Fuzzy Sugeno λ-measures and theirs applications to community detection problems.*

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***Abstract*—In this paper we propose a new framework for community detection problems. The starting point is a n-vector which defines some evidence about the elements of a finite set. This vector is used to build an interaction measure between the n elements of the set to which it refers. This interaction measure is represented by a Sugeno λ-measure to which we make it being also a fuzzy measure. Then, we obtain the weighted graph associated with this new capacity measure. To carry on with it, we make use of the Shapley value. We also introduce the notion of extended vector fuzzy graph, which relates a graph with the capacity measure introduced in this work. Finally, we use a community detection method, based on Louvain algorithm, to search a cluster structure in the weighted graph. This partition is based on the relations among the individuals obtained from the initial vector. Let us note that in the case that there exist some connections among the elements, apart from their affinity, we can combine this extra information with that given by the vector, in order to obtain groups with highly-knit elements among which there are strong relations.**

***Index Terms*—Fuzzy measure, Sugeno λ-measure, Community detection problem, Extended vector fuzzy graph.**

I. Introduction

Graphs are a mathematical tool really useful to model the relationships or connections existing between a set of objects. One of the most popular applications of graphs is devoted to community detection problems. These problems are solved with some unsupervised learning tasks [1], [2], called clustering methods. Given an input set of objects, the main objective of community detection algorithms [3] is to define a partition of subgroups within the input set. These subgroups are called communities, modules or clusters. This partition depends on the connections or relationships among the elements to be grouped. The key point is that those objects which are in the same cluster should be as similar as possible between them, whereas they should be as dissimilar as possible with respect to the elements which are in other group.

Clustering has become an essential tool of model development and exploratory data analysis in many areas, such as medical and biological disciplines, astronomy or engineering [4]–[7], as well as image processing or pattern recognition problems [8], [9]. There are many real-world networks in which the nodes are bound to have a modular structure [10], [11]. In last decades, many researchers have focused their attention on this problem, so a wide range of methods have been proposed to deal with this problem [12], [13].

Classical community detection algorithms are based on a graph which models the connections between the individuals. Obviously, the relationships between the objects, represented by the edges of the graph, should be the key point when searching a partition (Friedkin claimed that 'the more distant two vertices, the less they influence each other' [14]). However, it is not the only available information when modeling a problem. Some improvements where achieved in this field when the use of weighted fuzzy graphs was introduced in community detection problems background. Nevertheless, this tool is not enough to model real-life problems. Recent studies have proposed the incorporation of some additional information, not inherent to the structure of the graph, into community detection problems. One of these proposals is about modeling this additional information by means of a fuzzy measure [15] which defines some affinity relation between the individuals. This information about affinity relations between the elements indicates which nodes should be together in the same group.

Following this philosophy of considering some additional information, here we propose another option, which has never been addressed. It is based on the incorporation of a vector defining some evidence about the objects to the community detection problem. So, we will be able to consider situations which can not be approached with previous proposals, as this idea of adding individualized information about each vertex

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has never been developed. Furthermore, the complexity of the process is reduced, as only n values are needed, apart from the crisp graph. As a first step to deal with this problem, we propose a definition of the fuzzy measure related to the mentioned vector, which is also a Sugeno λ-measure.

Sugeno was one of the pioneers in the field of fuzzy sets [16]–[18]. He introduced a particular type of fuzzy measures, called Sugeno λ-measures [19]. For this particular measure, g, the following holds: knowing $g(A)$ and $g(B)$ for two disjoint sets, we can reconstruct the degree $g(A \cup B)$ [20].

So, having a graph and a n-vector giving some evidence about its vertices, here we propose to build a fuzzy measure from mentioned vector, based on Sugeno λ-measures. This type of information, which had not been exploited until now, is a rich data source when modeling real-life problems. The approach here addressed allows us to manage the additional information given by this vector. Particularly, we suggest the consideration of extended vector fuzzy graphs.

In order to have a simple visualization of the new fuzzy measure, we define the weighted graph associated with it. This graph, whose construction is based on the Shapley value [21], shows how each element is affected by the absence of other.

Once the fuzzy measure is summarized in that basic structure, (its associated weighted graph), it is not difficult to apply it in a wide set of problems. Particularly, here we proposed to use it in community detection problems.

Traditionally, community detection algorithms analyze the adjacency matrix of a graph to find a 'good' partition of its nodes, in which obtained clusters are composed of tightly-knit groups of nodes. However, it is obvious that a graph is not enough for modeling real-life situations, in which there is much more information regardless the connections among the individuals. Hence, here we propose a solution for dealing with the additional information about the graph given by a vector. Following the philosophy previously mentioned of working with extended fuzzy graphs [15], our proposal is based on a modification of Louvain algorithm [22], which can manage the additional information given by a vector to find communities.

Then, the obtained result is a partition such that nodes that are in the same cluster are densely connected, and also are related in some way by means of the information given by the additional information vector.

The remainder of the paper is organized as follows. In Section II, we define the fuzzy measure associated with a vector. We also describe some important concepts related to the fuzzy measures framework. In Section III we show the definition of the weighted graph associated with a fuzzy measure. In Section IV we explain the background of graphs with additional information. Here we introduce the definition of a new concept: extended vector fuzzy graph. Then, in Section V, we propose a particular application of the fuzzy measure previously mentioned in community detection problems. We illustrate this idea with an example. Then, we carry out with some conclusions in Section VI.

II. Fuzzy measures based on a vector

Let $G = (V, E)$ be a graph, where $V = \{1, \dots, n\}$ is a set of nodes, and E is a set of edges, $E = \{\{i, j\} |\ i, j \in V\}$. Let x be a n-vector, where $\forall i \in V$, $x_i \geq 0$ defines some weight or evidence related to the element $i \in V$. Hence, we consider the pair (G, x).

In this section we introduce some capacity measures. Particularly, given a vector x, we focus on these measures built from the vector x as fuzzy Sugeno λ-measures. Let us recall some basic definitions.

Definition II.1: **Fuzzy measure** [23]. Let $V = \{1, 2, ..., n\}$ be a set, and let $\mu : 2^V \longrightarrow [0, 1]$ be a function such that $\mu(\emptyset) = 0$ and $\mu(V) = 1$. Let us assume that $\forall A, B$ such that $A \subseteq B \subseteq V$, it holds that $\mu(A) \leq \mu(B)$. Then, the fuzzy set μ is called fuzzy measure.

Definition II.2: **Sugeno λ-measure** [19] Let $V = \{1, \dots, n\}$ be a finite set, and let $\lambda \in (-1, +\infty)$ be a parameter. A Sugeno λ-measure is a function $g : 2^V \longrightarrow [0, 1]$ such that $\forall A, B \subseteq V$, if $A \cap B = \emptyset$, then:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B)$$

Definition II.3: **1-additivity** [24] Let μ be a fuzzy measure defined over the set $V = \{1, \dots, n\}$. The fuzzy measure μ is said to be 1-additive if $\forall A \subseteq V$, μ can be defined as $\mu(A) = \sum_{i=1}^{n} a_i z_i$, where $z_i = 1$ if $i \in A$ and $z_i = 0$, otherwise.

In this paper we use Sugeno λ-measures (Definition II.2), to which we also force to be fuzzy measures (Definition II.1). We will refer to this type of measures as fuzzy Sugeno λ-measures. Then, the function $\mu : 2^V \longrightarrow [0, 1]$, is a Sugeno λ-measures if the following points hold:

- $\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B), \ \ \forall A, B \subseteq V$ with $A \cap B = \emptyset$.
- $\mu(A) \leq \mu(B)$, $\forall A \subseteq B \subseteq V$.
- $\mu(\emptyset) = 0$ and $\mu(V) = 1 \Longrightarrow \lambda + 1 = \prod_{i=1}^{n}(1 + \lambda\mu(i))$

Depending on the problem modeled by μ, we can differentiate several situations in the characterization of this fuzzy measure. To make easier the notation, we denote $\mu(i) := \mu(\{i\})$.

1) If $\sum_{i=1}^{n} \mu(i) = 1$, then $\lambda = 0$. Hence, $\forall S \subseteq V$, $\mu(S) = \sum_{i \in S} \mu(i)$. In this case, μ does not have any multiplicative component, so its an additive measure.
2) If $p = \sum_{i=1}^{n} \mu(i) < 1$, then $\lambda > 0$. Here p is the additive component of μ, and $(1-p)$ is its multiplicative component, representing positive interactions.
3) If $p = \sum_{i=1}^{n} \mu(i) > 1$, then $\lambda < 0$. Here p is the additive component of μ, and $(1-p)$ is its multiplicative component, representing negative interactions.

In the framework of this paper, let $x = (x_1 \dots, x_n)$ be a n-vector defining some evidence about the elements of V, where $x_i \geq 0, \forall i \in \{1, \dots, n\}$. We define μ_x, a fuzzy measure obtained from x according to Sugeno λ-measure. Then, for μ_x, the points mentioned in Definition II.1 and Definition II.2 have to hold. Let us propose a natural definition of μ_x.

Definition II.4: Let $x = (x_1, \ldots, x_n)$ be a vector, where $x_i \geq 0\ \forall i$. We assume that x defines some evidence about the elements of the set $V = \{1, \ldots, n\}$. Given a parameter $p \in (0, 1]$, a natural definition of μ_x is:

$$\mu_x(i) = \frac{px_i}{\sum_{k=1}^{n} x_k}, \qquad \forall i \in V$$

and

$$\mu_x(A \cup B) = \mu_x(A) + \mu_x(B) + \lambda\mu_x(A)\mu_x(B),$$

$\forall A, B \subseteq V$, with $A \cap B = \emptyset$ and $\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda\mu_x(i))$

Let us remark that the interpretation of the multiplicative component of μ_x depends on the value of p.

- If $p = 1$, μ_x is an additive fuzzy measure, and $\lambda = 0$.
- If $p \in (0, 1)$, the multiplicative component has a positive character, in the sense that adding more individuals to a set provides benefit. In this case, $\lambda > 0$.

Proposition II.1: Let $p = 1$ be a parameter. The function μ_x introduced in Definition II.4 is a fuzzy Sugeno λ-measure. Furthermore, μ_x is a 1-additive fuzzy measure.

Proof: Due to the assumption of $p = 1$, we have $\lambda = 0$. Then, $\forall A \subseteq V$, $\mu_x(A) = \frac{\sum_{l \in A} \mu_l}{\sum_{k=1}^{n} \mu_k}$. Then we prove that μ_x meets the points mentioned in Definitions II.1,II.2 and II.3.

- $\mu_x(\emptyset) = 0$ Trivial
- $\mu_x(V) = \frac{x_1}{\sum_{k=1}^{n} x_k} + \cdots + \frac{x_n}{\sum_{k=1}^{n} x_k} = 1$
- Let $A \subseteq B \subseteq V$. Then, $\mu_x(B) = \frac{\sum_{l \in B} x_l}{\sum_{k=1}^{n} x_k} = \frac{1}{\sum_{k=1}^{n} x_k}\left(\sum_{l \in A} x_l + \sum_{t \in B \backslash A} x_t\right) \geq \frac{\sum_{l \in A} x_l}{\sum_{k=1}^{n} x_k} = \mu_x(A)$

  Hence, μ_x is a fuzzy measure.
- **Sugeno λ-measure**. Trivial by definition.
- 1-**additivity**: this point is trivial if $\forall i \in \{1, \ldots, n\}$, we define $a_i = \mu_x(i)$.

■

Proposition II.2: Let $p \in (0, 1)$ be a parameter. Then, μ_x (introduced in Definition II.4) is a fuzzy Sugeno λ-measure.

Proof: Because of the assumption of $p \in (0, 1)$, we have $\lambda > 0$. Then, we demonstrate that μ_x meets all the points mentioned in Definitions II.1 and II.2.

- $\mu_x(\emptyset) = 0$ Trivial
- λ is defined according to Sugeno λ-measures definition, so the property $\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda\mu_x(i))$ holds for this parameter. Because of Definition II.1, $\mu_x(V) = 1$, provided that μ_x is defined according to Sugeno's formulation. Particularly in this case, μ_x is defined in this way, so $\mu_x(V) = 1$.
- Let $A \subseteq B \subseteq V$. Then, $\mu_x(B) = \mu_x(A) + \mu_x(B \backslash A) + \lambda\mu_x(A)\mu_x(B \backslash A)$. We have $\lambda > 0$, and $0 \leq \mu_x(A) \leq 1$ and $0 \leq \mu_x(B \backslash A) \leq 1$. Then, $\mu_x(A) \leq \mu_x(B)$.
  Hence, μ_x is a fuzzy measure.
- **Sugeno λ-measure**. Trivial by definition.

■

Proposition II.3: Let $p \in (0, 1]$ be a parameter. Then, the fuzzy measure μ_x introduced in Definition II.4 is a supperadditive fuzzy measure.

Proof: Let $A, B \subseteq V$ such that $A \cap B = \emptyset$. We have $\mu_x(A \cup B) = \mu_x(A) + \mu_x(B) + \lambda\mu_x(A)\mu_x(B)$. Also, $\lambda \geq 0$; $0 \leq \mu_x(A) \leq 1$; and $0 \leq \mu_x(B) \leq 1$. Then, $\mu_x(A \cup B) \geq \mu_x(A) + \mu_x(B)$, so μ_x is a supperaditive fuzzy measure. ■

To finish this Section, let us illustrate the construction of a fuzzy measure μ_x with a toy example, considering several values of p.

Example II.1: Let $V = \{1, 2, 3, 4\}$ be a set, and let $x = (2, 1, 10, 10)$ be a vector. μ_x is the fuzzy measure obtained from the vector x according to Definition II.4. Considering the grill of values $p =\ 1,\ 0.75,\ 0.5,\ 0.25$, in Table I, we show the values of μ_x depending on different values of p.

TABLE I: Characterization of μ_x for $p = 0.25,\ 0.5,\ 0.75,\ 1$.

	$p = 0.25$ $\lambda = 24.01$	$p = 0.5$ $\lambda = 5.21$	$p = 0.75$ $\lambda = 1.33$	$p = 1$ $\lambda = 0$
$\mu_x(\{1\})$	0,0217	0,0435	0,0652	0,0870
$\mu_x(\{2\})$	0,0109	0,0217	0,0326	0,0435
$\mu_x(\{3\})$	0,1087	0,2174	0,3261	0,4348
$\mu_x(\{4\})$	0,1087	0,2174	0,3261	0,4348
$\mu_x(\{1, 2\})$	0,0383	0,0701	0,1006	0,1304
$\mu_x(\{1, 3\})$	0,1872	0,3101	0,4195	0,5217
$\mu_x(\{1, 4\})$	0,1872	0,3101	0,4195	0,5217
$\mu_x(\{2, 3\})$	0,1479	0,2638	0,3728	0,4783
$\mu_x(\{2, 4\})$	0,1479	0,2638	0,3728	0,4783
$\mu_x(\{3, 4\})$	0,5011	0,6810	0,7933	0,8696
$\mu_x(\{1, 2, 3\})$	0,2469	0,3670	0,4703	0,5652
$\mu_x(\{1, 2, 4\})$	0,2469	0,3670	0,4703	0,5652
$\mu_x(\{1, 3, 4\})$	0.7844	0.8788	0.9272	0.9565
$\mu_x(\{2, 3, 4\})$	0,6427	0,7799	0,8603	0,9130
$\mu_x(\{1, 2, 3, 4\})$	1	1	1	1

From now on, we differentiate those cases in which μ_x is additive, ($p = 1$ and $\lambda = 0$), denoted as μ_x^a; and those in which μ_x has a multiplicative component with positive character, ($p \in (0, 1)$ and $\lambda > 0$), denoted as μ_x^m.

III. The weighted graph associated with a fuzzy Sugeno λ-measure.

Let x be a vector defining some evidence about the elements of a finite set, $V = \{1, \ldots, n\}$. Let μ_x be the function obtained from x according to Definition II.4 (let us assume we only consider these cases in which it is a fuzzy Sugeno λ-measure). To deal with μ_x in a simple way, we propose to calculate the weighted graph associated with it, G_{μ_x}. To define this weighted graph, we make use of the Shapley value [21] as follows: for each pair of nodes $\{i, j\}$, where $i, j \in V$, we calculate the weight F_{ij}, which appraises how each individual of the pair is affected by the absence of the other.

$$F_{ij} = \phi\left(Sh_i(\mu_x) - Sh_i^j(\mu_x), Sh_j(\mu_x) - Sh_j^i(\mu_x)\right) \quad (1)$$

where $\phi : [-1, 1] \longrightarrow [0, 1]$ is a bi-variate aggregation operator [25]; $Sh_i(\mu_x)$ and $Sh_i^j(\mu_x)$ are the Shapley values of i on μ_x in the presence of all the elements of V or in the presence of all the elements of V except j, respectively.

In the framework of Sugeno λ-measures, the calculation of Shapley value has exponential complexity when $\lambda \neq 0$.

Some sampling techniques have been proposed to solve this issue [26], [27]. However, these methods are not needed when additive fuzzy measures are considered, since in these cases the calculation of the Shapley value is immediate. For this reason, we focus on additive fuzzy measures. Particularly, now we work with Sugeno λ-measures in which $\lambda = 0$. This is, we work with the function μ_x^a. Then, after a brief remark about the Shapley value, we will provide some properties of μ_x^a.

Remark III.1: Let $\mu : 2^V \rightarrow [0,1]$ be a fuzzy measure, where $|V| = n$. For every $i \in V$, the Shapley value can be calculated as:

$Sh_i(\mu) = \sum_{S \subseteq V \setminus \{i\}} \frac{(n-|S|-1)!|S|!}{n!} (\mu(S \cup \{i\}) - \mu(S))$

An alternative definition of this index was proposed in [28]. Then, being $pred(i)$ the set of predecessors of the individual $i \in V$; $\pi(n)$ the set of all the possible permutations of the elements of V; and $o \in \pi(n)$ one of this orders, the Shapley value can be rewritten as:

$$Sh_i(\mu) = \frac{1}{n!} \sum_{o \in \pi(n)} [\mu(pred(i) + \{i\}) - \mu(pred(i))] \quad (2)$$

This formula is applied to prove the following proposition.

Proposition III.1: Let $\mu_x^a : 2^V \longrightarrow [0,1]$ be the fuzzy Sugeno λ-measure obtained from vector x according to Definition II.4, with $p = 1$. Then, $\forall i \in V$, its Shapley value related to μ_x when i is in a coalition with all the elements of V is:

$$Sh_i(\mu_x^a) = \frac{x_i}{\sum_{k=1}^n x_k}$$

Proof: Regarding Equation (2), $Sh_i(\mu_x^a) = \frac{1}{n!}\sum_{o\in\pi(n)}[\mu_x^a(pred(i)+\{i\}) - \mu_x^a(pred(i))] = \frac{1}{n!}\sum_{o\in\pi(n)}[(\mu_x^a(pred(i)) + \mu_x^a(i)) - \mu_x^a(pred(i))] = \frac{1}{n!}\sum_{o\in\pi(n)} \mu_x^a(i) = \frac{1}{n!}\sum_{o\in\pi(n)} \frac{x_i}{\sum_{k=1}^n x_k} = \frac{x_i}{\sum_{k=1}^n x_k}$. ■

Proposition III.2: Let $\mu_x^a : 2^V \longrightarrow [0,1]$ be the fuzzy Sugeno λ-measure obtained from vector x according to Definition II.4, with $p = 1$. Then, $\forall i \in V$, its Shapley value related to μ_x when i is in a coalition with all the elements of V except j is:

$$Sh_i^j(\mu_x^a) = \frac{x_i}{\sum_{\substack{k=1 \\ k\neq j}}^n x_k}$$

Proof: Let $x^* = \left(x_1, \ldots, x_{(j-1)}, x_{(j+1)}, \ldots, x_n\right)$. Then: $Sh_i^j(\mu_x^a) = Sh_i(\mu_{x^*}^a) = \frac{1}{n!}\sum_{o\in\pi(n)}[\mu_{x^*}^a(pred(i)+\{i\}) - \mu_{x^*}^a(pred(i))] = \frac{1}{n!}\sum_{o\in\pi(n)}[(\mu_{x^*}^a(pred(i)) + \mu_{x^*}^a(i)) - \mu_{x^*}^a(pred(i))] = \frac{1}{n!}\sum_{o\in\pi(n)} \mu_{x^*}^a(i) = \frac{1}{n!}\sum_{o\in\pi(n)} \frac{x_i}{\sum_{\substack{k=1 \\ k\neq j}}^n x_k} = \frac{x_i}{\sum_{\substack{k=1 \\ k\neq j}}^n x_k}$ ■

IV. Networks with additional information: The extended vector fuzzy graph

Given a finite set of elements, graphs are used to model the relationships or connections among its individuals with edges or arcs. Then, the pair $G = (V, E)$ is a graph, where $V = \{1, \ldots, n\}$ is a set of nodes and $E = \{\{i,j\} \mid i,j \in V\}$ is a set of links or edges. An alternative to represent a graph is by means of its adjacency matrix, A, which shows the direct connections among the nodes. A specific type of graphs are weighted graphs, whose links have a numerical weight.

On the other hand, there is a broader concept: fuzzy graphs. Fuzzy graphs, introduced by Rosenfeld [29] and based on the fuzzy relations among the individuals [18], are used to model situations in which there is some uncertainty.

Definition IV.1: [29] Let $V \neq \emptyset$ be a finite set. Considering the conjunction operator B, let the functions $\eta : V \longrightarrow [0,1]$, and $\psi : V \times V \longrightarrow [0,1]$ be such that for all $x, y \in V$, $\psi(x,y) \leq B(\eta(x), \eta(y))$. A **fuzzy graph** is the triplet $G = (V, \eta, \psi)$, where η is known as the fuzzy vertex set of G, and ψ is known as the fuzzy edge set of G.

The fuzzy vertex set it is usually assumed to be crisp. Then, we can simplify previous definition, under the assumption that a fuzzy graph is characterized by the pair $G = (V, \psi)$. Sometimes it is also assumed the existence of E, a crisp set of edges. In this context, the edges in E limit the value of each fuzzy arc, forcing it to be 0 if the related link does not exist in the crisp set of edges. Under both assumptions, we provide the following definition of a fuzzy graph.

Definition IV.2: Let $G = (V, E)$ be a graph, and let $\psi : E \rightarrow [0,1]$ be a fuzzy set. The triplet $\widetilde{G} = (V, E, \psi)$ is a **fuzzy graph**, also called **crisp graph** G **with fuzzy edges** ψ.

Two nodes that are not adjacent in the crisp graph G, can not have any membership degree in the fuzzy measure ψ. Then, the crisp graph G provides all the available information, considering the weight associated with each edge. For this reason, we can see fuzzy graphs as weighted graphs. In order to generalize fuzzy graphs, we suggest the use of extended fuzzy graphs [15], which are based on fuzzy measures [23].

Definition IV.3: [15] Let the pair $G = (V, E)$ be a graph, and let μ be a fuzzy measure defined over the set of nodes V, $\mu : 2^V \longrightarrow [0,1]$. The triplet $\widetilde{G} = (V, E, \mu)$ is an **extended fuzzy graph** or **crisp graph with fuzzy measure** μ.

The context of this study is about the pair (G, x), where G is a graph and x is a vector. We modify the notion of extended fuzzy graph to adapt it to this background.

Definition IV.4: Let $G = (V, E)$ be a graph, and let x be a vector defining some evidence about the elements of V. μ_x is the fuzzy measure obtained from x (see Definition II.4). The triplet $\widetilde{G}_x = (V, E, \mu_x)$ is an **extended vector fuzzy graph**.

V. Community detection problems with 1-additive fuzzy measures

Community detection problem consists of finding 'good' partitions for given graphs, that is, a group structure of nodes which are densely connected. These groups are called clusters, communities or modules. Community detection problem, which has lot of applications in many fields, has been widely analyzed in last decades [30], [31]. Classical algorithms use the connections of the graphs, (that is, the set of edges), to find modules of tightly-knit nodes. It is clear that the relationships between the individuals are a key factor when identifying communities in a graph. However, it should not be the only one. It is obvious that in real-life problems there is extra information beyond the connections among the elements. This extra information is a rich source of knowledge to find coherent groups. So, we think it would be reasonable to model the additional information and then apply it for solving problems, particularly, for searching partitions in a graph.

Previous studies have analyzed these situations, by adding the information of a fuzzy measure [15] to community detection problems. This idea is based on the use of extended fuzzy graphs $\widetilde{G} = (V, E, \mu)$. Its starting point is one of the most popular methods in community detection framework, Louvain algorithm ‘ [22]’. The key idea is about an alternative vision of this method, such that we differentiate between two input parameters: one is devoted to find ‘possible’ clusters, and the other is devoted to calculate the maximum of modularity.

The key point is to calculate G_μ, the weighted graph associated with μ, whose adjacency matrix is F. The proposal in [15] is about an aggregation of both matrices, A and F, $\mathcal{M} = \theta(A, F)$, where θ is an aggregation operator [32] defined over the set of quadratic n-matrices $\mathcal{X}$, used to aggregate two matrices into one. Then, the algorithm proposed to consider some additional information when finding communities is a quite basic method: it is based on the application of the Louvain algorithm with a modification, which consists on using the matrix $\mathcal{M}$ to calculate the modularity variation. Of course, just the adjacency matrix A is used to find the connections in the graph (two nodes can not be in the same cluster if they are not connected in the crisp edges set E). In that previous work it was proposed a linear combination of matrices A and F. However, any aggregation function θ could be used [25], [33], [34].

In this section we propose a specific application of the fuzzy measure obtained from a vector: community detection problems with additional information. Let $G = (V, E)$ be a graph, and let x be a vector defining some evidence about the elements of V. The issue is about considering the information given by x when detecting communities in a graph. Let us illustrate this problem with a toy example.

Example V.1: Let $G = (V, E)$ be a graph (Figure 1), and let $x = (9,\ 9.5,\ 8,\ 9.2,\ 8.7,\ 10,\ 1,\ 1.5,\ 2,\ 1.7,\ 2.5,\ 0.5)$ be a vector defining some information about the elements of V, for example the rating of a film given by 12 people.

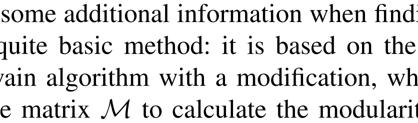

Fig. 1: Graph $G = (V, E)$

Any community detection algorithm which only considers the structure of the graph, particularly Louvain algorithm [22], identifies three clusters in this structure, so that the obtained partition is $P = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$. On the other hand, we recall the assumption that x provides information about the rating of a particular film given by 12 people. Considering that two individuals whose opinions are similar are prone to having great affinity, it could be logical that, not only considering the graph structure, but also the information given by x, obtained partition should be $P^x = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12\}\}$.

Knowing this alternative vision of Louvain algorithm which differentiates several information sources [15], in $Sugeno\_Louvain$ Algorithm we propose a method which can consider the additional information provided by a vector when finding communities in a graph. The first step of the algorithm is to calculate the weighted graph associated with μ_x, whose adjacency matrix is F (see Equation (1)). Let us remark that the performance of our algorithm, as well as Louvain's one, depends on the permutation of the elements of V considered, $o \in \pi(V)$. Then, according to the structure of the Louvain algorithm, we divide our algorithm into two phases. The first one starts with each node in an isolated community. This phase iterates moving nodes from one community to another (just considering the communities of the related neighbours) until a maximum of modularity is reached, as it is done in the respective phase of the Louvain algorithm. Here the difference lies in that in $Sugeno\_Louvain$ Algorithm, the matrix used to find the maximum of modularity is $\mathcal{M}$, which combines the different information sources available. Let us remark that the only neighbours considered are those provided by the adjacency matrix A. The philosophy of second phase is quite similar to the first one, but in this case, considering as nodes the communities obtained in the first phase. $Sugeno\_Louvain$ Algorithm can be summarized in following pseudocode, where o is a permutation of the elements of V; $\theta : \mathcal{X}^2 \longrightarrow \mathcal{X}$ is an aggregation operator [25], [33] used to aggregate two matrices into one; and $\Delta Q_k(j)$ is the variation of modularity obtained when moving k to j's community.

Algorithm 1 *Sugeno-Louvain* **input**=$(A,\ x)$ **output**=P

Define μ_x, the fuzzy measure related to vector x
Calculate G_{μ_x}, the weighted graph associated with μ_x. Its adjacency matrix is $\mathcal{F}$
$\mathcal{M} = \theta(A, \mathcal{F})$
Phase 1
$\forall i \in V$, let i be an isolated community
$o = (o^1, \ldots, o^n) = permutation(V)$
for $i = 1$ **to** n **do**
  search in A all the neighbours of o^i, $(e_1, \ldots, e_r)$
  for $j = 1$ **to** r **do**
    calculate $\Delta Q_{o^i}(e_j)$ in matrix $\mathcal{M}$
  end for
  $j^* = \{\ e_j \mid \Delta Q_{o^i}(j^*) = \max_{j\in\{1\ldots,r\}}\{\Delta Q_{o^i}(e_j)\}\ \}$
  if $\Delta Q_{o^i}(j^*) > 0$ **then**
    Move node o^i to j^* 's community
  else
    o^i remains in its community
  end if
end for
Phase 1 Ends
Phase 2
A^* is the aggregated matrix obtained from A, whose nodes are the communities found in Phase 1
$\mathcal{M}^*$ is the aggregated matrix obtained from $\mathcal{M}$, whose nodes are the communities found in Phase 1
While there is some change, apply Phase 1 and Phase 2, considering matrices A^* and $\mathcal{M}^*$
Phase 2 Ends

To avoid the exponential complexity of the calculation of the Shapley value for Sugeno λ-measures, we propose to calculate the fuzzy measure related to vector x according to Definition II.4, with $p = 1$, so that μ_x^a is an additive fuzzy measure. Particularly, we propose a specific application of $Sugeno\_Louvain$ Algorithm in Algorithm 2.

According to propositions III.1 and III.2, following properties hold for the additive fuzzy measure μ_x^a:

$$Sh_i(\mu_x^a) = \frac{x_i}{\sum_{k=1}^n x_k}$$
$$Sh_i^j(\mu_x^a) = \frac{x_i}{\sum_{\substack{k=1\\k\neq j}}^n x_k}$$

Then, when additive fuzzy measures are considered, the complexity of the $Sugeno\_Louvain$ Algorithm is reduced, as the only initial step is the calculation of the matrix F, which is immediate for additive fuzzy measures. Therefore, the complexity of 1-*Additive* $Sugeno\_Louvain$ Algorithm is the same as Louvain algorithm.

Algorithm 2 1-*Additive* $Sugeno\_Louvain$ **input=**$(A,\ x)$ **output=**P

$\mathcal{F}_{ij} = \min\{|\frac{x_i}{\sum_{k=1}^n x_k} - \frac{x_i}{\sum_{\substack{k=1\\k\neq j}}^n x_k}|,\ |\frac{x_j}{\sum_{k=1}^n x_k} - \frac{x_j}{\sum_{\substack{k=1\\k\neq i}}^n x_k}|\}$
$\mathcal{M} = \theta\,(A, \mathcal{F})$
Phase 1
$\forall i \in V$, let i be an isolated community
$o = \left(o^1, \ldots, o^n\right) = permutation(V)$
for $i = 1$ **to** n **do**
  search in A all the neighbours of o^i, $(e_1, \ldots, e_r)$
  for $j = 1$ **to** r **do**
    calculate $\Delta Q_{o^i}(e_j)$ in matrix $\mathcal{M}$
  end for
  $j^* = \{\ e_j \mid \Delta Q_{o^i}(j^*) = \max_{j\in\{1\ldots,r\}}\{\Delta Q_{o^i}(e_j)\}\ \}$
  if $\Delta Q_{o^i}(j^*) > 0$ **then**
    Move node o^i to j^* 's community
  else
    o^i remains in its community
  end if
end for
Phase 1 Ends
Phase 2
A^* is the aggregated matrix obtained from A, whose nodes are the communities found in Phase 1
$\mathcal{M}^*$ is the aggregated matrix obtained from $\mathcal{M}$, whose nodes are the communities found in Phase 1
While there is some change, apply Phase 1 and Phase 2, considering matrices A^* and $\mathcal{M}^*$
Phase 2 Ends

Let us recall the Example V.1. We show that $1-$ $Additive\ Sugeno\_Louvain$ Algorithm provides the partition which seems logical when considering the information of x.

Example V.2: Let $G = (V, E)$ be the graph in Figure 1, and let $x = (9,\ 9.5,\ 8,\ 9.2,\ 8.7,\ 10,\ 1,\ 1.5,\ 2,\ 1.7,\ 2.5,\ 0.5)$ be a vector defining some information about the elements of V. Let μ_x^a be the fuzzy measure obtained from x according to Definition II.4, with $p = 1$. Let $G_{\mu_x^a}$ be the weighted graph associated with μ_x^a, whose adjacency matrix is F, (Figure 2), where $F_{ij} = \phi\left(Sh_i(\mu_x^a) - Sh_i^j(\mu_x^a), Sh_j(\mu_x^a) - Sh_j^i(\mu_x^a)\right) = \min\{|\frac{x_i}{\sum_{k=1}^n x_k} - \frac{x_i}{\sum_{\substack{k=1\\k\neq j}}^n x_k}|,\ |\frac{x_j}{\sum_{k=1}^n x_k} - \frac{x_j}{\sum_{\substack{k=1\\k\neq i}}^n x_k}|\}$. Due to μ_x^a's additivity, the calculation of the Shapley value is immediate (see Proposition III.1 and Proposition III.2).

Considering the adjacency matrix of G, A any community detection algorithm based on modularity optimization, (for example Louvain algorithm [22], detects the partition $P = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\}\}$. However, if we pay attention to the extra information provided by x, the obtained partition should be $P^x = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9, 10, 11, 12\}\}$. We can see that partition P_x preserves better than P the information provided by vector x, in the sense that in P_x maintains together in a group nodes which have a high score in x, cluster $\{1, 2, 3, 4, 5, 6\}$, and maintains in another group nodes which have a low score in x, cluster $\{7, 8, 9, 10, 11, 12\}$. This situation is showed by matrix F (see Figure 2: the values related to nodes whose score in x is high, are the highest in the matrix (those related to $i, j = 1, \ldots, 6$, they are highlighted in the matrix).

$$\frac{1}{10000}\begin{pmatrix}
0 & \mathbf{246} & \mathbf{203} & \mathbf{238} & \mathbf{224} & \mathbf{259} & 22 & 34 & 45 & 38 & 57 & 11\\
\mathbf{246} & 0 & \mathbf{214} & \mathbf{252} & \mathbf{236} & \mathbf{276} & 23 & 36 & 48 & 41 & 61 & 11\\
\mathbf{203} & \mathbf{214} & 0 & \mathbf{208} & \mathbf{196} & \mathbf{226} & 20 & 30 & 40 & 34 & 51 & 9\\
\mathbf{238} & \mathbf{252} & \mathbf{208} & 0 & \mathbf{229} & \mathbf{265} & 23 & 34 & 46 & 39 & 59 & 11\\
\mathbf{224} & \mathbf{236} & \mathbf{196} & \mathbf{229} & 0 & \mathbf{249} & 21 & 33 & 44 & 37 & 55 & 10\\
\mathbf{259} & \mathbf{276} & \mathbf{226} & \mathbf{265} & \mathbf{249} & 0 & 25 & 37 & 51 & 43 & 64 & 12\\
22 & 23 & 20 & 23 & 21 & 25 & 0 & 3 & 5 & 4 & 6 & 1\\
34 & 36 & 30 & 34 & 33 & 37 & 3 & 0 & 7 & 6 & 9 & 1\\
45 & 48 & 40 & 46 & 44 & 51 & 5 & 7 & 0 & 8 & 12 & 2\\
38 & 41 & 34 & 39 & 37 & 43 & 4 & 6 & 8 & 0 & 10 & 2\\
57 & 61 & 51 & 59 & 55 & 64 & 6 & 9 & 12 & 10 & 0 & 3\\
11 & 11 & 9 & 11 & 10 & 12 & 1 & 1 & 2 & 2 & 3 & 0
\end{pmatrix}$$

Fig. 2: Matrix F

Then, let us show the performance of $1-$ $Additive\ Sugeno\_Louvain$ Algorithm in a well known example: the dolphins network.

Example V.3: The dolphin network [35], [36] has been widely analyzed in the literature. It is an undirected social network about frequent associations between some dolphins in a community living on Doubtful Sound, New Zealand. It comprises 62 nodes and 159 links. In Figure 3 we show the partition provided by Louvain Algorithm, which has 5 groups, $C_1, \ldots, C_5$. Particularly, node 2 is assigned to cluster cluster C_1, node 8 is assigned to cluster C_2, node 24 is assigned to cluster C_4, node 37 is assigned to cluster C_5 and node 41 is assigned to cluster C_3. The groups are: $C_1 = \{\mathbf{2}, 6, 7, 10, 14, 18, 23, 26, 27, 28, 32, 33, 42, 49, 55, 57, 58, 61\}$; $C_2 = \{1, 3, \mathbf{8}, 11, 20, 31, 43, 48\}$; $C_3 = \{13, 15, 17, 34, 35, 38, 39, \mathbf{41}, 44, 45, 47, 50, 51, 53, 54, 59, 62\}$; $C_4 = \{5, 12, 16, 19, 22, \mathbf{24}, 26, 30, 36, 46, 52, 56\}$; $C_5 = \{4, 9, 21, 29, \mathbf{37}, 40, 60\}$.

Let x be a vector defining some evidence about the individuals of the dolphins network, where $x_2 = x_8 = x_{24} = x_{37} = x_{41} = 1$, and $x_i = 0$ otherwise If we apply the $1-$ $Additive\ Sugeno\_Louvain$ Algorithm considering that vector, the obtained partition has 4 clusters, $C_1^x, \ldots, C_4^x$ (see Figure 4), in contrast to the solution with 5 groups provided by the aplication of Louvain Algorithm.

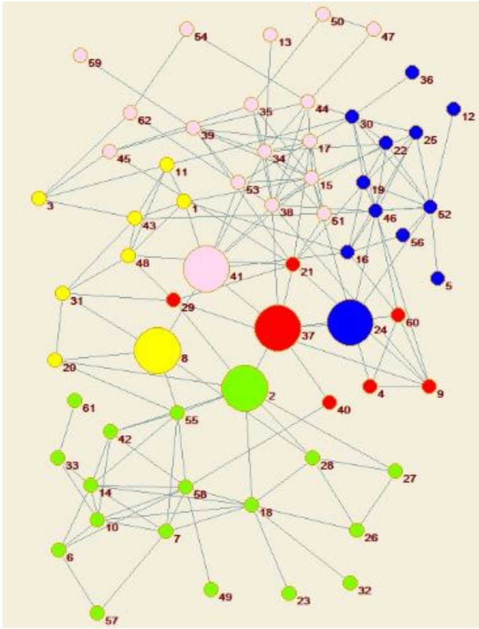

Fig. 3: Partition of dolphins network with Louvain Algorithm

Due to the consideration of the vector x, those nodes whose value in mentioned vector is 1, are likely to stay together in the same cluster, leading to changes in the structure of groups in which the rest of the nodes are organized. If the vector x is not considered, those nodes are assigned to different groups. Then, considering as input parameters the adjacency matrix of the dolphin network, and that vector x, the groups obtained with $1 - Additive\ Sugeno\_Louvain$ Algorithm are: $C_1^x = C_1 \backslash \{\mathbf{2}, 26, 27, 28\}$; $C_2^x = C_2 \cup \{\mathbf{2}, \mathbf{24}, 26, 27, 28, 29, 37, 40, \mathbf{41}\}$; $C_3^x = (C_3 \backslash \{\mathbf{41}\}) \cup \{21\}$; $C_4^x = (C_4 \backslash \{\mathbf{24}\}) \cup \{4, 9, 60\}$.

Let us remark that the complexity of the $1 - Additive\ Sugeno\_Louvain$ Algorithm does not grow when the scale of the graph increases. However, according to our best knowledge, this experimental verification does not apply on this paper, so we will soon work on a wide development of computational results which will be included in other articles.

VI. Conclusions and Further Research

In this work we deal with the problem of finding communities in a graph from which we have some additional information, independent of its topology, provided by a vector.

Hence, given a graph $G = (V, E)$, the starting point is the n-vector x, which defines some evidence related to the elements of V. Then, we introduce the definition of the function μ_x (see Definition II.4). It is the fuzzy measure related to x [37], [38] which fixes the properties of Sugeno λ-measures [19]. Depending on the problem, we differentiate several options in the definition of μ_x. Particularly, we propose two different characterization of it, μ_x^a and μ_x^m (Definition II.4, with $p = 1$ and $p \in (0, 1)$, respectively).

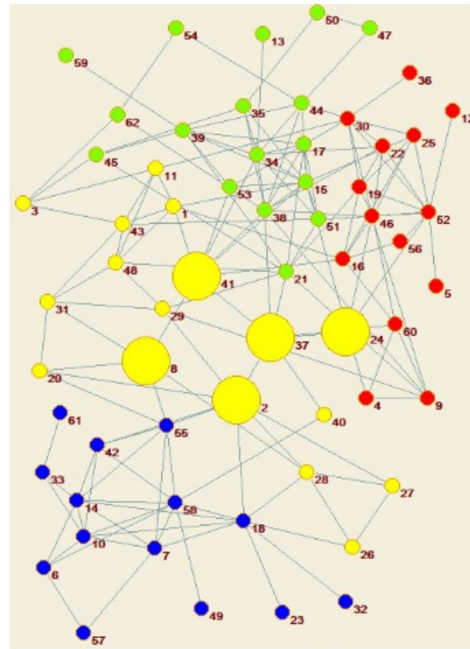

Fig. 4: Partition of dolphins network with $1 - Additive\ Sugeno\_Louvain$ Algorithm

Having defined the fuzzy measure related to vector x, we propose a modification of the notion of extended fuzzy graph, in order to adapt it the context we handle in this work. Hence, we introduce the concept of **e**xtended vector fuzzy graph, $\widetilde{G}_x = (V, E, \mu_x)$, where $G = (V, E)$ is a graph and μ_x is obtained from vector x, which defines some evidence about the elements of V.

Then, we propose a particular application of this fuzzy measure associated with a vector: community detection problems. So, we first define G_{μ_x}, the weighted graph associated with μ_x [15]. For the specific case of μ_x^a , which is an additive fuzzy measure, several properties of its related Shapley value [21] are enunciated. Finally, inspired by the performance of Louvain algorithm [22], we propose a community detection algorithm to find a partition in the extended vector fuzzy graph $\widetilde{G}_x = (V, E, \mu_x)$, called $Sugeno\_Louvain$ Algorithm. This cluster structure, apart from considering the direct connections of the graph G, given by E, is obtained considering the information provided by the vector x. Hence, this partition is consistent with all the available information, not only the direct connections among the elements but also the additional evidence given by x. We also provide a particular application of this method, devoted to additive fuzzy measures, such that our proposal has the same complexity as Louvain algorithm. It is called $1 - Additive\ Sugeno\_Louvain$ Algorithm, and it is summarized in Algorithm 2. To illustrate the performance of this method, we provide its performance in two cases. The first one is a toy example with which it is easy to understand the algorithm. Then, we work with a well-known example in networks literature, the dolphins network [35], [36], providing the partition obtained with Louvain Algorithm [22], and with

$1-Additive\ Sugeno\_Louvain$ Algorithm.

In this work we have focused on additive fuzzy measures obtained from a vector and its application in community detection problems. However, there is a wide framework to develop about non-additive fuzzy measures (see for example the work done in [39]). Our immediate further work is the development of an experimental study to test the efficiency of the approach here addressed as well as an analysis of the processing time and memory usage of the algorithms here proposed, among others points. We will also work in a theoretical analysis in order to analyze the behaviour of the multiplicative component of μ_x. Then, we propose to expand the background of this paper, by considering not only additive Sugeno λ-measures in which $\lambda \neq 0$. We will also analyze other possible application of the capacity measures here introduced, besides community detection problems.

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