

Proof: Since L is an MV-algebra, it holds that $\bigwedge_{i\in I}(a \otimes b_i) = a \otimes \bigwedge_{i\in I} b_i$. Form the proof of (iv) of Theorem 4.1, one can simply derive the desired equality. $\square$

Theorem 4.3: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure, and let K be an integral kernel. If $\mu(\{x \in X \mid K(x,y) = \top\}) = \top$ for any $y \in Y$, then $F^{\otimes}_{(K,\mu)}(a_X) = a_Y$.

Proof: Assume that $\mu(\{x \in X \mid K(x,y) = \top\}) = \top$ for any $y \in Y$, and let $a \in L$. Put $X_y = \{x \in X \mid K(x,y) = \top\}$. Then

$$\begin{aligned} F^{\otimes}_{(K,\mu)}(a_X)(y) &= \bigvee_{A\in\mathcal{F}} (\mu(A) \otimes \bigwedge_{x\in A} (K(x,y) \otimes (a \otimes \top_X(x)) \\ &= \bigvee_{A\in\mathcal{F}} (\mu(A) \otimes \bigwedge_{x\in A} (K(x,y) \otimes a) \\ &\geq \mu(X_y) \otimes \bigwedge_{x\in X_y} (K(x,y) \otimes a = a = a_Y(y). \end{aligned}$$

On the other side, we trivially have $\mu(A) \otimes \bigwedge_{x\in A}(K(x,y) \otimes a) \leq a$ for any $A \in \mathcal{F}$. Hence, we find $F^{\otimes}_{(K,\mu)}(a_X)(y) \leq a = a_Y(y)$, which proves the desired equality. $\square$

As a special case of the previous theorem, we obtain the preservation of the unit function by the $(K,\mu,\otimes)$-integral transform, i.e. $F^{\otimes}_{(K,\mu)}(\top_X) = \top_Y$, if $\mu(\{x \in X \mid K(x,y) = \top\}) = \top$ for any $y \in Y$. Unfortunately, the opposite implication of this statement is not true in general, i.e., $\mu(\{x \in X \mid K(x,y) = \top\}) = \top$ for any $y \in Y$ is also a necessary condition. The following theorem shows a sufficient condition for the fuzzy measures and integral kernels to get the equivalence in the previous theorem. Since the proof is too long to be presented here, we have to omit it because of the space limitation.

Theorem 4.4: Let L be linearly ordered, let μ be a upper-continuous fuzzy measure on $(X, \mathcal{F})$, and let $K(\cdot, y)$ be an $\mathcal{F}$-alg$(\mathcal{U}(L)$-measurable fuzzy set for any $y \in Y$. Then $F^{\otimes}_{(K,\mu)}(\top_X) = \top_Y$ if and only if

$$\mu(\{x \in X \mid K(x,y) = \top\}) = \top$$

for any $y \in Y$.

Let K be an integral kernel. We say that functions $f, g \in \mathcal{F}(X)$ are $(K,\otimes)$-*comonotonic* (or *comonotonically compatible with K and $\otimes$*) if $f \otimes K(\cdot,y)$ and $g \otimes K(\cdot,y)$ are comonotonic for any $y \in Y$.

Theorem 4.5: Let L be a linearly ordered complete Heyting algebra. Let $f, g, K(\cdot,y)$ be $\mathcal{F}$-alg$(\mathcal{U}(L)$-measurable for any $y \in Y$. If f and g are $(K,\wedge)$-comonotonic, then

$$F^{\wedge}_{(K,\mu)}(f \wedge g) = F^{\wedge}_{(K,\mu)}(f) \wedge F^{\wedge}_{(K,\mu)}(g) \tag{15}$$

$$F^{\wedge}_{(K,\mu)}(f \vee g) = F^{\wedge}_{(K,\mu)}(f) \vee F^{\wedge}_{(K,\mu)}(g) \tag{16}$$

Proof: It is a simple consequence of Theorems 2.3 and 3.4 and the fact that $f \wedge K(\cdot,y)$ and $g \wedge K(\cdot,y)$ are comonotonic for any $y \in L$. $\square$

From the previous theorem, one can see that a "linearity" condition for the $(K,\mu,\otimes)$-integral transform holds only under the satisfaction of a very special condition, namely, the infimum or supremum is preserved only for the $(K,\otimes)$-comonotonic functions.

b) $(K,\mu,\to)$-integral transform.: The next definition of the integral transform is motivated by the $F^{\downarrow}$-transform.

Definition 4.2: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure space, and let $K : X \times Y \to L$ be a semi-normal in the second component fuzzy relation. A map $F^{\to}_{(K,\mu)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ defined by

$$F^{\to}_{(K,\mu)}(f)(y) = \int^{\otimes} K(x,y) \to f(x)\, d\mu$$

is called a $(K,\mu,\to)$-integral transform.

Similarly to the $(K,\mu,\otimes)$-integral transform, we assume the semi-normality in the second argument for the integral transform, since the $(K,\mu,\to)$-integral transform of any function from $\mathcal{F}(X)$ is trivially equal to $\top$ for any point $y \in Y$ such that $K(x,y) = \bot$ for any $x \in X$.

In what follows, let us assume that $(X, \mathcal{F}, \mu)$ is a fuzzy measure space and $F^{\to}_{(K,\mu)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a $(K,\mu,\to)$-integral transform. The following theorem shows basic properties of $(K,\mu,\to)$-integral transform.

Theorem 4.6: For any $f, g \in \mathcal{F}(X)$ and $a \in L$, we have

(i) $F^{\to}_{(K,\mu)}(f) \leq F^{\to}_{(K,\mu)}(g)$ if $f \leq g$;
(ii) $F^{\to}_{(K,\mu)}(f \wedge g) \leq F^{\to}_{(K,\mu)}(f) \wedge F^{\to}_{(K,\mu)}(g)$;
(iii) $F^{\to}_{(K,\mu)}(f) \vee F^{\to}_{(K,\mu)}(g) \leq F^{\to}_{(K,\mu)}(f \vee g)$;
(iv) $a \otimes F^{\to}_{(K,\mu)}(f) \leq F^{\to}_{(K,\mu)}(a \otimes f)$;
(v) $F^{\to}_{(K,\mu)}(a \to f) \leq a \to F^{\to}_{(K,\mu)}(f)$,

Proof: It can be done by similar arguments as the proof of Theorem 4.1. $\square$

Theorem 4.7: Let $(X, \mathcal{F}, \mu)$ be a fuzzy measure, and let K be an integral kernel. Then $F^{\to}_{(K,\mu)}(\top_X) = \top_Y$. Moreover, if $\mu(\{x \in X \mid K(x,y) = \top\}) = \top$ and $\mu(Z) \leq a$ for any $Z \in \mathcal{F}$ such $Z \not\subseteq \{x \in X \mid K(x,y) = \top\}$ holds for any $y \in Y$ and a certain $a \in L$, then $F^{\to}_{(K,\mu)}(a_X) = a_Y$.

Proof: We have

$$F^{\to}_{(K,\mu)}(\top_X)(y) = \int^{\otimes} K(x,y) \to \top_X(x)\, d\mu = \top = \top_Y(y),$$

where we used $K(x,y) \to \top_X(x) = \top$, which holds in any residuated lattice.

Let the condition of the theorem is satisfied. Put $X_y^{\top} = \{x \in X \mid K(x,y) = \top\}$. Then

$$\begin{aligned} F^{\to}_{(K,\mu)}(a_X)(y &= \int^{\otimes} K(x,y) \to a_X(x)\, d\mu \\ &= \bigvee_{A\in\mathcal{F}} (\mu(A) \otimes \bigwedge_{x\in A} (K(x,y) \to a_X(x))), \\ &= \bigvee_{\substack{A\in\mathcal{F} \\ A\subseteq X_y^{\top}}} (\mu(A) \otimes \bigwedge_{x\in A} (K(x,y) \to a_X(x)))\vee \\ &\quad \bigvee_{\substack{A\in\mathcal{F} \\ A\not\subseteq X_y^{\top}}} (\mu(A) \otimes \bigwedge_{x\in A} (K(x,y) \to a_X(x))) = a_Y(y), \end{aligned}$$

where the inequality

$$\mu(A) \otimes \bigwedge_{x \in A} (K(x,y) \to a_X(x)) \leq \mu(A) \leq a$$

is used. $\square$

One can see that the different types of integral transforms requires different assumptions to preserve a constant function.

Let K be an integral kernel. We say that functions $f, g \in \mathcal{F}(X)$ a $(K, \to)$-*comonotonic* (or *comonotonicaly compatible with* K *and* $\to$) if $f \to K(\cdot, y)$ and $g \to K(\cdot, y)$ are comonotonic for any $y \in Y$. We finish the paper with the statement showing conditions under which the infimum or supremum of functions is preserved by the $(K, \mu, \to)$-fuzzy transform.

Theorem 4.8: Let L be a linearly ordered complete Heyting algebra which is dense. Let $f, g, K(\cdot, y)$ be $\mathcal{F}$-alg$(\mathcal{U}(L)$-measurable for any $y \in Y$. If f and g are $(K, \to)$-comonotonic, then

$$F^{\to}_{(K,\mu)}(f \wedge g) = F^{\to}_{(K,\mu)}(f) \wedge F^{\to}_{(K,\mu)}(g) \tag{17}$$

$$F^{\to}_{(K,\mu)}(f \vee g) = F^{\to}_{(K,\mu)}(f) \vee F^{\to}_{(K,\mu)}(g) \tag{18}$$

Proof: It is a simple consequence of Theorems 2.6 and 3.4, the fact that $f \to K(\cdot, y)$ and $g \to K(\cdot, y)$ are comonotonic for any $y \in L$ and $a \to (b \wedge c) \leq (a \to b) \wedge (a \to c)$, which holds in any linearly ordered residuated lattice. $\square$

V. Conclusion

In this paper, we introduced two types of integral transforms of residuated lattice valued functions, which are based on the multiplication based fuzzy integral ($\otimes$-fuzzy integral) and an integral kernel function. We investigated the measurability of various constructions of residuated lattice valued functions. Note that the measurability of functions leads to the beneficial formulation of the $\otimes$-fuzzy integral, as was shown in Theorem 3.2. Further, we investigated the property of the preservation of the infimum and supremum for so-called comonotonic functions. We showed a partial result for comonotonic functions, whose function values belong to a linearly ordered complete Heyting algebra. Note that a generalization of this result is a non-trivial task for a further research. Finally, we provided a basic analysis of the proposed integral transforms focusing on the monotonicity property, preservation of a constant function, or preservation of the infimum and supremum. A deeper analysis of the integral transforms properties, as well as the introduction of other types of integral transforms based on so-called residuum based fuzzy integrals (desintegrals) [15], [17], [25], is a subject of our future research.

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