A proposal of the notions of ordered and strengthened ordered directional monotonicity for interval-valued functions based on admissible orders

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***Abstract*—Two of the main research lines in the theory of aggregation functions is the extension to more general domains and the relaxation of the monotonicity conditions. In this work, we discuss the state-of-the-art of the main introduced relaxed forms of monotonicity that can be found in the literature, i.e., weak, directional, ordered directional and strengthened ordered directional monotonicity. We pay special attention to the extension of a relaxed form of monotonicity to the interval-valued setting and we propose the concepts of ordered and strengthened ordered directional monotonicity for this general setting. Moreover, we study the main properties of the functions that satisfy the introduced properties and present some construction methods.**

***Index Terms*—Aggregation function, Directional monotonicity, Ordered directional monotonicity, Strengthened ordered directional monotonicity, Interval-valued function**

I. Introduction

The idea of aggregation deals with the problem of representing the information given by n sources by a single value. In the specific case of numerical values in $[0, 1]$, a function that takes n numerical inputs, outputs a number in the same interval, satisfies two specific boundary conditions and is increasing with respect to all its arguments is called aggregation function. This family of functions has been greatly studied, in part, because of their relevance in many applications [1]–[4].

Among the cutting edge research lines in the theory of aggregation functions, we find the extension of aggregation functions to more general scales and the relaxation of the monotonicity conditions [5], [6].

Regarding the extension to more general scales, the case of interval-valued functions stands out as it is one of the predominant manners of handling uncertainty and measurement errors. Thus, interval-valued aggregation functions have caught the attention of many researchers [7]–[11]. Certainly, there also exist many works dealing with the extension of aggregation functions to other settings [12]–[14].

The relaxation of the monotonicity conditions are motivated by the existence of many functions that are valid to fuse data but do not satisfy the requirement of monotonicity, e.g., the Gini and Lehmer means [15]. However, the idea of aggregation suggest that, if not the usual, some monotonicity-type condition should be required to fuse information. Consequently, in an attempt of creating a framework of functions that are valid to aggregate data but do not necessarily meet all the properties of aggregation functions, Wilkin and Beliakov proposed the notion of weak monotonicity [16]. This concept was then generalized to directional monotonicity [17], which studies the increase of functions $f : [0, 1]^n \to [0, 1]$ along real rays in $\mathbb{R}^n$. From the applied perspective, directional monotonicity has enabled great achievements in the field of fuzzy rule-based classification algorithms [18], [19].

Furthermore, there exist more proposals regarding the relaxation of the monotonicity condition of aggregation functions. In particular, ordered directional monotonicity [20] and strengthened ordered directional monotonicity [21] study a monotonicity condition along rays that vary depending on the specific input vector. Besides, ordered directional monotonicity has been applied in the task of edge detection [22], a computer vision problem.

More recently, works regarding both trends in the aggregation theory have been published. In [23], directional monotonicity for functions that fuse a collection of types of fuzzy values is proposed and, in [24], the case of interval-valued functions is further studied.

In this work, we review the state-of-the-art of the cited weaker forms of monotonicity in the aggregation framework and we propose the introduction of the notions of ordered

This work is supported by the project TIN2016-77356-P (AEI/FEDER, UE), by the Public University of Navarra under the projects PJUPNA13 and PJUPNA1926 and by the Slovak grant APVV-18-0052.

978-1-7281-6932-3/20/$31.00 ©2020 IEEE

directional monotonicity and strengthened ordered directional monotonicity for the interval-valued setting. We also study the particularities of each concept, provide some construction methods and the most relevant properties.

This work is organized as follows: in Section II we introduce the notation and expound some preliminary concepts about interval-valued aggregation functions and admissible orders. In Section III we review all the relaxed forms of monotonicity, such as weak, directional, ordered directional and strengthened ordered directional monotonicity, as well as their main properties. In Section IV we show the definition and basic features of directional monotonicity in the interval-valued setting and in Section V we propose the concepts of ordered directional and strengthened ordered directional monotonicity for the interval-valued setting. Additionally, we study their main properties and discuss some construction methods. We end this work by some concluding remarks and our goals for future work in Section VI.

II. Preliminaries

A. General notation

In this work, we use the letter n to refer to a positive integer, we denote points in the unit hypercube by bold letters, $\mathbf{x} = (x_1, \dots, x_n) \in [0,1]^n$, and real vectors, connoting directions in the space, as $\vec{r} \in \mathbb{R}^n$. Similary, when dealing with intervals, we denote a vector of intervals with capital bold letters, $\mathbf{X} = \big([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}]\big) \in L([0,1])^n$, and vectors in $(\mathbb{R}^2)^n$ by arrowed bold letters, $\vec{\mathbf{v}} = ((a_1, b_1), \dots, (a_n, b_n)) \in (\mathbb{R}^2)^n$.

Additionally, we need to permute the components of n-tuples and, in that account, we denote by $\mathcal{S}_n$ the set of all permutations of n elements. If $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ or $\vec{r} \in \mathbb{R}^n$, then $\mathbf{x}_\sigma$ refers to the point $(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in [0,1]^n$ and, similarly, $\vec{r}_\sigma = (r_{\sigma(1)}, \dots, r_{\sigma(n)}) \in \mathbb{R}^n$. In the case of intervals, we use the same notation, i.e., $\mathbf{X}_\sigma = \Big([\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}], \dots, [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]\Big) \in L([0,1])^n$ and $\vec{\mathbf{v}}_\sigma = \big((a_{\sigma(1)}, b_{\sigma(1)}), \dots, (a_{\sigma(n)}, b_{\sigma(n)})\big) \in (\mathbb{R}^2)^n$.

B. Interval-valued aggregation functions

Let us recall the notion of aggregation function [1], [2].

Definition 1: A function $A\colon [0,1]^n \to [0,1]$ is an aggregation function if it satisfies the following:

1) $A(0, \dots, 0) = 0$;
2) $A(1, \dots, 1) = 1$;
3) A is increasing[1] with respect to all its arguments.

In this work, we are interested in the set of all closed intervals contained in $[0,1]$, which we denote by

$$L([0,1]) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}.$$

Note that given $[a,b]$, $[c,d] \in L([0,1])$, we can define the following order relation: $[a,b] \leq_L [c,d]$ if and only if $a \leq c$ and $b \leq d$, which is known as standard partial order. The pair $(L([0,1]), \leq)$ is a bounded lattice whose top and bottom elements are $1_L = [1,1]$ and $0_L = [0,0]$, respectively.

Similarly, we can extend the standard partial order to the set $L([0,1])^n = L([0,1]) \times \ldots \times L([0,1])$ componentwise, i.e., given $\mathbf{X} = ([\underline{x_1}, \overline{x_1}], \dots, [\underline{x_n}, \overline{x_n}])$ and $\mathbf{Y} = ([\underline{y_1}, \overline{y_1}], \dots, [\underline{y_n}, \overline{y_n}])$, we set

$$\mathbf{X} \leq_{L^n} \mathbf{Y} \text{ if and only if } [\underline{x_i}, \overline{x_i}] \leq_L [\underline{y_i}, \overline{y_i}] \text{ for all } i \in \{1, \dots, n\}.$$

Thus, we recall the definition of an interval-valued, IV, aggregation function.

Definition 2: A function $A : L([0,1])^n \to L([0,1])$ is an IV aggregation function if it satisfies the following:

1) $A(0_L, \dots, 0_L) = 0_L$;
2) $A(1_L, \dots, 1_L) = 1_L$;
3) A is increasing with respect to each argument considering the order $\leq_L$.

C. Admissible orders

Although the principal order relation we use in $L([0,1])$ is the standard partial order $\leq_L$, the existence of incomparable intervals complicates the definition of some of the concepts that we propose in this work. Thus, we make use of admissible orders [25], a family of total orders that refine the partial order in $L([0,1])$.

Definition 3: An order $\preceq$ on $L([0,1])$ is admissible if $\preceq$ is a total order on $L([0,1])$, and for all $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}] \in L([0,1])$, if $[\underline{x}, \overline{x}] \leq_L [\underline{y}, \overline{y}]$ then $[\underline{x}, \overline{x}] \preceq [\underline{y}, \overline{y}]$.

Lexicographical orders and the one defined in [26] are examples of admissible orders.

Example 1: The following are admissible orders on $L([0,1])$. (i)

1) $[\underline{x}, \overline{x}] \preceq_{Lex1} [\underline{y}, \overline{y}]$ if $\underline{x} < \underline{y}$ or ($\underline{x} = \underline{y}$ and $\overline{x} \leq \overline{y}$).
2) $[\underline{x}, \overline{x}] \preceq_{Lex2} [\underline{y}, \overline{y}]$ if $\overline{x} < \overline{y}$ or ($\overline{x} = \overline{y}$ and $\underline{x} \leq \underline{y}$).
3) $[\underline{x}, \overline{x}] \preceq_{XY} [\underline{y}, \overline{y}]$ if $\underline{x} + \overline{x} < \underline{y} + \overline{y}$ or ($\underline{x} + \overline{x} = \underline{y} + \overline{y}$ and $\overline{y} - \underline{y} \leq \overline{x} - \underline{x}$).

Furthermore, admissible orders can be constructed in terms of two generating functions defined on $K([0,1]) = \{(x,y) \in [0,1]^2 \mid x \leq y\}$: an admissible order $\preceq$ on $L([0,1])$ is said to be generated by two continuous functions $f, g : K([0,1]) \to \mathbb{R}$ if it holds that, for all $[\underline{x}, \overline{x}], [\underline{y}, \overline{y}] \in L([0,1])$,

$$[\underline{x}, \overline{x}] \preceq [\underline{y}, \overline{y}] \text{ if and only if } [f(\underline{x}, \overline{x}), g(\underline{x}, \overline{x})] \preceq_{Lex1} [f(\underline{y}, \overline{y}), g(\underline{y}, \overline{y})].$$

Please see [25] for details of the properties that the cited functions ought to satisfy.

Example 2: Taking that construction method into account, we can define a family of admissible orders based on the function $K_\alpha : K([0,1]) \to [0,1]$ given by $K_\alpha(x,y) = (1-\alpha)x + \alpha y$ for some constant $\alpha \in [0,1]$. Indeed, given $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$, the relation $\preceq_{\alpha,\beta}$ defined by, for $[\underline{x}, \overline{x}]$, $[\underline{y}, \overline{y}] \in L([0,1])$,

$$[\underline{x}, \overline{x}] \preceq_{\alpha,\beta} [\underline{y}, \overline{y}] \text{ if and only if } K_\alpha(\underline{x}, \overline{x}) < K_\alpha(\underline{y}, \overline{y}) \text{ or } (K_\alpha(\underline{x}, \overline{x}) = K_\alpha(\underline{y}, \overline{y}) \text{ and } K_\beta(\underline{x}, \overline{x}) \leq K_\beta(\underline{y}, \overline{y})),$$

[1] We use the term increasing to refer to the property of non-decreasingness, i.e., if $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0,1]^n$ such that $x_i \leq y_i$ for all $i \in \{1, \dots, n\}$, then $A(x_1, \dots, x_n) \leq A(y_1, \dots, y_n)$.

is an admissible order on $L([0,1])$.

The admissible orders $\preceq_{Lex1}$, $\preceq_{Lex2}$ and $\preceq_{XY}$ are particular cases of $\preceq_{\alpha,\beta}$ orders; $\preceq_{0,1}$, $\preceq_{1,0}$ and $\preceq_{0.5,1}$, respectively.

III. Relaxed forms of monotonicity

In this work we handle the following relaxed forms of monotonicity: weak monotonicity, directional monotonicity, ordered directional monotonicity and strengthened ordered directional monotonicity.

Definition 4 ([16]): Let $f : [0,1]^n \to [0,1]$ be a function. We say that f is weakly increasing (resp. weakly decreasing), if for all $c > 0$ and $(x_1,\dots,x_n) \in [0,1]^n$ such that $0 \leq x_i + c \leq 1$ for all $i \in \{1,\dots,n\}$, it holds that $f(x_1,\dots,x_n) \leq f(x_1+c,\dots,x_n+c)$ (resp. $f(x_1,\dots,x_n) \geq f(x_1+c,\dots,x_n+c)$).

The rationale behind weak monotonicity is to define a property that is less restrictive than standard monotonicity but captures the fact that if all the inputs increase the same amount then the output should also increase. Geometrically, this property can be seen as an increase defined by the vector $\vec{1} = (1,\dots,1)$ and it can be generalized taking an arbitrary vector $\vec{0} \neq \vec{r} \in \mathbb{R}^n$.

Definition 5 ([17]): Let $\vec{0} \neq \vec{r} \in \mathbb{R}^n$ and $f : [0,1]^n \to [0,1]$ be a function. We say that f is $\vec{r}$-increasing (resp. $\vec{r}$-decreasing), if for all $c > 0$ and $\mathbf{x} = (x_1,\dots,x_n) \in [0,1]^n$ such that $\mathbf{x} + c\vec{r} \in [0,1]^n$, it holds that $f(\mathbf{x}) \leq f(\mathbf{x}+c\vec{r})$ (resp. $f(\mathbf{x}) \geq f(\mathbf{x}+c\vec{r})$).

A function f that is simultaneously $\vec{r}$-increasing and $\vec{r}$-decreasing is said to be $\vec{r}$-constant.

These two forms of monotonicity are based on monotonicity along a fixed ray $\vec{r} \in \mathbb{R}^n$, but there exist some relaxed forms of monotonicity for which the direction of increasingness is variable depending on the specific point of the domain. In particular, the next definition presents the notion of ordered directional (OD) monotonicity [20].

Definition 6 ([20]): Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$ and $f : [0,1]^n \to [0,1]$. We say that f is ordered directionally (OD) $\vec{r}$-increasing (resp. OD $\vec{r}$-decreasing) if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ such that

$$1 \geq x_{\sigma(1)} + cr_1 \geq \cdots \geq x_{\sigma(n)} + cr_n \geq 0, \quad (1)$$

it holds that

$$f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$$

(resp. $f(\mathbf{x}) \geq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$), where σ^{-1} is the inverse permutation of σ.

A function f that is simultaneously OD $\vec{r}$-increasing and OD $\vec{r}$-decreasing is said to be OD $\vec{r}$-constant.

Modifying the requirement (1), the concept of strengthened ordered directional (SOD) monotonicity was presented.

Definition 7 ([21]): Let $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$ and $f : [0,1]^n \to [0,1]$. We say that f is strengthened ordered directionally (SOD) $\vec{r}$-increasing (resp. SOD $\vec{r}$-decreasing) if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{x} \in [0,1]^n$ with $x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)}$ such that $\mathbf{x}_\sigma + c\vec{r} \in [0,1]^n$, it holds that

$$f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$$

(resp. $f(\mathbf{x}) \leq f(\mathbf{x} + c\vec{r}_{\sigma^{-1}})$).

A function f that is simultaneously SOD $\vec{r}$-increasing and SOD $\vec{r}$-decreasing is said to be SOD $\vec{r}$-constant.

The following are examples of functions that satisfy the presented monotonicity conditions.

Example 3:

Let $L : [0,1]^2 \to [0,1]$ be the function given by

$$L(x,y) = \frac{x^2+y^2}{x+y},$$

with the convention $\frac{0}{0} = 0$. This function, known as Lehmber mean, only increases along the direction given by the vector $(1,1)$ [17]. Therefore, it is weakly increasing but not increasing.

The next example is an instance of a restricted equivalence function [27], a family of functions that have been applied [28], [29]. Moreover, it illustrates the differences between OD and SOD monotonicity.

Example 4: Let $f : [0,1]^2 \to [0,1]$ be the function given by

$$f(x,y) = 1 - |x-y|.$$

In [30] it is shown that f is SOD $\vec{r}$-increasing if and only if $\vec{r} = (r,r) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and f is OD $\vec{r}$-increasing if and only if $\vec{r} = (r_1,r_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $r_1 \leq r_2$.

A relevant fact about the mentioned forms of monotonicity is that the set of directions along which a function increases is closed under convex combination, in the sense of the following three theorems.

Theorem 1 ([17]): Let $\vec{r},\vec{s} \in \mathbb{R}^n \setminus \{\vec{0}\}$, $a,b > 0$ and $\mathbf{x} \in [0,1]^n$ and $c > 0$ such that whenever $\mathbf{x}$ and $\mathbf{x} + c(a\vec{r} + b\vec{s}) \in [0,1]^n$, it holds that $\mathbf{x} + ca\vec{r} \in [0,1]^n$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n$. Thus, if a function $f : [0,1]^n \to [0,1]$ is both $\vec{r}$-increasing and $\vec{s}$-increasing, then f is also $(a\vec{r} + b\vec{s})$-increasing.

Theorem 2 ([20]): Let $\vec{r},\vec{s} \in \mathbb{R}^n \setminus \{\vec{0}\}$, $a,b > 0$, $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in \mathcal{S}_n$ such that whenever $1 \geq x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)} \geq 0$ and

$$1 \geq x_{\sigma(1)} + c(ar_1 + bs_1) \geq \ldots \geq x_{\sigma(n)} + c(ar_n + bs_n) \geq 0,$$

it holds that

$$1 \geq x_{\sigma(1)} + car_1 \geq \ldots \geq x_{\sigma(n)} + car_n \geq 0,$$

or

$$1 \geq x_{\sigma(1)} + cbs_1 \geq \ldots \geq x_{\sigma(n)} + cbs_n \geq 0.$$

Thus, if a function $f : [0,1]^n \to [0,1]$ is both OD $\vec{r}$-increasing and OD $\vec{s}$-increasing, then f is also OD $(a\vec{r} + b\vec{s})$-increasing.

Theorem 3 ([21]): Let $\vec{r},\vec{s} \in \mathbb{R}^n$, $a,b > 0$, $\mathbf{x} \in [0,1]^n$, $c > 0$ and $\sigma \in \mathcal{S}_n$ such that whenever $1 \geq x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)} \geq 0$ and $\mathbf{x}_\sigma + c(a\vec{r} + b\vec{s}) \in [0,1]^n$, it holds that $\mathbf{x} + ca\vec{r} \in [0,1]^n$ or $\mathbf{x} + cb\vec{s} \in [0,1]^n$. Thus, if a function $f : [0,1]^n \to [0,1]$ is both SOD $\vec{r}$-increasing and SOD $\vec{s}$-increasing, then f is also SOD $(a\vec{r} + b\vec{s})$-increasing.

Consequently, standard monotonicity of a function $f : [0,1]^n \to [0,1]$ can be characterized in terms of each of the defined forms of monotonicity.

Theorem 4 ([21]): Let $f\colon [0,1]^n \to [0,1]$ and $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be such that $\vec{e}_i = (0, \ldots, 0, \underset{\widehat{i}}{1}, 0, \ldots, 0) \in \mathbb{R}^n$ for each $i \in \{1, \ldots, n\}$. Then, the following are equivalent:

1) f is increasing;
2) f is $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$;
3) f is OD $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$;
4) f is SOD $\vec{e}_i$-increasing for all $i \in \{1, \ldots, n\}$.

It is also equivalent for a function $f : [0,1]^n \to [0,1]$ to be weakly increasing, OD and SOD $\vec{1}$-increasing.

IV. Weak and directional monotonicity of IV functions

In this section we show the developments made in [23], for the general framework of Riesz spaces, and in [24], for interval-valued functions $F : L([0,1])^n \to L([0,1])$, regarding weak and directional monotonicity. Thus, this section sets the bases over which we define the concepts of OD and SOD monotonicity for IV functions.

We recall the definition of standard and directional monotonicity for IV functions.

Definition 8: We say that a function $F : L([0,1])^n \to L([0,1])$ is increasing (resp. decreasing) if for all $\mathbf{X}, \mathbf{Y} \in L([0,1])^n$ such that $\mathbf{X} \leq_{L^n} \mathbf{Y}$, it holds that $F(\mathbf{X}) \leq_L F(\mathbf{Y})$ (resp. $F(\mathbf{X}) \geq_L F(\mathbf{Y})$).

Definition 9: Let $\vec{\mathbf{v}} = ((a_1, b_1), \ldots, (a_n, b_n)) \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ for some $i \in \{1, \ldots, n\}$. We say that a function $F : L([0,1])^n \to L([0,1])$ is $\vec{\mathbf{v}}$-increasing (resp. $\vec{\mathbf{v}}$-decreasing) if for all $\mathbf{X} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{X} + c\vec{\mathbf{v}} \in L([0,1])^n$, it holds that $F(\mathbf{X}) \leq_L F(\mathbf{X} + c\vec{\mathbf{v}})$ (resp. $F(\mathbf{X}) \geq_L F(\mathbf{X} + c\vec{\mathbf{v}})$).

If the function F is both $\vec{\mathbf{v}}$-increasing and $\vec{\mathbf{v}}$-decreasing, then we say that F is $\vec{\mathbf{v}}$-constant.

Note that, as in the case of functions $f : [0,1]^n \to [0,1]$, whose directions of increasingness lie in $\mathbb{R}^n$ rather than in $[0,1]^n$, the directions of increasingness of IV functions F lie in the vector-lattice $(\mathbb{R}^2)^n$. The rationale of this fact is explained in [23].

Furthermore, from Definition 9, we can define the concept of weak monotonicity of IV functions.

Definition 10: Let $\vec{0} \neq (a,b) \in \mathbb{R}^2$. We say that a function $F : L([0,1])^n \to L([0,1])$ is (a,b)-weakly increasing (resp. (a,b)-weakly decreasing) if for all $\mathbf{X} \in L([0,1])^n$ and $c > 0$ such that $\mathbf{X} + c\left((a,b), \ldots, (a,b)\right) \in L([0,1])^n$, it holds that $F(\mathbf{X}) \leq_L F\left(\mathbf{X} + c\left((a,b), \ldots, (a,b)\right)\right)$ (resp. $F(\mathbf{X}) \geq_L F\left(\mathbf{X} + c\left((a,b), \ldots, (a,b)\right)\right)$).

Standard weak increasingness coincides with c-weak increasingness for any $c > 0$. The following is an example of a weakly monotone IV function.

Example 5: Let $F : L([0,1])^2 \to L([0,1])$ be given by

$$F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) = \frac{1}{2}\left[\underline{x_1} + \underline{x_2}, \max\left(\underline{x_1} + \overline{x_2}, \overline{x_1} + \underline{x_2}\right)\right].$$

Note that, for $a, b \in [0,1]$ and $c > 0$, it holds that

$$\begin{aligned} F(([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c((a,b),(a,b))) \\ = F([\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}]) + c\left[a, \frac{a+b}{2}\right]. \end{aligned}$$

Therefore, F is (a,b)-weakly increasing if and only if $a > 0$ and $a + b \geq 0$, or $a = 0$ and $b > 0$.

A. Relevant properties

In this subsection, we show that the relevant properties that standard directionally monotone functions satisfy are also valid properties for directionally monotone IV functions. Specifically, they are the adaptations of Theorems 1 and 4.

Theorem 5 ([24]): Let $a, b > 0$ and $\vec{\mathbf{v}}, \vec{\mathbf{u}} \in (\mathbb{R}^2)^n \setminus \{\vec{\mathbf{0}}\}$ such that for all $\mathbf{X} \in L([0,1])^n$ and $c > 0$ that satisfy $\mathbf{X} + c(a\vec{\mathbf{v}} + b\vec{\mathbf{u}}) \in L([0,1])^n$, it holds that either $\mathbf{X} + ca\vec{\mathbf{v}} \in L([0,1])^n$ or $\mathbf{X} + cb\vec{\mathbf{u}} \in L([0,1])^n$. Then, if a function $F : L([0,1])^n \to L([0,1])$ is both $\vec{\mathbf{v}}$-increasing (resp. $\vec{\mathbf{v}}$-decreasing) and $\vec{\mathbf{u}}$-increasing (resp. $\vec{\mathbf{u}}$-decreasing), then F is $(a\vec{\mathbf{v}} + b\vec{\mathbf{u}})$-increasing (resp. $(a\vec{\mathbf{v}} + b\vec{\mathbf{u}})$-decreasing).

To translate Theorem 4 to the interval-valued setting, we need the family of vectors $\vec{\mathbf{e}}_i$ that are filled with 0s except for the ith position, that is occupied with a 1, for all $i \in \{1, \ldots, 2n\}$. We call this set of vectors the canonical basis of $(\mathbb{R}^2)^n$. In the two dimensional case, the family of vectors $\{\vec{\mathbf{e}}_i\}_{i=1}^4$ is given by the following:

$$\begin{aligned} \vec{\mathbf{e}}_1 &= ((1,0),(0,0)); & \vec{\mathbf{e}}_2 &= ((0,1),(0,0)); \\ \vec{\mathbf{e}}_3 &= ((0,0),(1,0)); & \vec{\mathbf{e}}_4 &= ((0,0),(0,1)). \end{aligned}$$

Theorem 6 ([24]): Let $F : L([0,1])^n \to L([0,1])$ and let $\{\vec{\mathbf{e}}_i\}_{i=1}^{2n}$ be the canonical basis of $(\mathbb{R}^2)^n$. Then, F is increasing (resp. decreasing) if and only if F is $\vec{\mathbf{e}}_i$-increasing (resp. $\vec{\mathbf{e}}_i$-decreasing) for all $i \in \{1, \ldots, 2n\}$.

B. Special class of IV functions

There exists a special class of IV functions regarding their directional monotonicity, in the sense that it can be expressed as standard directional monotonicity of certain functions defined on $[0,1]^n$. This family of IV functions are called representable [31].

Definition 11: We say that a function $F : L([0,1])^n \to L([0,1])$ is representable if there exist two functions $f, g : [0,1]^n \to [0,1]$ satisfying

$$f(x_1, \ldots, x_n) \leq g(y_1, \ldots, y_n),$$

whenever $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$ and such that

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = [f(\underline{x_1}, \ldots, \underline{x_n}), g(\overline{x_1}, \ldots, \overline{x_n})].$$

We say that f and g are the component functions of F and we write $F \equiv (f, g)$.

The particularity of representable functions is that the directions along which they increase are fixed by the directions along which the component functions increase.

Theorem 7 ([24]): Let $F \equiv (f, g)$ be representable and let $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ be such that $\vec{a}, \vec{b} \neq \vec{0}$. Thus, F is $((a_1,b_1),\ldots,(a_n,b_n))$-increasing (resp. $((a_1,b_1),\ldots,(a_n,b_n))$-decreasing) if and only if f is $\vec{a}$-increasing (resp. $\vec{a}$-decreasing) and g is $\vec{b}$-increasing (resp. $\vec{b}$-decreasing).

Theorem 7 assists in the construction of examples of directionally monotone IV functions, as it is possible to use all the examples in the literature regarding standard directionally monotone functions [16], [17], [32]–[34].

Example 6: In [17], it is proven that the arithmetic mean is $\vec{r}$-increasing if and only if $\vec{r} \in \mathbb{R}^n \setminus \{\vec{0}\}$ is such that $\sum_{i=1}^n r_i \geq 0$. Similarly, it is easy to check that the maximum is $\vec{s}$ increasing only if $\vec{s} \in \mathbb{R}^n \setminus \{\vec{0}\}$ satisfies that $s_i \geq 0$ for all $i \in \{1,\ldots,n\}$.

Thus, the function $F : L([0,1])^n \to L([0,1])$, given by

$$F([\underline{x_1}, \overline{x_1}], \ldots, [\underline{x_n}, \overline{x_n}]) = \left[\frac{1}{n}\sum_{i=1}^n \underline{x_i}, \max(\overline{x_1}, \ldots, \overline{x_n})\right],$$

is only $\vec{\mathbf{v}}$-increasing for directions

$$\vec{\mathbf{v}} = ((a_1,b_1),\ldots,(a_n,b_n)) \in (\mathbb{R}^2)^n \setminus \{\vec{0}\}$$

such that

- $\sum_{i=1}^n a_i \geq 0$; and
- $b_i \geq 0$ for all $i \in \{1,\ldots,n\}$.

V. OD and SOD monotonicity of IV functions

In this section we introduce the concepts of OD and SOD monotonicity for IV functions, as well as present their principal properties.

The main problem of extending OD (and SOD) monotonicity to the IV framework is the fact of ordering the inputs of the function, which, in this scenario, are intervals and, therefore, there exist incomparable elements. The solution we propose to address this problem is to make use of admissible orders, which are total orders that refine the standard partial order for intervals, i.e., when ordering the input intervals in a decreasing manner, the relative position of the comparable intervals is preserved.

Definition 12: Let $\vec{\mathbf{v}} = ((a_1,b_1),\ldots,(a_n,b_n)) \in (\mathbb{R}^2)^n$ such that $(a_i,b_i) \neq \vec{0}$ for some $i \in \{1,\ldots,n\}$ and let $\preceq$ be an admissible order. We say that a function $F : L([0,1])^n \to L([0,1])$ is OD $\vec{\mathbf{v}}$-increasing (resp. OD $\vec{\mathbf{v}}$-decreasing) with respect to $\preceq$ if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{X} \in L([0,1])^n$ with $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ such that

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + c(a_1,b_1)\\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + c(a_n,b_n)\\ &\succeq [0,0],\end{aligned}$$

it holds that

$$F(\mathbf{X}) \leq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}})$$

(resp. $F(\mathbf{X}) \geq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}})$), where σ^{-1} is the inverse permutation of σ.

A function F that is simultaneously OD $\vec{\mathbf{r}}$-increasing and OD $\vec{\mathbf{r}}$-decreasing is said to be OD $\vec{\mathbf{r}}$-constant.

Definition 13: Let $\vec{\mathbf{v}} = ((a_1,b_1),\ldots,(a_n,b_n)) \in (\mathbb{R}^2)^n$ such that $(a_i,b_i) \neq \vec{0}$ for some $i \in \{1,\ldots,n\}$ and let $\preceq$ be an admissible order. We say that a function $F : L([0,1])^n \to L([0,1])$ is SOD $\vec{\mathbf{v}}$-increasing (resp. SOD $\vec{\mathbf{v}}$-decreasing) with respect to $\preceq$ if for all $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{X} \in L([0,1])^n$ with $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ such that $\mathbf{X}_\sigma + c\vec{\mathbf{v}} \in L([0,1])^n$, it holds that

$$F(\mathbf{X}) \leq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}})$$

(resp. $F(\mathbf{X}) \geq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}})$).

As before, a function F that is simultaneously SOD $\vec{\mathbf{r}}$-increasing and SOD $\vec{\mathbf{r}}$-decreasing is said to be SOD $\vec{\mathbf{r}}$-constant.

A. Properties

In this section, we present some properties that the IV functions that meet the proposed types of monotonicity satisfy.

Proposition 1: Let $\vec{\mathbf{v}} \in (\mathbb{R}^2)^n$ such that $(a_i,b_i) \neq \vec{0}$ for some $i \in \{1,\ldots,n\}$. If a function $F : L([0,1])^n \to L([0,1])$ is SOD $\vec{\mathbf{v}}$-increasing with respect to $\preceq$, then it is OD $\vec{\mathbf{v}}$-increasing with respect to $\preceq$.

Proof: Since F is SOD $\vec{\mathbf{v}}$-increasing with respect to $\preceq$, then, given $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{X} \in L([0,1])^n$ with $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ such that $\mathbf{X}_\sigma + c\vec{\mathbf{v}} \in L([0,1])^n$, it holds that

$$F(\mathbf{X}) \leq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}}).$$

In particular, if, additionally, it holds that

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + c(a_1,b_1)\\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + c(a_n,b_n)\\ &\succeq [0,0],\end{aligned}$$

then we also obtain that

$$F(\mathbf{X}) \leq F(\mathbf{X} + c\vec{\mathbf{r}}_{\sigma^{-1}}).$$

Hence, F is OD $\vec{\mathbf{v}}$-increasing with respect to $\preceq$. ■

In some settings, the concepts of OD monotonicity and SOD monotonicity are equivalent. For the following result, note that the admissible orders $\preceq_{\alpha,\beta}$ work on $L([0,1])$ and also on $[0,1] \times [0,1]$.

Proposition 2: Let $\alpha, \beta \in [0,1]$ such that $\alpha \neq \beta$ and let $\vec{\mathbf{v}} \in (\mathbb{R}^2)^n$ such that $(a_i,b_i) \neq \vec{0}$ for some $i \in \{1,\ldots,n\}$ and such that

$$(a_1,b_1) \succeq_{\alpha,\beta} \ldots \succeq_{\alpha,\beta} (a_n,b_n).$$

Then, a function $F : L([0,1])^n \to L([0,1])$ is OD $\vec{\mathbf{v}}$-increasing with respect to $\preceq_{\alpha,\beta}$ if and only if F is SOD $\vec{\mathbf{v}}$-increasing with respect to $\preceq_{\alpha,\beta}$.

Proof: The results follow the fact that the order $\preceq_{\alpha,\beta}$ is compatible with the $+$ operation of $L([0,1])$ and, hence, if $[\underline{x_1}, \overline{x_1}] \preceq_{\alpha,\beta} [\underline{x_2}, \overline{x_2}]$ and $[\underline{x_3}, \overline{x_3}] \preceq_{\alpha,\beta} [\underline{x_4}, \overline{x_4}]$ for any $[\underline{x_1}, \overline{x_1}], [\underline{x_2}, \overline{x_2}], [\underline{x_3}, \overline{x_3}], [\underline{x_4}, \overline{x_4}] \in L([0,1])$, it holds that

$$[\underline{x_1}, \overline{x_1}] + [\underline{x_3}, \overline{x_3}] \preceq_{\alpha,\beta} [\underline{x_2}, \overline{x_2}] + [\underline{x_4}, \overline{x_4}].$$

In this setting, it is clear that both monotonicity conditions are equivalent. ■

There are also similar results to those of directionally monotone IV functions regarding construction methods, i.e., we can construct a new OD (or SOD) monotone IV function starting from one.

Proposition 3: Let $\vec{\mathbf{v}} \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ for some $i \in \{1, \ldots, n\}$ and let $F : L([0,1])^n \to L([0,1])$ be an OD (resp. SOD) $\vec{\mathbf{v}}$-increasing function. If $\varphi : L([0,1]) \to L([0,1])$ is an increasing function, then the function $(\varphi \circ F)$ is OD (resp. SOD) $\vec{\mathbf{v}}$-increasing.

Proof: Let F be OD $\vec{\mathbf{v}}$-increasing and φ increasing. Let $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{X} \in L([0,1])^n$ with $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ such that

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + c(a_1, b_1) \\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + c(a_n, b_n) \\ &\succeq [0,0].\end{aligned}$$

Then

$$\begin{aligned}(\varphi \circ F)(\mathbf{X} + c\vec{\mathbf{v}}) &= \varphi(F(\mathbf{X} + c\vec{\mathbf{v}})) \\ &\geq_L \varphi(F(\mathbf{X})) \\ &= (\varphi \circ F)(\mathbf{X}).\end{aligned}$$

Therefore, $(\varphi \circ F)$ is OD $\vec{\mathbf{v}}$-increasing. The case of SOD monotonicity is analogous. ■

Another method to obtain new IV functions satisfying these properties is by aggregating a set of such IV functions.

Proposition 4: Let $\vec{\mathbf{v}} \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ for some $i \in \{1, \ldots, n\}$ and let $F_1, \ldots, F_k : L([0,1])^n \to L([0,1])$ be OD (resp. SOD) $\vec{\mathbf{v}}$-increasing functions. If $A : L([0,1])^k \to L([0,1])$ is an increasing function, then the function $A(F_1, \ldots, F_k)$ is OD (resp. SOD) $\vec{\mathbf{v}}$-increasing.

Proof: Let $F_1, \ldots, F_k$ be OD $\vec{\mathbf{v}}$-increasing functions and A increasing. Let $c > 0$, $\sigma \in \mathcal{S}_n$ and $\mathbf{X} \in L([0,1])^n$ with $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ such that

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + c(a_1, b_1) \\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + c(a_n, b_n) \\ &\succeq [0,0].\end{aligned}$$

Then,

$$\begin{aligned}A(F_1, \ldots, F_k)(\mathbf{X} + c\vec{\mathbf{v}}) &= A(F_1(\mathbf{X} + c\vec{\mathbf{v}}), \ldots, F_k(\mathbf{X} + c\vec{\mathbf{v}})) \\ &\geq_L A(F_1(\mathbf{X}), \ldots, F_k(\mathbf{X})) \\ &= A(F_1, \ldots, F_k)(\mathbf{X}).\end{aligned}$$

The case of SOD monotonicity is analogous. ■

Finally, as Theorem 5 is the extension of Theorem 1 to the interval-valued setting, let us present the extensions of Theorems 2 and 3.

Theorem 8: Let $\vec{\mathbf{v}} = ((a_1, b_1), \ldots, (a_n, b_n)), \vec{\mathbf{w}} = ((c_1, d_1), \ldots, (c_n, d_n)) \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ and $(c_j, d_j) \neq \vec{0}$ for some $i, j \in \{1, \ldots, n\}$, let $a, b > 0$, $\mathbf{X} \in L([0,1])^n$, $c > 0$ and $\sigma \in \mathcal{S}_n$ such that whenever $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ and

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + c(a(a_1, b_1) + b(c_1, d_1)) \\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + c(a(a_n, b_n) + b(c_n, d_n)) \\ &\succeq [0,0],\end{aligned}$$

it holds that

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + ca(a_1, b_1) \\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + ca(a_n, b_n) \\ &\succeq [0,0],\end{aligned}$$

or

$$\begin{aligned}[1,1] &\succeq [\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] + cb(c_1, d_1) \\ &\succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}] + cb(c_n, d_n) \\ &\succeq [0,0].\end{aligned} \tag{2}$$

Thus, if a function $F : L([0,1])^n \to L([0,1])$ is both OD $\vec{\mathbf{v}}$-increasing w.r.t. $\preceq$ and OD $\vec{\mathbf{w}}$-increasing w.r.t. $\preceq$, then F is also OD $(a\vec{\mathbf{v}} + b\vec{\mathbf{w}})$-increasing w.r.t. $\preceq$.

Proof: Let $a, b > 0$, $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in (\mathbb{R}^2)^n$, $c > 0$ and $\mathbf{X} \in L([0,1])^n$ satisfying all the requirements. Without loss of generality, we assume that (2) holds. Thus, we get that

$$F(\mathbf{X} + c(a\vec{\mathbf{v}} + b\vec{\mathbf{w}})) \geq_L F(\mathbf{X} + cb\vec{\mathbf{w}}) \geq_L F(\mathbf{X}).$$

■

Theorem 9 ([21]): Let $\vec{\mathbf{v}} = ((a_1, b_1), \ldots, (a_n, b_n)), \vec{\mathbf{w}} = ((c_1, d_1), \ldots, (c_n, d_n)) \in (\mathbb{R}^2)^n$ such that $(a_i, b_i) \neq \vec{0}$ and $(c_j, d_j) \neq \vec{0}$ for some $i, j \in \{1, \ldots, n\}$, let $a, b > 0$, $\mathbf{X} \in L([0,1])^n$, $c > 0$ and $\sigma \in \mathcal{S}_n$ such that whenever $[\underline{x_{\sigma(1)}}, \overline{x_{\sigma(1)}}] \succeq \cdots \succeq [\underline{x_{\sigma(n)}}, \overline{x_{\sigma(n)}}]$ and

$$\mathbf{X} + c(a\vec{\mathbf{v}} + b\vec{\mathbf{w}}) \in L([0,1])^n,$$

it holds that either

$$\mathbf{X} + ca\vec{\mathbf{v}} \in L([0,1])^n,$$

or

$$\mathbf{X} + cb\vec{\mathbf{w}} \in L([0,1])^n.$$

Thus, if a function $F : L([0,1])^n \to L([0,1])$ is both SOD $\vec{\mathbf{v}}$-increasing w.r.t. $\preceq$ and SOD $\vec{\mathbf{w}}$-increasing w.r.t. $\preceq$, then F is also SOD $(a\vec{\mathbf{v}} + b\vec{\mathbf{w}})$-increasing w.r.t. $\preceq$.

Proof: Similar to the proof of Theorem 8. ■

VI. Conclusion

We have reviewed the state-of-the-art of the trend in the aggregation theory that is the relaxation of the monotonicity constraint, including the concept of directional monotonicity for interval-valued functions. Moreover, we have proposed the concepts of ordered directional monotonicity and strengthened ordered directional monotonicity for the interval-valued setting by means of an admissible order. We have studied some differences of the introduced concepts, as well as some construction methods. We have shown that the proposed forms of monotonicity for interval-valued functions maintain the relevant properties of the functions defined on the unit hypercube that satisfy standard ordered directional and strengthened ordered directional monotonicity.

Our intention for future work includes a deep theoretical study of the notions of OD and SOD monotonicity for the interval-valued case, exploring the possibility of making use of the concept of admissible permutation, introduced by Paternain et al. [35], to consider all the ways of ordering a vector formed of intervals while preserving the standard partial order. Additionally, we intend to find meaningful examples of interval-valued OD and SOD monotone functions with the intention of applying them in the task of edge detection as in [20].

References

[1] G. Beliakov, H. Bustince, and T. Calvo, *A Practical Guide to Averaging Functions*, ser. Studies in Fuzziness and Soft Computing. Springer International Publishing, 2016.

[2] M. Grabisch, J. Marichal, R. Mesiar, and E. Pap, *Aggregation functions*. Cambridge University Press, 2009.

[3] L. De Miguel, M. Sesma-Sara, M. Elkano, M. Asiain, and H. Bustince, "An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators," *Information Fusion*, vol. 37, pp. 126–131, 2017.

[4] D. Paternain, J. Fernandez, H. Bustince, R. Mesiar, and G. Beliakov, "Construction of image reduction operators using averaging aggregation functions," *Fuzzy Sets and Systems*, vol. 261, pp. 87–111, 2015.

[5] R. Mesiar, A. Kolesárová, and A. Stupňanová, "Quo vadis aggregation?" *International Journal of General Systems*, vol. 47, no. 2, pp. 97–117, 2018.

[6] M. Gagolewski, *Data fusion: theory, methods, and applications*. Institute of Computer Science Polish Academy of Sciences, 2015.

[7] H. Bustince, M. Galar, B. Bedregal, A. Kolesarova, and R. Mesiar, "A new approach to interval-valued Choquet integrals and the problem of ordering in interval-valued fuzzy set applications," *IEEE Transactions on Fuzzy systems*, vol. 21, no. 6, pp. 1150–1162, 2013.

[8] H. Bustince, E. Barrenechea, M. Pagola, and J. Fernández, "Interval-valued fuzzy sets constructed from matrices: Application to edge detection," *Fuzzy Sets and Systems*, vol. 160, no. 13, pp. 1819–1840, 2009.

[9] G. Beliakov, H. Bustince, S. James, T. Calvo, and J. Fernandez, "Aggregation for Atanassov's intuitionistic and interval valued fuzzy sets: The median operator," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 3, pp. 487–498, 2012.

[10] G. Deschrijver, "Generalized arithmetic operators and their relationship to t-norms in interval-valued fuzzy set theory," *Fuzzy Sets and Systems*, vol. 160, no. 21, pp. 3080–3102, 2009.

[11] D. Paternain, L. Jin, R. Mesiar, L. Vavríkova, and H. Bustince, "OWA operators based on admissible permutations," in *2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*, 2019, pp. 1–5.

[12] M. Demirci, "Aggregation operators on partially ordered sets and their categorical foundations," *Kybernetika*, vol. 42, no. 3, pp. 261–277, 2006.

[13] M. Komorníková and R. Mesiar, "Aggregation functions on bounded partially ordered sets and their classification," *Fuzzy Sets and Systems*, vol. 175, no. 1, pp. 48–56, 2011.

[14] R. Mesiar and E. Pap, "Aggregation of infinite sequences," *Information Sciences*, vol. 178, no. 18, pp. 3557–3564, 2008.

[15] P. S. Bullen, *Handbook of means and their inequalities*. Springer Science & Business Media, 2013, vol. 560.

[16] T. Wilkin and G. Beliakov, "Weakly monotonic averaging functions," *International Journal of Intelligent Systems*, vol. 30, no. 2, pp. 144–169, 2015.

[17] H. Bustince, J. Fernandez, A. Kolesárová, and R. Mesiar, "Directional monotonicity of fusion functions," *European Journal of Operational Research*, vol. 244, no. 1, pp. 300–308, 2015.

[18] G. Lucca, J. Sanz, G. Dimuro, B. Bedregal, M. J. Asiain, M. Elkano, and H. Bustince, "CC-integrals: Choquet-like copula-based aggregation functions and its application in fuzzy rule-based classification systems," *Knowledge-Based Systems*, vol. 119, pp. 32–43, 2017.

[19] G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, H. Bustince, and R. Mesiar, "CF-integrals: A new family of pre-aggregation functions with application to fuzzy rule-based classification systems," *Information Sciences*, vol. 435, pp. 94–110, 2018.

[20] H. Bustince, E. Barrenechea, M. Sesma-Sara, J. Lafuente, G. P. Dimuro, R. Mesiar, and A. Kolesárová, "Ordered directionally monotone functions. Justification and application," *IEEE Transactions on Fuzzy Systems*, vol. 26, no. 4, pp. 2237–2250, 2018.

[21] M. Sesma-Sara, J. Lafuente, A. Roldán, R. Mesiar, and H. Bustince, "Strengthened ordered directionally monotone functions. Links between the different notions of monotonicity," *Fuzzy Sets and Systems*, vol. 357, pp. 151–172, 2019.

[22] M. Sesma-Sara, H. Bustince, E. Barrenechea, J. Lafuente, A. Kolesárová, and R. Mesiar, "Edge detection based on ordered directionally monotone functions," in *Advances in Fuzzy Logic and Technology 2017*. Springer, 2017, pp. 301–307.

[23] M. Sesma-Sara, R. Mesiar, and H. Bustince, "Weak and directional monotonicity of functions on Riesz spaces to fuse uncertain data," *Fuzzy Sets and Systems*, vol. 386, pp. 145–160, 2020.

[24] M. Sesma-Sara, L. De Miguel, R. Mesiar, J. Fernandez, and H. Bustince, "Interval-valued pre-aggregation functions: a study of directional monotonicity or interval-valued functions," in *2019 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)*. IEEE, 2019, pp. 1–6.

[25] H. Bustince, J. Fernández, A. Kolesárová, and R. Mesiar, "Generation of linear orders for intervals by means of aggregation functions," *Fuzzy Sets and Systems*, vol. 220, pp. 69–77, 2013.

[26] Z. Xu and R. R. Yager, "Some geometric aggregation operators based on intuitionistic fuzzy sets," *International journal of general systems*, vol. 35, no. 4, pp. 417–433, 2006.

[27] H. Bustince, E. Barrenechea, and M. Pagola, "Restricted equivalence functions," *Fuzzy Sets and Systems*, vol. 157, no. 17, pp. 2333–2346, 2006.

[28] M. Galar, J. Fernandez, G. Beliakov, and H. Bustince, "Interval-valued fuzzy sets applied to stereo matching of color images," *IEEE Transactions on Image Processing*, vol. 20, no. 7, pp. 1949–1961, 2011.

[29] M. Sesma-Sara, L. De Miguel, M. Pagola, A. Burusco, R. Mesiar, and H. Bustince, "New measures for comparing matrices and their application to image processing," *Applied Mathematical Modelling*, vol. 61, pp. 498–520, 2018.

[30] M. Sesma-Sara, L. De Miguel, R. Mesiar, J. Fernandez, and H. Bustince, "Local properties of strengthened ordered directional and other forms of monotonicity," in *International Fuzzy Systems Association World Congress*. Springer, 2019, pp. 33–43.

[31] G. Deschrijver and C. Cornelis, "Representability in interval-valued fuzzy set theory," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 15, no. 3, pp. 345–361, 2007.

[32] G. Beliakov, T. Calvo, and T. Wilkin, "Three types of monotonicity of averaging functions," *Knowledge-Based Systems*, vol. 72, pp. 114–122, 2014.

[33] G. Beliakov and J. Špirková, "Weak monotonicity of Lehmer and Gini means," *Fuzzy Sets and Systems*, vol. 299, pp. 26–40, 2016.

[34] G. Beliakov, T. Calvo, and T. Wilkin, "On the weak monotonicity of Gini means and other mixture functions," *Information Sciences*, vol. 300, pp. 70–84, 2015.

[35] D. Paternain, L. De Miguel, G. Ochoa, I. Lizasoain, R. Mesiar, and H. Bustince, "The interval-valued Choquet integral based on admissible permutations," *IEEE Transactions on Fuzzy Systems*, vol. 27, no. 8, pp. 1638–1647, 2019.