# Fuzzy Number Value or Defuzzified Value; Which One Does It Better?

Amir Pourabdollah

***Abstract*—This paper proposes a cross-domain statistical method for comparing defuzzification algorithms based on incorporating the concepts of fuzzy sets' pairwise operators of similarities and distances. The method measures the correlation between the changes in defuzzified value between two fuzzy sets and comparing it with their similarity/distance. Two support-based defuzzification methods known for fuzzy controllers (COG and MOM) along with two level-based methods known for ranking fuzzy numbers (ALC and VAL) are chosen and compared using the proposed method. The results indicate that the chosen methods in fuzzy number domain more strongly captures the embedded information in the fuzzy sets than those of fuzzy rule-based system. The contributions of this paper are firstly proposing a heuristic method for comparing any two different fuzzification algorithms, and secondly highlighting the potentials for cross-domain utilisation of those algorithms where possible.**

## I. Introduction

The term defuzzification generally denotes the method of mapping a fuzzy set to a representative number. Despite having many common properties, the concept of defuzzification is handled differently among fuzzy rule-base system community and fuzzy arithmeticians. As such, many defuzzification methods are specialised in isolation between the two groups. A large number of algorithms are developed on each side over the years, some, but not many of which are commonly used. The fact that fuzzy numbers are special cases of fuzzy sets (restricted by convexity, uni-modality and normality) has limited some cross-domain reuse of the defuzzification methods. Moreover, because there are many methods on each side, there are not much attempts to use a method of one group in the context of the other. These facts indicate that there are still rooms for uncovering the common grounds.

This paper is not going to propose yet another defuzzification or fuzzy number ranking method. Instead, it proposes an statistical solution on how the two sides' defuzzification/ranking algorithms can be compared quantitatively regardless of their contextual differences. In fact, it can be used for comparing any two arbitrary defuzzification methods. Comparing a method of the first group with the other may initially seem pointless as they may be developed to represent fuzzy sets for different purposes, one for reducing an output fuzzy set of a rule-based system to a crisp number, and the other for arithmetic purposes such as ranking a group of fuzzy numbers. Most of the algorithms in the first group are support-based (i.e., based on averaging the MF) while the second group methods are mainly level-based (i.e., based on averaging $\alpha$-levels) [1].

Amir Pourabdollah is with the School of Science and Technology, Nottingham Trent University, Nottingham, UK. (email: amir.pourabdollah@ntu.ac.uk)

This context-independence comparison is achieved through studying how much a method can capture the embedded information in a fuzzy set and encoded in different aspects of its MF. In the absence of any definite application-independent measure of comparing defuzzification algorithms, a heuristic method is developed in this paper based on the pairwise properties of fuzzy sets, i.e., similarity index and directional distance. Centre of Gravity (COG or Centroid) and Mean of Maxima (MOM) are chosen as two common defuzzification methods of rule-based system, as well as Averaging Level Cuts (ALC) and Fuzzy Number Value (VAL) representing the second group. A correlation test is designed for assessing how much each defuzzification method is statistically bound to any of the pairwise operations on fuzzy sets. This test can indicate the amount of fuzzy set information captured by the algorithm solely based on the MF appearance. It is noticeable that in rule-based fuzzy logic systems, capturing the maximum information or uncertainties embedded in a fuzzy set is not limited to the defuzzification element, but also can be done in other parts such as the inference engine [2], [3].

In summary, the contributions of this paper are 1) providing a general methodology for statistically comparing any two defuzzification methods across different application domains; and 2) applying the developed methodology to exemplar cross-domain defuzzification algorithms and comparing their performances, thus highlighting the potentials for cross-domain utilisation of those algorithms where possible.

In the rest of this paper, first the background information including the definitions will be provided in Section II. Then in Section III, the proposed comparison method will be explained. Section IV is for presenting and discussing the experiment results, which is followed by the paper conclusion and the future works in Section V.

## II. Backgrounds and Definitions

In this section, the concepts that are used in the rest of the paper are formally defined. The definitions start from the basic concepts, the picked up defuzzification and valuation methods followed by the pairwise similarity/distance measurements of the fuzzy sets.

*1) A Multidisciplinary Approach to Defuzzification:* In the context of fuzzy rule-based system, defuzzification is an essential building block towards the end of the system where a crisp output is needed (e.g., for the voltage needed to drive a motor). Since the output fuzzy sets are the outcome of a highly non-linear inference system, it is quite common that the output fuzzy sets (before defuzzification) are highly irregular in shape. These sets can be non-convex, non-normal and multi-model, as they are not meant to be human-interpretable.

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In fuzzy arithmetic, a fuzzy number [4] is a generalisation of ordinary numbers to uncertain values by representing the number as a fuzzy set. Such a fuzzy set is meant to be human-interpretable, so that by definition, the fuzzy set that represents a fuzzy number is restricted to be normal, convex and uni-modal[1]. Defuzzification in this context is the process of choosing a representative ordinary value, and is mostly used for ranking the fuzzy numbers. Many support-based and level-based defuzzification algorithms are developed for fuzzy numbers, a couple of which will be reviewed in the this section.

In this paper, the defuzzification of the output fuzzy sets in rule-based systems is considered conceptually different from the defuzzification of fuzzy numbers. This is not because of the underlying algorithms (which is essentially the same, and could be even interchangeable), but because of the defuzzification purpose (*representation-only* versus *valuation*). Therefore for clarity in this paper, the term "defuzzification" is used for the context of fuzzy rule-based system only. For fuzzy numbers, the process is called *valuation*. Because of having common grounds for comparing defuzzification and valuation in this paper, the scope is also limited to convex normal uni-model fuzzy sets.

It is noticeable here that some known valuation methods have been used for rule-based system (and vice-versa), though being relatively rare. For example in [6], $\alpha$-levels are used for defuzzification when the explicit MFs are not known, but $\alpha$-cuts are. An approximated MF is then generated by piece-wise linear functions derived from the known $\alpha$-cuts; finally, the approximated MF is defuzzified by known defuzzification methods such as COG.

In the following sub-sections, the choices of two common defuzzification and two common valuation methods will be briefly reviewed.

*2) The Chosen Defuzzification Methods:* The majority of defuzzification methods used for fuzzy rule-based systems are support-based, i.e., based on different forms of averaging the membership function (MF). Some known defuzzification algorithms in this group are termed as COG, MOM/-FOM/LOM (Mean/First/Last of Maxima), MOS/FOS/LOS (Mean/First/Last of Support) and BADD (Basic Defuzzification Distributions). There are many reviews, comparisons and extensions over such methods in the literature (see for example, [1]). In this paper, in order to limit the scope, COG and MOM are focused, as they are arguably the most widely-used defuzzification methods for fuzzy rule-based systems [7], [8].

The Centre of Gravity (COG), also known as centroid, is defined using the MF of a fuzzy set represented as $\mu(x)$. In discrete model, where the $x$-axis is quantised as a collection of $x_i$'s, the COG is defined as:

$$COG = \frac{\sum x_i \mu(x_i)}{\sum \mu(x_i)} \tag{1}$$

Mean/Middle of Maxima (MOM) is another widely-used defuzzification method due to its calculation simplicity. MOM is simply defined as the mean of the x-values whose membership degrees are maximum [1]. This maximum for normal sets is 1. If the set has a single peak (i.e., a uni-modal set) the MOM is equivalent to the corresponding x-value to the peak.

[1]The uni-modality condition is lifted in some papers, such as in [5].

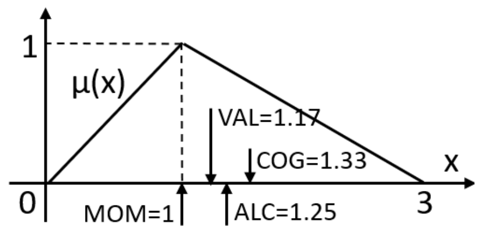



Fig. 1. An example of the calculated four different values by COG, MOM, ALC and VAL algorithms for the case of a fuzzy set with triangular NF (not an accurate scale

$$MOM = Mean(x_i \mid \mu(x_i) = max) \tag{2}$$

*3) The Chosen Valuation Methods for Fuzzy Numbers:* Most of the valuation algorithms used for ranking fuzzy numbers are level-based, i.e., based on alpha-cut representation of fuzzy sets. An $\alpha$-cut is simply a subset of a fuzzy set whose membership grades are greater than or equal to $\alpha \in [0, 1]$ [9].

$$A_\alpha = \{x_i \mid \mu(x_i) \geq \alpha\}; \alpha \in [0, 1] \tag{3}$$

A method of flat averaging of all midpoints of the alpha-cuts, i.e., without weighting, is introduced by Yager in [10], called ALC in [11] too. The ALC algorithm is also equivalent to calculating the *Expected Value (EV)* of a fuzzy number as introduced by Oussalah in [11] and Heilpern in [12].

$$ALC = Mean(Mid(x_i) \mid x_i \in A_\alpha \ ; \ \alpha \in [0, 1]) \tag{4}$$

It is intuitive to consider the individual contributions of each $\alpha$-cut, in which the $x$ values that appear in more $\alpha$-cuts should have a greater contribution in defuzzification than the others. Based on this idea, the *value* of a fuzzy number is defined by Delgado et al. in [5], based on using $\alpha$-levels as weighting factors in averaging the $\alpha$-cut midpoints. The value of a fuzzy number, by the following definition, is called VAL in this paper.

$$VAL = \frac{\sum \alpha Mid(x_i \in A_\alpha)}{\sum \alpha} \quad \alpha \in [0, 1] \tag{5}$$

ALC and VAL, as defined here, are two dominant *valuation* methods in this group, and will be used later in this paper. Similarly, the ideas of interval-valued defuzzification and the mean of fuzzy numbers initially suggested by Dubois and Prade [13] is another area of research for defuzzifying a fuzzy number having a level-based approach.

An example of the calculated four different values by COG, MOM, ALC and VAL algorithms for the case of a fuzzy set with triangular MF is shown in Fig. 1

*4) Pairwise Comparisons of Fuzzy Sets:* In this paper, the comparison between the chosen defuzzification and valuation algorithms are carried out based on measuring their correlations to a couple of known pairwise comparisons of fuzzy sets (details in the next section). Therefore, they are formally introduced here.

Measuring the similarity between two fuzzy sets has been widely known in the literature [14]. One of the most common similarity measures on fuzzy sets is the Jaccard's Similarity Index [15], an adaptation of the initial Jaccard's coefficient [16] for fuzzy sets. This index (called SI hereafter) assigns a value between 0 (completely dissimilar sets) to 1 (completely similar sets) to a pair of fuzzy sets. For two fuzzy sets $A$ and $B$, SI is defined as:

$$SI(A,B) = \frac{\sum \min(\mu_A(x_i), \mu_B(x_i))}{\sum \max(\mu_A(x_i), \mu_B(x_i))} \tag{6}$$

*5) Directional Distance:* Another group of pairwise operators between two fuzzy sets is measuring the *distance* between them in the same unit as the $x$-axis of the MFs of the two compared sets. Many such methods exist (e.g., [17]–[19]). The focus of this paper is on measuring Directional Distance (abbreviated here as DD) [20], which is able to measure the distance between two fuzzy sets as a signed value (in two directions). A positive DD indicates how much the second set's MF is on the right side of the first set's, and the negative DD shows that the second set is on the left side of the first. A zero DD means the two sets are shaped around a single value along the $x$-axis.

The definition of the DD between two FSs is based on the distance between their $\alpha$-cuts. According to [20], if the $\alpha$-cuts of two arbitrary FSs $X$ and $Y$ are defined as intervals $[\mu_X]_\alpha = [\overline{x}_\alpha, \underline{x}_\alpha]$ and $[\mu_Y]_\alpha = [\overline{y}_\alpha, \underline{y}_\alpha]$ respectively, their DD is defined as:

$$DD(X,Y) = \frac{\sum \left(\alpha.h([\mu_X]_\alpha, [\mu_Y]_\alpha)\right)}{\sum \alpha}; \quad \alpha \in [0,1] \tag{7}$$

$$h([\mu_X]_\alpha, [\mu_Y]_\alpha) = \begin{cases} \overline{x}_\alpha - \overline{y}_\alpha, & if \quad |\overline{x}_\alpha - \overline{y}_\alpha| > |\underline{x}_\alpha - \underline{y}_\alpha| \\ \underline{x}_\alpha - \underline{y}_\alpha, & otherwise. \end{cases} \tag{8}$$

where $h$ is a modified version of Hausdorff distance [19] between two intervals.

In the next section, based on the above definitions, the proposed method for comparing the picked defuzzification and valuation methods is provided.

## III. Methodology

In the following sub-sections, the proposed solution for statistically comparing different diffuzification/valuation algorithms are explained.

*1) Comparison Algorithm:* In order to evaluate the *goodness* of a particular defuzzification algorithm, one may measure its performance when utilised in a fuzzy logic system or the other one may assess it against a set of defuzzification axioms (e.g., [21]). Similarly for fuzzy number, there are axioms developed for assessing the ranking performances (e.g., in [22]).

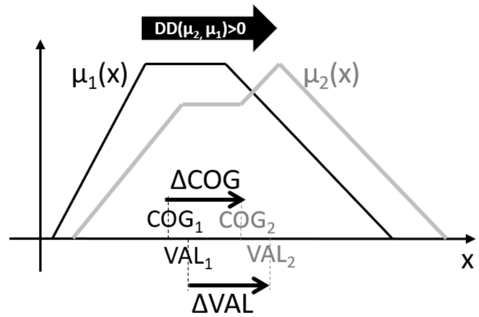



Fig. 2. The relation between the directional distance of two fuzzy sets vs. the signed difference in their defuzzified values.

For a cross-comparison between the algorithms in the two subject areas (fuzzy arithmetic vs. rule-based systems), it is arguable if the above axiomatic or performance assessment methods are reasonable since the assessment settings are usually different. The alternative assessment can be a comparison method relying on the appearance characteristics of the fuzzy set's MF independently from how the fuzzy set is used for a specific purpose. The pairwise operators discussed in the previous section have such potential, thus can be considered.

The general idea is to choose a defuzzification/valuation method and a pairwise distance/similarity operator, and check their statistical correlations when applied over a large number of fuzzy set pairs. More precisely, the correlation is checked between the *change* in the defuzzification/valuation result and the distance/similarity outcome. It is assumed that for any two fuzzy sets, the differences between their defuzzification/valuation results should follow (i.e., be a monotonic function of) the two sets' similarity and/or distance.

For example, more similarity means less defuzzification difference, so imagine that for two sets, their SI is measured as 10%, the calculated COGs show 12% difference, and the calculated VALs are 15% different. In this case, COG has done it better since its change has better reflected the similarity. If the DD is measured for the same sets, the winner defuzzification method would be the one that keeps changing more closely with the DD. An illustrative example is shown in Fig. 2.

In other words, the theory is that if a defuzzification/-valuation algorithm shows a closer link to the appearance characteristics of its corresponding fuzzy set, it indicates a better capture of the information embedded in the MF by that method, thus making a better representative value out of the set.

Based on the above idea, a statistical test is designed for comparing between the picked defuzzification/valuation algorithms (COG, MOM, ALC and VAL) based on assessing how they are bound with the picked pairwise operators (SI and DD). The assessment algorithm includes the following tasks:

- Generate a large number of randomised fuzzy sets. For compatibility with fuzzy numbers, the generated sets are normal and convex.

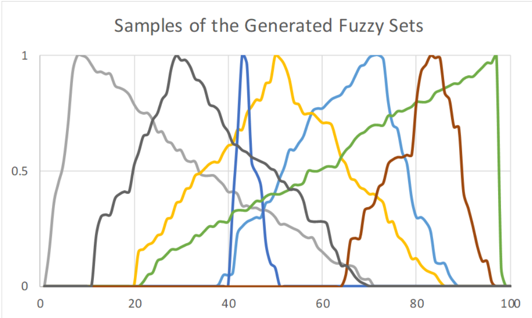



Fig. 3. A number of the sample generated random sets

- For each generated fuzzy set, compute its COG, MOM, ALC and VAL.
- For each possible pair of the generated fuzzy sets, compute their SI and DD. For fuzzy sets number i and j, these are called $SI_{ij}$ and $DD_{ij}$.
- For each of the above fuzzy set pairs, calculate the differences between their COGs, MOMs, ALCs and VALs. These are called $\Delta COG_{ij}$, $\Delta MOM_{ij}$, $\Delta ALC_{ij}$ and $\Delta VAL_{ij}$.
- Over all the fuzzy set pairs (the $i$th and the $j$th set), calculate statistical correlations between any of ($\Delta COG_{ij}$, $\Delta MOM_{ij}$, $\Delta ALC_{ij}$, $\Delta VAL_{ij}$) values and any of ($SI_{ij}$, $DD_{ij}$) values. The stronger the correlation, the better the algorithm in respect to the corresponding pairwise operator.

*2) Generating Random Fuzzy Sets and pairing them:* A computer program is developed to generate 50 convex normal sets. Having 50 sets generated, 1225 combinations are possible for the required set pairs[2], which are statistically sufficient for this experiment.

For generating each set, the program chooses three random numbers in an arbitrary range over the $x$-axis (here 1 to 100), as being the left, mid and right points of the MF. The generated set is supposed to be non-symmetric, therefore the mid point is not actually in the middle of the left and the right points. Then the MF randomly (but monotonically) grows from 0 to 1 between the left and the mid points, then falls randomly towards 0 as it reaches to the right point. A number of the sample generated random sets are shown in Fig. 3

*3) Pairwise Calculations:* The 50 fuzzy sets are considered to have randomly distributed defuzzified values, similarities and differences. Each of the 50 sets is defuzzified using the 4 discussed methods. Having discretized fuzzy sets (here in 100 levels of discretization over $x$- and $y$-axes), a computer program is designed to calculate the necessary results as explained previously. For each of the 1225 pairs, the pairwise differences between the defuzzified values in each method are calculated. Finally, the DD and SI are calculated for each pair according to their definitions in (6) and (7).

*4) Correlation Measurements:* Two widely-known statistical correlation tests are considered: Pearson's Bivariate Correlation Coefficient [23] and Distance Correlation [24]. The first one returns a value between -1 to 1 indicating how linear two variables follow each other's changes, in which 1 means a completely linear direct relation, 0 means no relation and -1 means a completely inverse linear relationship. The bivariate correlation is defined as:

$$Correlation(X,Y) = \frac{\sum_{i=0}^{n}[(x_i - \overline{x})(y_i - \overline{y})]}{\sqrt{\sum_{i=1}^{n}[(x_i - \overline{x})^2(y_i - \overline{y})^2]}}; \quad (9)$$

where $X = \{x_1...x_n\}$ and $Y = \{y_1...y_n\}$ are two sample sets of $n$ points with average values $\overline{x}$ and $\overline{y}$.

For example in the case of measuring the correlation between In the $\Delta COG_{ij}$ and $DD_{ij}$ among the 50 generated fuzzy sets,

$$X = \{\Delta COG_{ij} \mid i = 1...50,\ j = i+1\ ...50\} \quad (10)$$

$$Y = \{DD_{ij} \mid i = 1...50,\ j = i+1\ ...50\} \quad (11)$$

The other similar method, Distance Correlation also delivers the similar outcomes but it is not limited to determining linear relationship, so that it can investigate a wider range of cases where two variables follow each other's changes. More details on its calculation formula can be found in [24].

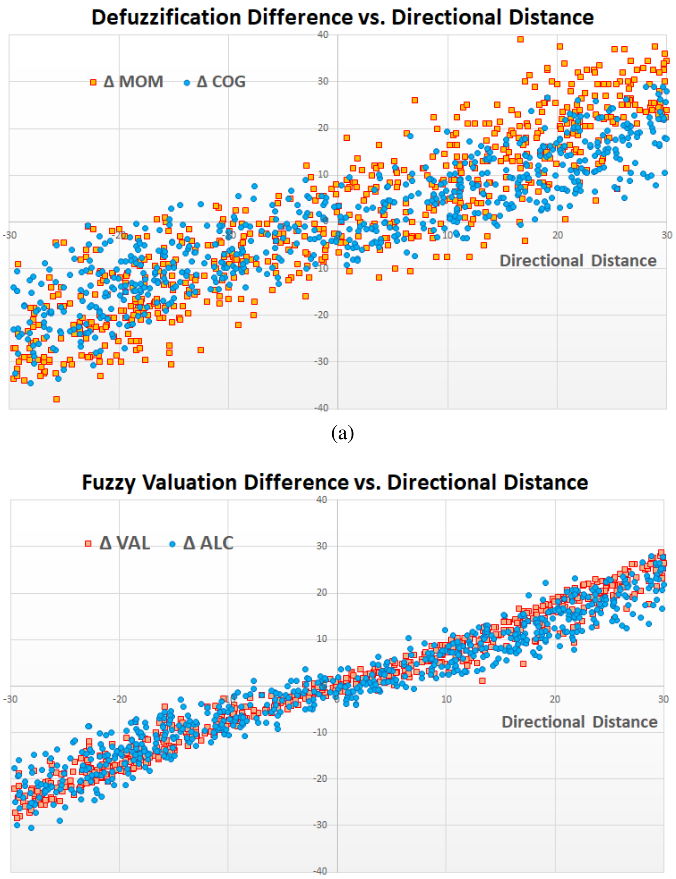
## IV. Experiment Results and Discussions

Fig. 4 and 5 illustrate the experiment results shown in scatter graphs. The following observations are made from Figs 4-5:

- In regards to Fig. 4 (a,b), all 4 methods follow the DD but in different levels of linearity and variance. VAL shows the closest match with less perturbation, then ALC, then COG and finally MOM.
- For Fig. 5 (a,b), an inverse following of the SI is observed for all 4 methods, as expected, but again with different levels of concentration and linearity. COG in this case has performed better than the others, followed by ALC, VAL and MOM.
- From Fig.4 (a,b), more variances in respect to DD are observed for both defuzzification methods (COG and MOM) than the so-called valuation methods (VAL and ALC). This statistically show that the valuation methods have captured more information from a fuzzy set's MF than the other group. For instance, the higher the variance, the higher the chance of having two pairs of fuzzy sets, in which their directional distance is almost the same, but the difference between their defuzzified values is relatively high. This fact is the case in the SI dependency graph (Fig.5 (a,b)) only between COG and ALC.
- There is a concentrated trace of points along $y$-axis in Fig. 5 (a,b). This is the results of many set pairs having zero similarities. It is noticeable that if two sets are disjoint their similarity is zero according to SI definition. This means that SI test has excluded a relatively high number of samples disregarding the information embedded in the wide range of differences between their defuzzified/valuated outcomes. This observation gives more credit to the results of DD-test than SI-test.

The statistical analysis of the results are shown in Tables II and II. The following facts can be observed from the tables:

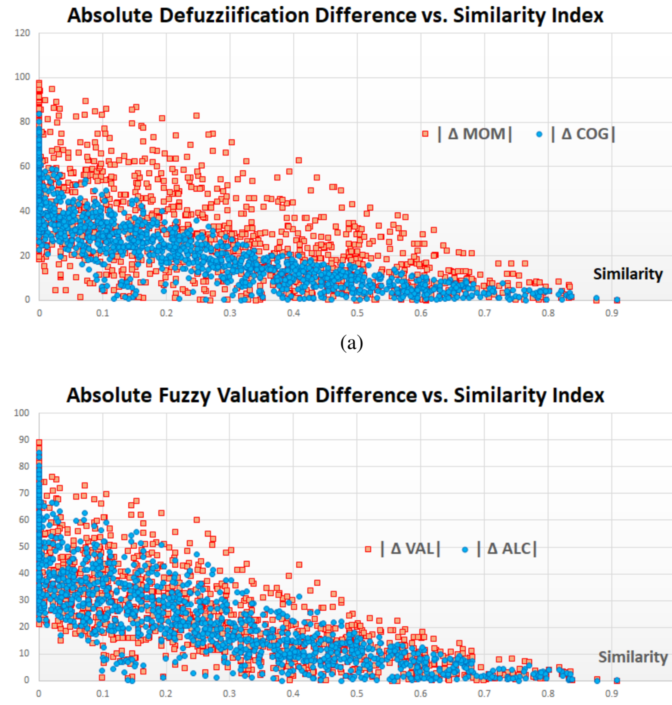
[2]No repetition or self-paired are allowed, thus $(50 \times 49)/2 = 1225$.




(a)

(b)

Fig. 4. The graphs show (a) the dependencies between two defuzzification methods MOM and COG to Directional Distance; (b) the dependencies between two fuzzy number valuation methods VAL and ALC to the same Directional Distance.




(a)

(b)

Fig. 5. The graphs show (a) the dependencies between two defuzzification methods MOM and COG to the Jaccard's Similarity Index; (b) the dependencies between two fuzzy number valuation methods VAL and ALC to the same Similarity Index. In both, the absolute value of the differences are considered since they are being correlated to an unsigned similarity index.

- Both tables show some similar statistical facts. This shows that the biravriate correlation and distance correlation have measured almost the same correlation behaviour. Since the bivariate correlation measurement is limited to linear relationships, this shows that the majority of the discovered correlations are of linear type.
- The best method in accordance to DD correlation is VAL and the worst is MOM. Both valuations methods have outperformed the both defuzzification methods.
- With respect to SI, the stronger correlation is for COG from the defuzzification group of methods. Note that in this table 1 and -1 are both showing high correlations. MOM is still the worst and the two other methods in the valuation group remained at the middle. Unlike the DD case, the difference between the best and the worst performances is relatively large up to about 27%.
- Within the group of valuations, in respect to DD, VAL has always outperformed ALC, though with very small differences. In respect to SI, interestingly, VAL has outperformed ALC.
- Within the defuzzification group, COG has always outperformed MOM.

In summary, for ranking the picked algorithms out of the shown graphs and tables, DD and SI tests have a slightly different statistical outcomes, however DD measurement seems to be more statistically credible. The results are mainly in favour of the valuation algorithms than the defuzzification algorithms, leading to rank the algorithms in the following order: (1) VAL; (2) ALC; (3) COG; and (4) MOM.

TABLE I
THE CALCULATED BIVARIATE CORRELATIONS BETWEEN DIFFERENT DEFUZZIFICATION/VALUATION METHODS, DIRECTIONAL DISTANCE AND SIMILARITY INDEX.

| Variable | Bivar-Corr. to Directional Distance | Bivar-Corr. to Jaccard's Similarity Index |
|---|---|---|
| $\Delta VAL$ | 0.9962 (best) | -0.74144 (-8.52%) |
| $\Delta ALC$ | 0.9953 (-0.10%) | -0.78741 (-2.85%) |
| $\Delta COG$ | 0.9768 (-1.95%) | -0.8105 (best) |
| $\Delta MOM$ | 0.9823 (-1.40%) | -0.6195 (-23.56%) |

TABLE II
THE CALCULATED DISTANCE CORRELATIONS BETWEEN DIFFERENT DEFUZZIFICATION/VALUATION METHODS, DIRECTIONAL DISTANCE AND SIMILARITY INDEX.

| Method | Dist-Corr. to Directional Distance | Dist-Corr. to Jaccard's Similarity Index |
|---|---|---|
| $\Delta VAL$ | 0.9953 (best) | 0.7514 (-12.17%) |
| $\Delta ALC$ | 0.9906 (-0.47%) | 0.8149 (-4.74%) |
| $\Delta COG$ | 0.9693 (-2.61%) | -0.8555 (best) |
| $\Delta MOM$ | 0.9823 (-2.09%) | -0.6108 (-28.61%) |

## V. Conclusions and Future Works

This paper is a comparative study on the utilities of two groups of algorithms in mapping a fuzzy set to a number: Defuzzification (in the context of fuzzy rule-based systems) and valuation (a term used here referring to such a mapping in the context of fuzzy numbers and their rankings). Two mostly used algorithms of the first group are COG and MOM, along with two algorithms of the second groups being fuzzy number value VAL and ALC. The analysis utilises two pairwise fuzzy sets operations DD (Directional Distance) and SI (Jaccard's Similarity Index) in order to provide a common testing platforms known in the two contexts. The analysis is around a theory that the better algorithm more closely follow the changes in the distance/similarity between any two sets since this test indicates the amount of fuzzy set information captured by the algorithm.

As such, an statistical test is designed to rank the algorithms when their performances are tested over a large number of arbitrary fuzzy sets pairs. Each method performance is defined as the correlation strength between the changes in the defuzzified/valuated results within a pair against the directional distance and the similarity between the two sets. 50 randomly generated sets leading to 1225 possible pairs, and two correlation measures (bivriate correlation coefficient and distance correlation) were used. The results show that is general, the VAL algorithm outperformed the other three algorithms, followed by ALC, COG and MOM, respectively.

This research does not make recommendations on prioritising a particular method over the other just because it outperformed in this statistical analysis, nor suggests a new defuzzification method. Other factors such as computational complexities and the application context are to be incorporated for making decisions in selecting a particular defuzzification method. However, this research, statistically suggests that the main valuation methods developed for fuzzy numbers have a greater potential of capturing the maximum possible information from fuzzy sets.

Although some axiomatic measures have been developed for evaluating defuzzification methods (e.g., [21], [22]), there is no definite application-independent indicator to be used for comparing the utilities of the defuzzification methods nor a measure of capturing the information associated in a fuzzy set by a such methods. Therefore, the contribution of this paper in using SI and DD as means of such comparison shall be considered as heuristic, and its results as indicative only.

The other practical outcome of this research is to show the potential of sharing the defuzzification algorithms between the communities of rule-based systems and fuzzy arithmetic. As such, a main follow-up path after this research is to test the applicability and utility of valuation algorithm in fuzzy arithmetic (e.g., VAL) for fuzzy rule-based system (e.g., in a fuzzy controller) and assess its performance, computational complexities and other practical implications. Moreover, adding some defuzzification methods for a more comprehensive analysis, and also performing different types of fuzzy number sampling are the other possible future works.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