On the Dominance Relation Between Ordinal Sums of Quasi-Overlap Functions

1st Ivan Mezzomo
Departamento de Ciências Naturais,
Matemática e Estatística - DCME
Universidade Federal Rural do Semi-Árido - UFERSA
Mossoró, Brazil
imezzomo@ufersa.edu.br

2nd Heloisa Frazão
Departamento de Ciência e Tecnologia - DCT
Universidade Federal do Rio Grande do Norte - UFRN
Caraúbas, Brazil
heloisafrazao@ufersa.edu.br

3nd Benjamín Bedregal
Departmento de Informatica e Matemática Aplicada - DIMAp
Universidade Federal do Rio Grande do Norte - UFRN
Natal, Brazil
bedregal@dimap.ufrn.br

4rd Matheus da Silva Menezes
Departamento de Ciências Naturais,
Matemática e Estatística - DCME
Universidade Federal Rural do Semi-Árido - UFERSA
Mossoró, Brazil
matheus@ufersa.edu.br

***Abstract*—In this paper, we consider the notions of overlap functions and dominance relation of conjunctors to define dominance relations on overlap functions, and we prove some results involving these concepts and automorphisms. Moreover, we weaken the notion of overlap functions, excluding the continuity, called quasi-overlap functions, and we prove that the ordinal sums of quasi-overlap functions also are quasi-overlap functions.**

***Index Terms*—Quasi-overlap functions, dominance relation, ordinal sums, automorphisms.**

I. Introduction

The notion of overlap functions was introduced by Bustince et. al. [9], for an application on classification problems in image processing where the overlap functions are used on the identification of the objects in a given image. Overlap functions are a particular case of continuous aggregation functions [5], that is, it can be considered with a specific class of binary aggregation function.

The class of overlap functions is reacher than the class of t-norms in the sense of t-norms there are one idempotent t-norm and two homogeneous t-norms and overlap functions there is an uncountable number of idempotent, as well as homogeneous overlap functions. There are many papers making comparisons among properties of overlap and t-norms, as can be seen in [3], [9], [10], [23].

Recently, Paiva et. al. [33], introduced a more general definition of overlap functions, called of quasi-overlap functions, which arise of abolishes the continuity condition and they investigated the main properties of (quasi-)overlaps on bounded lattices, namely, convex sum, migrativity, homogeneity, idempotency and cancellation law.

This work is supported by Brazilian National Council for Scientific and Technological Development - CNPq (Proc. 404382/2016-9 and 307781/2016-0) and UFERSA.

The idea of ordinal sums of semigroups are given by Climescu (1946) [13], Clifford (1954) [11] and Clifford and Preston (1961) [12]. In 1963, Schweizer and Sklar [40] characterized the idea of ordinal sums of semigroups from t-norms and t-conorms. After this, Ling (1965) [26] and Frank (1979) [21] gave important contributions for ordinal sums from t-norms and t-conorms. In 2000, Klement, Mesiar and Pap [24] introduced a new family of t-(co)norms called the ordinal sum of the summands (a_i, b_i, T_i) $((a_i, b_i, S_i))$ of t-(co)norms where $(T_i)_{i\in I}$ $((S_i)_{i\in I})$ be a family of t-(co)norms and $(]a_i, b_i[)_{i\in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0, 1]$. However, the ordinal sums of several other important fuzzy connectives also has been studied, such as, for example, the ordinal sums of copulas [32], overlap functions [15], uninorms [30], [31], fuzzy implications [19], [42] and fuzzy negations [4].

In 1976, Tardiff [43] introduced the dominance relation in the framework of probabilistic metric spaces as a binary relation on the class of all triangle functions and in 1983, Schweizer and Sklar [41] generalized to operations on a partially ordered set. Since then, several papers were developed using the notion of dominance relation in the construction of Cartesian products of probabilistic metric spaces, in the preservation of several properties and was also introduced in the framework of aggregation operators, see [1], [6], [7], [29], [37], [38], and dominance relation on the class of conjunctors, containing as particular cases the subclasses of quasi-copulas, copulas and t-norms [39].

In this work, we define the notion of dominance relation between two quasi-overlap functions and prove some results, among them, that automorphism preserves the dominance relation of quasi-overlap functions. Also, we consider the notions of quasi-overlap functions and ordinal sums of overlap functions to define an ordinal sum for quasi-overlap functions

978-1-7281-6932-3/20/$31.00 ©2020 IEEE

and we prove that there exists a dominance relation between ordinal sums of quasi-overlap functions if and only if there are a dominance relation of their respective summands.

This paper is organized as follows: Section 2 provides a review of concepts as aggegation functions, t-noms, automorphisms, (quasi-)overlap functions. Section 3 contains the defintion the dominance relation of quasi-overlap functions and we prove that automorphism preserves the dominance relation of quasi-overlap functions. In section 4, we use the definitions of quasi-overlap functions and ordinal sums of overlap functions to define an ordinal sum for quasi-overlap functions and prove results involving these concepts and dominance relation. In section 5, we have the final considerations and further works.

II. Preliminaries

Let $n \in \mathbb{N}$ such that $n \geq 2$. A function $A : [0,1]^n \rightarrow [0,1]$ is an n-ary aggregation operator if, for each $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$, A satisfies the following conditions:

A1. $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$;

A2. If $x_i \leq y_i$, for each $i = 1,\ldots,n$, then $A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n)$.

In particular, we can consider a specific class of binary aggregation function namely overlap functions, which are related in some sense with t-norms.

Definition 2.1: A function $T : [0,1]^2 \rightarrow [0,1]$ is a t-norm if, for all $x, y, z \in [0,1]$, the following axioms are satisfied:
1. Symmetric: $T(x,y) = T(y,x)$;
2. Associative: $T(x, T(y,z)) = T(T(x,y), z)$;
3. Monotonic: If $x \leq y$, then $T(x,z) \leq T(y,z)$;
4. One identity: $T(x,1) = x$.

A t-norm T is called positive if satifies the condiction: $T(x,y) = 0$ iff $x = 0$ or $y = 0$.

Example 2.1: Some examples of t-norms:
1. Gödel t-norm: $T_G(x,y) = \min(x,y)$;
2. Product t-norm: $T_P(x,y) = x.y$;
3. Łukasiewicz t-norm: $T_L(x,y) = \max(0, x+y-1)$;
4. Drastic t-norm:]

$$T_D(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,1[^2; \\ \min(x,y) & \text{otherwise.} \end{cases}$$

■

Proposition 2.1: [24] Let $(T_i)_{i \in I}$ be a family of t-norms and $(]a_i, b_i[)_{i \in I}$ be a family of nonempty, pairwise disjoint open subintervals of $[0,1]$. Then the function $T_I : [0,1]^2 \rightarrow [0,1]$ defined by

$$T_I(x,y) = \begin{cases} a_i + (b_i - a_i)\ T_i\left(\dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i}\right) \\ \qquad\qquad\qquad \text{if } (x,y) \in [a_i, b_i]^2; \\ \min(x,y), \text{ otherwise} \end{cases} \quad (1)$$

is a t-norm which is called the ordinal sum of the summands $(a_i, b_i, T_i)_{i \in I}$.

Definition 2.2: A function $\rho : [0,1] \rightarrow [0,1]$ is an automorphism if it is bijective and increasing.

Automorphisms are closed under composition if $\rho, \rho' \in Aut([0,1])$, then $\rho \circ \rho' \in Aut([0,1])$, where $\rho \circ \rho'(x) = \rho(\rho'(x))$. In addition, the inverse ρ^{-1} of an order automorphism ρ is also an order automorphism.

A. Overlap functions

Definition 2.3: A bivariante function $O : [0,1]^2 \rightarrow [0,1]$ is said to be a overlap function if it satisfies the following condictions:

O1. O is commutative;
O2. $O(x,y) = 0$ iff $xy = 0$;
O3. $O(x,y) = 1$ iff $xy = 1$;
O4. O is increasing;
O5. O is continuous.

Example 2.2: [15], [17] For each positive real number $p > 0$, the function $O_p(x,y) = x^p y^p$, $O_{m^p}(x,y) = \min(x^p, y^p)$ and $O_{mM^p}(x,y) = \min(x,y) \cdot \max(x^p, y^p)$ are examples of overlap functions.

Proposition 2.2: [3, Proposition 5.3] Let $\varphi \in Aut[0,1]$. O is an overlap function iff O^φ is also an overlap function such that, for all $x, y \in [0,1]$, the following holds

$$O^\varphi(x,y) = \varphi^{-1}(O(\varphi(x), \varphi(y))). \quad (2)$$

B. Quasi-overlap functions

The notion of overlap functions was introduced by Bustince et. al. [9] as a non-associative and continuous function, to solve the problem of fuzziness on the process of image classification. The requirement of continuity is justified, to avoid that the overlap function to be a uninorm. However, it is easy to see that if a uninorm is an overlap function, then it is necessarily a t-norm. In addition, in some contexts, continuity is not an indispensable property, as we can see in Paiva et. al. [34], for finite lattices.

In fact, there are several applications where continuity of aggregation functions is not required, for example in several methods of decision-making based on aggregation functions, Fuzzy Rule-Based Classification Systems (FRBCS), and digital image processing. In decision-making problems, we can find several methods using aggregation functions, see [10] and [36], which not need to be continuous. Aggregation functions were also used to reduce images, see [20], [35], interval t-norms and t-conorms were used in an edge detection method [14]. In both cases, the continuity of the aggregation functions is not required.

In [28], Choquet integrals were extended through a pair of bivariate (not necessarily continuous) functions satisfying a domination condition. These extended Choquet integrals were used in a FRBCS with success, in that some of these functions achieved similar performance to the FURIA algorithm [22], which is considered the best FRBCS in present days. Note that, in [27], a non-continuous function F_{NA1} was used to generate a CF-integral (a type of extension of the Choquet integral) and also used in a FRBCS with a good performance, although a little less efficient than the FURIA algorithm.

Therefore, in some situations, we have needed to weaken the notion of overlap functions, dropped the exigence of continuity. The notion of quasi-overlap functions on a bounded lattice was defined in [33], as a natural generalization of overlap functions and interval-valued overlaps functions (as in [2]), but it considering the continuity condition. So, we define quasi-overlap functions on [0,1] by:

Definition 2.4: Let $O : [0,1]^2 \to [0,1]$ be a bivariante function. If O satisfies properties O1 – O4, it is called a quasi-overlap function.

Example 2.3: For each $\alpha \in (0,1)$, the following functions are quasi-overlaps, but they are not overlap functions:

$$O_\alpha(x,y) = \begin{cases} \alpha & \text{, if } xy \in (0,1) \\ xy & \text{, otherwise} \end{cases}$$

$$O^{(\alpha)}(x,y) = \begin{cases} \alpha + (1-\alpha)O(x,y) & \text{, if } xy \in (0,1) \\ xy & \text{, otherwise} \end{cases}$$

where O is an overlap function.

Remark 2.1: Note that Proposition 2.2 can be easy generalized for quasi-overlap functions.

III. Dominance relation of quasi-overlap functions

In [39, Definition 6], the dominance relations between two conjunctors were defined. However, note that not every quasi-overlap function is a conjunctor. On the other hand, in [28], Choquet integrals were extended through a pair of bivariate functions satisfying a domination condition. Since these pair of function do not need to be continuous, then we can consider quasi-overlap functions, and therefore, it results in an important investigated the dominance relations of quasi-overlap functions.

Definition 3.1: Let O_1 and O_2 be two quasi-overlap functions. We say that O_1 dominates O_2, denoted by $O_1 \gg O_2$, if for all $x, y, u, v \in [0,1]$, it hold that

$$O_1(O_2(x,y), O_2(u,v)) \geq O_2(O_1(x,u), O_1(y,v)). \quad (3)$$

Observe that the dominance over a quasi-ovelap function is not an antisymmetric relation. In fact, for each positive real numbers m and n, we have that $O_m(O_n(x,y), O_n(u,v)) = O_n(O_m(x,u), O_m(y,v))$ for each $x, y, u, v \in [0,1]$ and therefore $O_m \gg O_n$ and $O_n \gg O_m$.

Example 3.1: Let $\alpha, \beta \in (0,1)$. Then $O_\alpha \gg O_\beta$ iff $\alpha \geq \beta$. In fact, if $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = 0$ then

$$\begin{aligned} O_\beta(x,y) \cdot O_\beta(u,v) = 0 &\Rightarrow O_\beta(x,y) = 0 \text{ or } O_\beta(u,v) = 0 \\ &\Rightarrow xyuv = 0 \\ &\Rightarrow O_\alpha(x,u) = 0 \text{ or } O_\alpha(y,v) = 0 \\ &\Rightarrow O_\beta(O_\alpha(x,y), O_\alpha(u,v)) = 0 \end{aligned}$$

and therefore, if $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = 0$ then $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = O_\beta(O_\alpha(x,y), O_\alpha(u,v))$.

If $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = 1$ then obviously $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = O_\beta(O_\alpha(x,y), O_\alpha(u,v))$.

Finally, if $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = \alpha$ then $O_\beta(x,y)O_\beta(u,v) \in (0,1)$ and hence, $xy \in (0,1)$ and $uv \in (0,1)$ or equivalently, $x, y, u, v \in (0,1)$. Therefore, $O_\alpha(O_\beta(x,y), O_\beta(u,v)) = \alpha \geq \beta = O_\beta(O_\alpha(x,y), O_\alpha(u,v))$.

Proposition 3.1: Let O_1, O_2, O_3 and O be quasi-overlap functions. If $O_i \gg O$, for any $i = 1, 2, 3$, then the binary operation $O^* : [0,1]^2 \to [0,1]$ defined by

$$O^*(x,y) = O_3(O_1(x,y), O_2(x,y)) \quad (4)$$

is also a quasi-overlap function such that $O^* \gg O$, for all $x, y \in [0,1]$.

Proof: First, we will prove that O^* is a quasi-overlap function. Let O_1, O_2, O_3 and O be quasi-overlap functions and $x, y \in [0,1]$.

O1.

$$\begin{aligned} O^*(x,y) &= O_3(O_1(x,y), O_2(x,y)) \\ &= O_3(O_1(y,x), O_2(y,x)) \\ &= O^*(y,x). \end{aligned}$$

O2.

$$\begin{aligned} O^*(x,y) = 0 &\Leftrightarrow O_3(O_1(x,y), O_2(x,y)) = 0 \\ &\Leftrightarrow O_1(x,y) \cdot O_2(x,y) = 0 \\ &\Leftrightarrow O_1(x,y) = 0 \text{ or } O_2(x,y) = 0 \\ &\Leftrightarrow xy = 0. \end{aligned}$$

O3.

$$\begin{aligned} O^*(x,y) = 1 &\Leftrightarrow O_3(O_1(x,y), O_2(x,y)) = 1 \\ &\Leftrightarrow O_1(x,y) \cdot O_2(x,y) = 1 \\ &\Leftrightarrow O_1(x,y) = 1 \text{ and } O_2(x,y) = 1 \\ &\Leftrightarrow xy = 1. \end{aligned}$$

O4. Since O_1, O_2, O_3 are increasing functions, then O^* is also an increasing function.

Now, we will prove that $O^* \gg O$. So,

$$\begin{aligned} & O^*(O(x,y), O(u,v)) \\ = \;& O_3(O_1(O(x,y), O(u,v)), O_2(O(x,y), O(u,v))) \\ \geq \;& O_3(O(O_1(x,u), O_1(y,v)), O(O_2(x,u), O_2(y,v))) \\ \geq \;& O(O_3(O_1(x,u), O_2(x,u)), O_3(O_1(y,v), O_2(y,v))) \\ = \;& O(O^*(x,u), O^*(y,v)). \end{aligned}$$

■

Corollary 3.1: O^* is an overlap function if O_1, O_2 and O_3 are overlap functions.
PROOF: Straightforward. ■

In the following proposition, we prove that the automorphism preserves the dominance relation of quasi-overlap functions.

Proposition 3.2: Let O_1 and O_2 be two quasi-overlap functions and $\varphi \in Aut([0,1])$. If $O_1 \gg O_2$, then $O_1^\varphi \gg O_2^\varphi$.
PROOF: Suppose $O_1 \gg O_2$ and $x, y, u, v \in [0,1]$, then

$$\begin{aligned}
& O_1^\varphi(O_2^\varphi(x,y), O_2^\varphi(u,v)) \\
= \;& O_1^\varphi(\varphi^{-1}(O_2(\varphi(x),\varphi(y))), \varphi^{-1}(O_2(\varphi(u),\varphi(v)))) \\
= \;& \varphi^{-1}(O_1(\varphi(\varphi^{-1}(O_2(\varphi(x),\varphi(y)))), \\
& \varphi(\varphi^{-1}(O_2(\varphi(u),\varphi(v)))))) \\
= \;& \varphi^{-1}(O_1(O_2(\varphi(x),\varphi(y)), O_2(\varphi(u),\varphi(v)))) \\
\geq \;& \varphi^{-1}(O_2(O_1(\varphi(x),\varphi(u)), O_1(\varphi(y),\varphi(v)))) \\
= \;& \varphi^{-1}(O_2(\varphi(\varphi^{-1}(O_1(\varphi(x),\varphi(u)))), \\
& \varphi(\varphi^{-1}(O_1(\varphi(y),\varphi(v)))))) \\
= \;& O_2^\varphi(\varphi^{-1}(O_1(\varphi(x),\varphi(u))), \varphi^{-1}(O_1(\varphi(y),\varphi(v)))) \\
= \;& O_2^\varphi(O_1^\varphi(x,u), O_1^\varphi(y,v)).
\end{aligned}$$

Therefore, $O_1^\varphi \gg O_2^\varphi$. ■

IV. DOMINANCE RELATION BETWEEN ORDINAL SUM OF QUASI-OVERLAP FUNCTIONS

In [15], the ordinal sum of overlap functions was defined, as given below.

Definition 4.1: [15, Definition 5.1] Let I be a countable set of indexes, $(O_i)_{i\in I}$ be a family of overlap funtions and $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The ordinal sum of $(O_i)_{i\in I}$ is a bivariante function $(\langle a_i, b_i, O_i\rangle) : [0,1]^2 \to [0,1]$, defined by

$$(\langle a_i, b_i, O_i\rangle)(x,y) = \begin{cases} a_i + (b_i - a_i)\cdot O_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) \\ \qquad \text{if } (x,y) \in [a_i, b_i], \\ \min\{f_A(x), f_A(y)\} \quad \text{otherwise.} \end{cases}$$

where $f_A : [0,1] \to [0,1]$ is given by

$$f_A(x) = \begin{cases} a_i + (b_i - a_i)\cdot O_i\left(\dfrac{x-a_i}{b_i-a_i}, 1\right) \\ \qquad \text{if } \exists\ i \in I\text{: } x \in [a_i, b_i], \\ x \quad \text{otherwise.} \end{cases}$$

Theorem 4.1: [15, Theorem 5.1] For a function $O : [0,1]^2 \to [0,1]$ the following statements are equivalent:
$i)$ O is an overlap function;
$ii)$ O is representable as an ordinal sum of overlap functions $(O_i)_{i\in I}$.

It is clear that if we consider quasi-overlap functions instead of overlap functions in Definition 4.1, the ordinal sum of these quasi-overlap functions is also a quasi-overlap function. However, this definition has unnecessary complexity, motivating the introduction of the following ordinal sums for quasi-overlap functions.

Definition 4.2: Let I be a countable set of indexes, $(O_i)_{i\in I}$ be a family of quasi-overlap functions and $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The ordinal sum of $(O_i)_{i\in I}$ is a bivariante function $\mathcal{O} = (\langle a_i, b_i, O_i\rangle) : [0,1]^2 \to [0,1]$, defined by

$$\mathcal{O}(x,y) = \begin{cases} a_i + (b_i - a_i)\cdot O_i\left(\dfrac{x-a_i}{b_i-a_i}, \dfrac{y-a_i}{b_i-a_i}\right) \text{ if } (x,y) \in [a_i, b_i] \\ \min\{x,y\} \quad \text{otherwise.} \end{cases}$$

Now, we will prove that if the summands $(O_i)_{i\in I}$ is a family of quasi-overlap funtions, then the ordinal sum defined by Eq. (5) is also a quasi-overlap function.

Proposition 4.1: If $(O_i)_{i\in I}$ is a family of quasi-overlap funtions and $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, then $\mathcal{O} = (\langle a_i, b_i, O_i\rangle)$ is also a quasi-overlap function.
PROOF: Let $(O_i)_{i\in I}$ be a family of quasi-overlap funtions and $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then

O1. Straightforward by commutativity of $\mathcal{O}$ and minimum function.

O2. Suppose $xy = 0$. Without loss of generality, suppose that $x = 0$. Thus if $y \in [a_i, b_i]$ and $a_i = 0$, then $\mathcal{O}(x,y) = b_i \cdot O_i\left(\dfrac{x}{b_i}, \dfrac{y}{b_i}\right) = 0$. Otherwise, $\mathcal{O}(x,y) = \min\{x,y\} = 0$.

Conversely, suppose $\mathcal{O}(x,y) = 0$.

If $x, y \in [a_i, b_i]$, then

$$a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) = 0$$

and so, $a_i = 0$ and $b_i \cdot O_i\left(\dfrac{x}{b_i}, \dfrac{y}{b_i}\right) = 0$. Thus, $x = 0$ or $y = 0$. Otherwise, $0 = \mathcal{O}(x,y) = \min\{x,y\}$ and then, $x = 0$ or $y = 0$.

O3. Suppose $xy = 1$, we have that $x = y = 1$. So, if $x, y \in [a_i, b_i]$, then $b_i = 1$ and

$$\mathcal{O}(x,y) = a_i + (1 - a_i)\cdot O_i(1,1) = a_i + (1 - a_i) = 1.$$

Otherwise, $\mathcal{O}(x,y) = \min\{x,y\} = 1$, because $x = y = 1$.

Conversely, suppose $\mathcal{O}(x,y) = 1$.

If $x, y \in [a_i, b_i]$, then

$$a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) = 1.$$

So, $b_i = 1$ and $O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) = 1$. Therefore, $\frac{x-a_i}{1-a_i} = 1$ and $\frac{y-a_i}{1-a_i} = 1$. Hence, $x = 1$ and $y = 1$. Otherwise, $1 = \mathcal{O}(x,y) = \min\{x,y\}$ and then, $x = 1$ and $y = 1$.

O4. Suppose $x, y, z \in [0,1]$ such that $y \leq z$.
Case 1: If $x, y, z \in [a_i, b_i]$, for some $i \in I$. So,

$$O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) \leq O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{z-a_i}{b_i-a_i}\right)$$

and thus,

$$\begin{aligned}\mathcal{O}(x,y) &= a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)\\ &\leq a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{z-a_i}{b_i-a_i}\right)\\ &= \mathcal{O}(x,z).\end{aligned}$$

Case 2: If $x, y \in [a_i, b_i]$, for some $i \in I$, and $z \notin [a_i, b_i]$. Since $y \leq z$ we have that $x \leq b_i \leq z$. So,

$$O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) \leq \frac{x-a_i}{b_i-a_i}$$

and thus,

$$\begin{aligned}\mathcal{O}(x,y) &= a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right)\\ &\leq a_i + (b_i - a_i)\cdot \frac{x-a_i}{b_i-a_i}\\ &= x\\ &= \min\{x,z\}\\ &= \mathcal{O}(x,z).\end{aligned}$$

Case 3: If $x, z \in [a_i, b_i]$, for some $i \in I$, and $y \notin [a_i, b_i]$. Since $y \leq z$ we have that $y \leq a_i \leq z$. So,

$$\begin{aligned}\mathcal{O}(x,y) &= \min\{x,y\}\\ &= y\\ &\leq a_i\\ &\leq a_i + (b_i - a_i)\cdot O_i\left(\frac{x-a_i}{b_i-a_i}, \frac{z-a_i}{b_i-a_i}\right)\\ &= \mathcal{O}(x,z).\end{aligned}$$

Case 4: If $x \notin [a_i, b_i]$ or $y, z \notin [a_i, b_i]$, then $\mathcal{O}(x,y) = \min\{x,y\} \leq \min\{x,z\} = \mathcal{O}(x,z)$.
Therefore, $\mathcal{O}$ is a quasi-overlap function. ■

Proposition 4.2: Let $(O_i)_{i\in I}$ be a family of quasi-overlap funtions and $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. $\mathcal{O} = (\langle a_i, b_i, O_i\rangle)$ has 1 as neutral element iff either for each $i \in I$, $b_i \neq 1$ or there exists $i \in I$, such that $b_i = 1$ and O_i has 1 as neutral element.

PROOF: ($\Rightarrow$) Suppose that $\mathcal{O}$ has 1 as neutral element. If there is $i \in I$ such that $b_i = 1$. Then for each $x \in [0,1]$, consider $x' = x(1-a_i) + a_i \in [a_i, 1]$. Since $x' = \mathcal{O}(x', 1)$ then

$$\begin{aligned}x &= \frac{\mathcal{O}(x',1) - a_i}{1-a_i}\\ &= \frac{a_i + (1-a_i)O_i\left(\frac{x'-a_i}{1-a_i}, \frac{1-a_i}{1-a_i}\right) - a_i}{1-a_i}\\ &= \frac{(1-a_i)O_i(x,1)}{1-a_i}\\ &= O_i(x,1)\end{aligned}$$

Otherwise, there is no $i \in I$ such that $b_i = 1$ and therefore if $\mathcal{O}$ has 1 as neutral element then either for each $i \in I$, $b_i \neq 1$ or there exists $i \in I$, such that $b_i = 1$ and O_i has 1 as neutral element.

($\Leftarrow$) Suppose that there exists $i \in I$, such that $b_i = 1$ and O_i has neutral element and let $x \in [0,1]$. If $x \in [a_i, 1]$ then

$$\begin{aligned}\mathcal{O}(x,1) &= a_i + (1-a_i)\cdot O_i\left(\frac{x-a_i}{1-a_i}, \frac{1-a_i}{1-a_i}\right)\\ &= a_i + (1-a_i)\cdot O_i\left(\frac{x-a_i}{1-a_i}, 1\right)\\ &= a_i + (1-a_i)\cdot \frac{x-a_i}{1-a_i}\\ &= x.\end{aligned}$$

if $x \notin [a_i, 1]$ then, since $1 \in [a_j, b_j]$ for some $j \in I$ iff $j = i$, we have that $\mathcal{O}(x,1) = \min\{x,1\} = x$. ■

In the following proposition, we will prove the relationship among dominance relation of ordinal sums of quasi-overlap functions and their summands.

Proposition 4.3: Let $(O_i)_{i\in I}$ be a family of quasi-overlap funtions, $(]a_i, b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, $\mathcal{O}_1 = (\langle a_i, b_i, O_{1,i}\rangle)$ and $\mathcal{O}_2 = (\langle a_i, b_i, O_{2,i}\rangle)$. If for each $i \in I$ the quasi-overlap functions $O_{1,i}$ and $O_{2,i}$ has 1 as neutral element then $\mathcal{O}_1 \gg \mathcal{O}_2$ iff $O_{1,i} \gg O_{2,i}$, for all $i \in I$.
PROOF: ($\Rightarrow$) Suppose that $\mathcal{O}_1 \gg \mathcal{O}_2$, then we want to prove that $O_{1,i} \gg O_{2,i}$, for all $i \in I$. Consider the function $\phi_i : [a_i, b_i] \to [0,1]$ defined by $\phi_i(x) = \frac{x-a_i}{b_i-a_i}$. Since ϕ_i is an increasing bijection, then there exist uinque $x', y', u', v' \in [a_i, b_i]$ such that $\phi_i(x') = x$, $\phi_i(y') = y$, $\phi_i(u') = u$ and $\phi_i(v') = v$. Since $\mathcal{O}_1 \gg \mathcal{O}_2$ then, for all $x, y, u, v \in [0,1]$, we have that

$$\mathcal{O}_1(\mathcal{O}_2(x,y), \mathcal{O}_2(u,v)) \geq \mathcal{O}_2(\mathcal{O}_1(x,u), \mathcal{O}_1(y,v)). \quad (5)$$

In particular, it can be equivalently expressed by

$$\begin{aligned}&\mathcal{O}_1(\phi_i^{-1}(\mathcal{O}_2(\phi_i(x'), \phi_i(y'))), \phi_i^{-1}(\mathcal{O}_2(\phi_i(u'), \phi_i(v'))))\\ \geq\ &\mathcal{O}_2(\phi_i^{-1}(\mathcal{O}_1(\phi_i(x'), \phi_i(u'))), \phi_i^{-1}(\mathcal{O}_1(\phi_i(y'), \phi_i(v'))))\end{aligned}$$

Note that, since

$$\begin{array}{rcl}\phi_i^{-1}(O_{2,i}(\phi_i(x'),\phi_i(y'))) & \in & [a_i,b_i],\\ \phi_i^{-1}(O_{2,i}(\phi_i(u'),\phi_i(v'))) & \in & [a_i,b_i],\\ \phi_i^{-1}(O_{1,i}(\phi_i(x'),\phi_i(u'))) & \in & [a_i,b_i],\\ \phi_i^{-1}(O_{1,i}(\phi_i(y'),\phi_i(v'))) & \in & [a_i,b_i],\end{array}$$

for all $x',y',u',v'\in[a_i,b_i]$, then

$$\begin{array}{rl} & \phi_i^{-1}(O_{1,i}(O_{2,i}(\phi_i(x'),\phi_i(y')),O_{2,i}(\phi_i(u'),\phi_i(v')))\\ \geq & \phi_i^{-1}(O_{2,i}(O_{1,i}(\phi_i(x'),\phi_i(u')),O_{1,i}(\phi_i(y'),\phi_i(v'))),\end{array}$$

that it is in turn equivalent to

$$\begin{array}{rl} & \phi_i^{-1}(O_{1,i}(O_{2,i}(x,y),O_{2,i}(u,v))\geq\\ & \phi_i^{-1}(O_{2,i}(O_{1,i}(x,u),O_{1,i}(y,v))\\ \Rightarrow & \phi(\phi_i^{-1}(O_{1,i}(O_{2,i}(x,y),O_{2,i}(u,v)))\geq\\ & \phi(\phi_i^{-1}(O_{2,i}(O_{1,i}(x,u),O_{1,i}(y,v)))\\ \Rightarrow & O_{1,i}(O_{2,i}(x,y),O_{2,i}(u,v))\geq O_{2,i}(O_{1,i}(x,u),O_{1,i}(y,v)).\end{array}$$

So, $O_{1,i}\gg O_{2,i}$, for all $i\in I$.

($\Leftarrow$) Since $O_{1,i}\gg O_{2,i}$, for all $i\in I$, then the Eq. (5) is fulfilled for all $x,y,u,v\in[0,1]$ by the isomorphism property.

Now, consider $z,w\in[0,1]$ such that $\min\{z,w\}\in[a_i,b_i]$, for some $i\in I$, it hold that

$$\mathcal{O}_1(z,w)=\mathcal{O}_1(\min\{z,b_i\},\min\{w,b_i\})\quad\text{and}$$
$$\mathcal{O}_2(z,w)=\mathcal{O}_2(\min\{z,b_i\},\min\{w,b_i\}).$$

Consider $x,y,u,v\in[0,1]$ such that $\min\{x,y,u,v\}=x$. Then, we have the following cases:

Case 1: Suppose $x\in[a_i,b_i]$ for some $i\in I$. Since $x\in[a_i,b_i]$, $\min\{y,b_i\}\in[a_i,b_i]$, $\min\{u,b_i\}\in[a_i,b_i]$, $\min\{v,b_i\}\in[a_i,b_i]$ and by hyphotesis $O_{1,i}\gg O_{2,i}$, for all $i\in I$, then we have that

$$\begin{array}{rl} & \mathcal{O}_1(\mathcal{O}_2(x,y),\mathcal{O}_2(u,v))\\ = & \mathcal{O}_1(\mathcal{O}_2(x,\min\{y,b_i\}),\mathcal{O}_2(\min\{u,b_i\},\min\{v,b_i\})).\end{array}$$

Note that

$$\begin{array}{rl} & \mathcal{O}_2(x,\min\{y,b_i\})\\ = & a_i+(b_i-a_i)\cdot O_{2,i}\left(\dfrac{x-a_i}{b_i-a_i},\dfrac{\min\{y,b_i\}-a_i}{b_i-a_i}\right)\in[a_i,b_i]\end{array}$$

and, analogously $\mathcal{O}_2(\min\{u,b_i\},\min\{v,b_i\})\in[a_i,b_i]$. So,

$$\begin{array}{rl} & \mathcal{O}_1(\mathcal{O}_2(x,\min\{y,b_i\}),\mathcal{O}_2(\min\{u,b_i\},\min\{v,b_i\}))\\ = & a_i+(b_i-a_i)O_{1,i}\left(O_{2,i}\left(\dfrac{x-a_i}{b_i-a_i},\dfrac{\min\{y,b_i\}-a_i}{b_i-a_i}\right),\right.\\ & \left.O_{2,i}\left(\dfrac{\min\{u,b_i\}-a_i}{b_i-a_i},\dfrac{\min\{v,b_i\}-a_i}{b_i-a_i}\right)\right).\end{array}\tag{6}$$

Since $O_{1,i}\gg O_{2,i}$, for all $i\in I$, then

$$\begin{array}{rl} & \mathcal{O}_1(\mathcal{O}_2(x,\min\{y,b_i\}),\mathcal{O}_2(\min\{u,b_i\},\min\{v,b_i\}))\\ \geq & \mathcal{O}_2(\mathcal{O}_1(x,\min\{u,b_i\}),\mathcal{O}_1(\min\{y,b_i\},\min\{v,b_i\}))\\ = & \mathcal{O}_2(\mathcal{O}_1(x,u),\mathcal{O}_1(y,v)).\end{array}$$

Conversely, if $\min\{y,v\}\notin[a_i,b_i]$, then $\mathcal{O}_1(y,v)\geq b_i$. Since $\mathcal{O}_1(x,\min\{u,b_i\})\leq b_i$ and analogous to Eq (6), we have that

$$\begin{array}{rl} & \mathcal{O}_1(\mathcal{O}_2(x,y),\mathcal{O}_2(u,v))\\ \geq & \mathcal{O}_1(\mathcal{O}_2(x,y),\mathcal{O}_2(\min\{u,b_i\},v))\\ = & \mathcal{O}_1(\min\{x,y\}),\min\{\min\{u,b_i\},v\})\\ = & \min\{\mathcal{O}_1(x,\min\{u,b_i\}),\mathcal{O}_1(x,v),\mathcal{O}_1(y,\min\{u,b_i\}),\\ & \mathcal{O}_1(y,v)\}\\ = & \mathcal{O}_1(x,\min\{u,b_i\})\\ = & \min\{\mathcal{O}_1(x,\min\{u,b_i\}),\mathcal{O}_1(y,v)\}\\ = & \mathcal{O}_2(\mathcal{O}_1(x,\min\{u,b_i\}),\mathcal{O}_1(y,v))\\ = & \mathcal{O}_2(\mathcal{O}_1(x,u),\mathcal{O}_1(y,v)).\end{array}$$

Case 2: Suppose $x\notin[a_i,b_i]$ for all $i\in I$, then $\mathcal{O}_1(x,*)=\mathcal{O}_2(x,*)$. Note that $\mathcal{O}_1(y,v)\geq x$ and $\mathcal{O}_2(u,v)\geq x$. Thus,

$$\begin{array}{rcl}\mathcal{O}_1(\mathcal{O}_2(x,y),\mathcal{O}_2(u,v)) & = & \mathcal{O}_1(x,\mathcal{O}_2(u,v))\\ & = & \min\{x,\mathcal{O}_2(u,v)\}\\ & = & x\\ & = & \min\{x,\mathcal{O}_1(y,v)\}\\ & = & \mathcal{O}_2(x,\mathcal{O}_1(y,v))\\ & = & \mathcal{O}_2(\mathcal{O}_1(x,u),\mathcal{O}_1(y,v)).\end{array}$$

Therefore, $\mathcal{O}_1\gg\mathcal{O}_2$. ■

Corollary 4.1: Let ψ an automorphism, $(O_i)_{i\in I}$ be a family of quasi-overlap functions, $(]a_i,b_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, $\mathcal{O}_1=(\langle a_i,b_i,O_{1,i}\rangle)$ and $\mathcal{O}_2=(\langle a_i,b_i,O_{2,i}\rangle)$. If $O_{1,i}\gg O_{2,i}$, then $\mathcal{O}_1^{\varphi}\gg\mathcal{O}_2^{\varphi}$.

PROOF: Straightforward from Propositions 4.3 and 3.2. ■

V. CONCLUSION

In this paper, we consider the notions of quasi-overlap functions, dominance relation of conjunctors and ordinal sum of overlap functions to define dominace relation of overlap functions and ordinal sum of quasi-overlap functions and prove some results about them.

In particular, we define a dominance relation of quasi-overlap functions and we prove that automorphism preserves the dominance relation of quasi-overlap functions. Moreover, we use the definitions of quasi-overlap functions and ordinal sums of quasi-overlap functions to define an ordinal sum for quasi-overlap functions and prove results involving these concepts and dominance relation.

As future work, we will investigate how and when the dominance between (quasi-)overlap functions determines the dominance between implications generated from such (quasi-) overlap functions, such as in [16], [18].

REFERENCES

[1] B. De Baets and R. Mesiar, *T-partitions*, Fuzzy Systems and Systems 97 (1998) 211 - 223.

[2] B.C. Bedregal, H. Bustince, E.S. Palmeira, G.P. Dimuro and J. Fernández, *Generalized interval-valued OWA operators with interval weights derived from interval-valued overlap functions*, Int. J. Approx. Reason. 90 (2017) 1–16.

[3] B.C. Bedregal, G.P. Dimuro, H. Bustince and E. Barrenechea, *New results on overlap and grouping functions*, Information Sciences 249 (2013) 148-170.

[4] B. Bedregal and I. Mezzomo, *Ordinal sums and multiplicative generators of the De Morgan triples*, Journal of Intelligent and Fuzzy Systems 34 (2018) 2159–2170.

[5] G. Beliakov, A. Pradera, T. Calvo, Aggregation functions: A Guide for Practitioners, Springer, Berlin, 2007.

[6] U. Bodenhofer, *A Similarity-Based Generalization of Fuzzy Orderings*, (Schriftenreihe der Johannes-Kepler-Universität Linz C 44) Universitätsverlag Rudolf Trauner, Linz 1999.

[7] U. Bodenhofer, *Representations and constructions of similarity-based fuzzy orderings*, Fuzzy Sets and Systems 137 (1) (2003) 113 - 136.

[8] H. Bustince, P. Burillo and F. Soria, *Automorphisms, negations and implication operators*, Fuzzy Sets and Systems 134 (2003) 209 - 229.

[9] H. Bustince, J. Fernandez, R. Mesiar, J. Monteiro and R. Orduna, *Overlap Functions*, Nonlinear Analysis: Theory, Methods & Applications 72 (3-4) (2010) 1488-1499.

[10] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier and F. Herrera, *Grouping, overlaps, and generalized bientropic functions for fuzzy modeling of pairwise comparisons*, IEEE Trans. Fuzzy Syst., 20 (3) (2012) 405 - 415.

[11] A.H. Clifford, *Naturally totally ordered commutative semigroups*. Amer. J. Math. 76 (1954) 631 - 646.

[12] A.H. Clifford and G.B. Preston *The algebraic theory of semigroups*. American Mathematical Society, Providence (1961).

[13] A.C. Climescu, *Sur l'equation fonctionelle de lássociativité*. Bull. École Polytechn Iassy 1 (1946) 1 - 16.

[14] L. L. Costa, B. Bedregal, H. Bustince, E. Barrenechea and M. P. da Rocha, *An interval extension of homogeneous and pseudo-homogeneous t-norms and t-conorms*, Information Sciences 355-356 (2016) 328-347.

[15] G.P. Dimuro, B. Bedregal, *Archimedean overlap functions: The ordinal sum and the cancellation, idempotency and limiting properties*, Fuzzy Sets and Systems 252 (2014) 39 - 54.

[16] G.P. Dimuro, B. Bedregal, *On residual implications derived from overlap functions*, Information Sciences 312 (2015) 78-88.

[17] G.P. Dimuro, B. Bedregal, H. Bustince, M.J. Asiain, R. Mesiar, *On additive generators of overlap functions*, Fuzzy Sets and Systems 287 (2016) 76-96.

[18] G.P. Dimuro, B. Bedregal, H. Bustince, A. Jurio, M. Baczyński and K. Mis, *On Fuzzy Implications Derived from Overlap and Grouping Functions: The case of the QL-operations*, International Journal of Approximate Reasoning 82 (2017) 170-192.

[19] P. Drygaś, A. Król, *Generating fuzzy implications by ordinal sums*, Tatra Mountains Mathematical Publications 66 (1) (2016) 39–50.

[20] A. D. S. Farias, V. S. Costa, R. H. N. Santiago and B. Bedregal, *The image reduction process based on Generalized Mixture functions*, Proceeding of NAFIPS 2016, 1 - 6.

[21] M.L. Frank, *On the simultaneous associativity of F(x, y) and x+y-F(x, y)*. Aequationes Math. 19 (1979) 194 - 226.

[22] J. Hühn and E. Hüllermeier, *Furia: An algorithm for unordered fuzzy rule induction*, Data Mining Knowl. Discovery 19 (3) (2009) 293 - 319.

[23] A. Jurio, H. Bustince, M. Pagola and A. Pradera, *Some properties of overlap and grouping functions and their application to image thresholding*, Fuzzy Sets and Sysems, 229 (2013) 69 - 90.

[24] E.P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[25] A. A. de Lima, B. Bedregal and I. Mezzomo, *Ordinal sums of the main classes of fuzzy negations and the natural negations of t-norms, t-conorms and fuzzy implications*, International Journal of Approximate Reasoning 116 (2020) 19 - 32.

[26] C.M. Ling, *Representation of associative functions*. Publ. Math. Debrecen 12 (1965) 189 - 212.

[27] G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, H. Bustince, and R. Mesiar, *CF-integrals: A new family of pre-aggregation functions with application to fuzzy rule-based classification systems*, Information Sciences 435 (2018) 94-110.

[28] G. Lucca, G. P. Dimuro, J. Fernández, H. Bustince, B. Bedregal and J. A. Sanz, *Improving the Performance of Fuzzy Rule-Based Classification Systems Based on a Nonaveraging Generalization of CC-Integrals Named C_{F1F2}-Integrals*, IEEE Transaction on Fuzzy Systems 27 (1) (2019) 124-134.

[29] S. Mesiar and S. Saminger, *Domination of ordered weighted averaging operators over t-norms*, Soft Computing 8 (2004) 562 - 570.

[30] A. Mesiarová-Zemánková, *Ordinal sum construction for uninorms and generalized uninorms*, International Journal of Approximate Reasoning 76 (2016) 1 - 17.

[31] A. Mesiarová-Zemánková, *Characterization of uninorms with continuous underlying t-norm and t-conorm by means of the ordinal sum construction*, International Journal of Approximate Reasoning 83 (2017) 176 - 192.

[32] R.B. Nelsen, *An Introduction to Copulas* (2nd Ed.), Springer Science+Business Media, Inc., New York, 2006.

[33] R. Paiva, E. Palmeira, R. Santiago and B. Bedregal, *Lattice-valued Overlap and Quasi-Overlap Functions*, CoRR abs/1902.00133 (2019).

[34] R. Paiva, B. Bedregal and R. Santiago, *Residuated implications derived from quasi-overla functions on lattices*, Submitted.

[35] D. Paternain, J. Fernández, H. Bustince, R. Mesiar and G. Beliakov, *Construction of image reduction operators using averaging aggregation functions*, Fuzzy Sets and Systems 261 (2015) 87-111.

[36] I. A. da Silva, B. Bedregal, C. G. da Costa, E. S. Palmeira and M. P. da Rocha, *Pseudo-uninorms and Atanassov's intuitionistic pseudo-uninorms*, Journal of Intelligent and Fuzzy Systems 29 (1) (2015) 267-281.

[37] S. Saminger, R. Mesiar and U. Bodenhofer, *Domination of aggregation operators and preservation of transitivity*, Internat. Journal of Uncertain. Fuzziness Knowledge-Based Systems 10 (2002) 11 - 35.

[38] S. Saminger, *Aggregation in Evaluation of Computer-Assisted Assessment*, (Schriftenreihe der Johannes-Kepler-Universität Linz C 44) Universitätsverlag Rudolf Trauner, Linz 2005.

[39] S. Saminger, B. De Baets and H. De Meyer, *On the Dominance Relation Between Ordinal Sums of Conjunctors*, Kybernetika 42 (3) (2006) 337 - 350.

[40] B. B. Schweizer and A. Sklar, *Associative functions and abstract semigroups*, Publicationes Mathematicae Debrecen 10 (1961) 69 - 81.

[41] B. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland New York 1983.

[42] Y. Su, A. Xie, H.W. Liu, *On ordinal sum implications*, Information Sciences 293 (2015) 251–262.

[43] R. M. Tardiff, *Topologies for probabilistic metric space*, Pacific J. Math. 65 (1976) 233 - 251.

[44] S. Zadroyzny and J. Kacprzyk, *Bipolar queries: an approach and its various interpretations*, in: Proceeding of IFSA - EUSFLAT 2009, Lisbon, Portugal (2009) 1288 - 1293.