Switched Control for Local Stabilization of Discrete-time Uncertain Takagi-Sugeno Fuzzy Systems with Relaxed Estimate of the Domain of Attraction

Gilberto R. dos Santos
UFMS – Federal University of Mato Grosso do Sul,
Campus of Pantanal,
Corumbá – MS, Brazil, 79.304–902
gilberto.rodrigues@ufms.br

Diogo R. de Oliveira
IFMS – Federal Institute of Education,
Science and Technology of Mato Grosso do Sul,
Campus Três Lagoas
Três Lagoas – MS, Brazil, 79.641–162
diogo.ramalho@ifms.edu.br

Marcelo C. M. Teixeira and Edvaldo Assunção and Rodrigo Cardim
UNESP – São Paulo State University,
School of Engineering,
Department of Electrical Engineering
Ilha Solteira – SP, Brazil, 15.385–000
{marcelo.minhoto,edvaldo.assuncao,rodrigo.cardim}@unesp.br

Adalberto Z. N. Lazarini
UNESP – São Paulo State University,
School of Engineering,
Department of Electrical Engineering
Ilha Solteira – SP, Brazil, 15.385–000
adalberto.lazarini@yahoo.com

***Abstract*—This paper addresses the local stabilization problem and the computation of invariant subsets of the domain of attraction for uncertain nonlinear discrete-time systems. The proposed procedures use Takagi-Sugeno (T-S) fuzzy models that have uncertain membership functions and known local linear models. Based on a non-quadratic Lyapunov function, the proposed method uses a switched control law and the design conditions are given in terms of a optimization problem subject to Linear Matrix Inequalities (LMIs) constraints. The procedure provides a new and effective way to enlarge the estimation of the domain of attraction (DOA). Finally, a numerical example illustrates the effectiveness of the proposed method and compares it with procedures found in the literature.**

***Index Terms*—Switched control; Discrete-time uncertain nonlinear system; Takagi-Sugeno (T-S) fuzzy model; Uncertain membership functions; Domain of attraction (DOA); Linear matrix inequality (LMI).**

I. Introduction

The description of nonlinear systems by Takagi-Sugeno (T-S) fuzzy models [1], [2] is an effective tool for modeling and designing controllers for nonlinear systems. The procedure provides the representation of the nonlinear systems by a convex combination of known linear local models weighted by known membership functions. Using the representation of the nonlinear systems by T-S fuzzy models, Lyapunov functions and constraints expressed on the form of Linear Matrix Inequalities (LMIs) [3], many control designs for nonlinear systems that use the membership functions in the implementation of the control law have been developed over the past decades [4]–[11].

The procedure presented in [12], [13] allows the application of the T-S fuzzy models for describing uncertain nonlinear systems, such that the nonlinearities and the bounded uncertainties of the system are represented by known linear local models and unknown membership functions [13]–[17]. In [13] is proposed a procedure that does not use the membership functions in the implementation of the control law that uses the state vector and auxiliary matrices to choose a state-feedback gain, at each instant of time, minimizing the time derivative of the Lyapunov function and the design conditions ensure the stability to the controlled system. To deal with the local stabilization problem of uncertain nonlinear systems subject to actuator saturation, the design conditions proposed in [13] were extended in [14]. The procedure presented in [14] proposes a smoothing switched control law, used to avoid the chattering in the control input. This smoothing switched control law was also used in [15] to deal with the local stability problem for uncertain nonlinear systems subject to persistent norm-bounded disturbance. A local $\mathcal{H}_\infty$ switched controller design for uncertain T-S fuzzy systems subject to actuator saturation was proposed in [16]. Finally, [17] proposed a switched control law to deal

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with the stabilization problem of discrete-time uncertain nonlinear systems described by T-S fuzzy models.

The T-S fuzzy model describes with accuracy the original dynamics of the nonlinear system in a bounded state space region, defined as operating region [18]. An alternative to guarantee that the system operates within the region of operation consists in employing the concept of positive invariant set, because in a positively invariant set all the trajectories starting from inside of it will remain in it [19], [20]. Thus, regions of allowable initial conditions ensure local stability and control system constraints, and can be used as estimation of the Domain of Attraction (DOA) [20]. In [10], [11], [16], [21], for example, the local stability problem for nonlinear systems is addressed, proposing methods that stabilize the controlled system and ensure that the state vector trajectories remain within the operating region in which the T-S fuzzy model is valid. The procedure for calculating the exact DOA is often a very difficult or even an impossible task [22]. Therefore an important issue in determining the inner estimate of the DOA is how to obtain larger and larger estimates for the DOA, because the larger estimates imply more accurate ones [23]. Recently this problem has been addressed by some authors [10], [11], [21].

In this context, this manuscript proposes a procedure that ensures the local stabilization of discrete-time uncertain nonlinear systems using the switched control law [17]. Besides that, the proposed conditions aim to obtain a less conservative estimation of the DOA. In order to obtain less conservative LMI conditions and avoid using the membership functions in the implementation of the switched control law, the control design adapts the non-quadratic Lyapunov function proposed in [7].

The manuscript is organized as follows: the discrete T-S fuzzy system, the non-quadratic Lyapunov function, the switched control law and preliminary and auxiliary results are introduced in Section II. Section III presents different methods available in the literature, to maximize the estimation of the DOA and then a new method to obtain a less conservative estimation of the DOA is proposed. Therefore, the main result of this manuscript is presented. Finally, in Section IV, a numerical example illustrates the effectiveness of the proposed method and compares it with procedures found in the literature.

For simplicity, the following notation will be adopted: $\mathbb{K}_r = \{1,2,\ldots,r\}$, $r \in \mathbb{N}$; $\text{co}\{a_1,\cdots,a_r\}$ is the convex hull of the vectors a_i, $i \in \mathbb{K}_r$; $\text{diag}\{a_1,a_2,\cdots,a_r\}$ represents a block-diagonal matrix in which the diagonal elements are $a_1,a_2,\cdots,a_r$; I_n represents the $n \times n$ identity matrix; $Y > 0$ ($Y < 0$) means that the matrix Y is symmetric positive definite (negative definite); a star $(*)$ in a symmetric matrix denotes the transposed element in the symmetric position; $Tr(Y)$ means the trace of a matrix Y; given $h(z(k)) = \begin{bmatrix} h_1(z(k)) & \cdots & h_r(z(k)) \end{bmatrix}^T \in \mathbb{R}^r$, $\Delta h_i(z(k)) = h_i(z(k+1)) - h_i(z(k))$. Finally,

$$Y_{z(k)} = \sum_{i=1}^{r} h_i(z(k))Y_i,\ Y_{z(k+1)} = \sum_{i=1}^{r} h_i(z(k+1))Y_i. \tag{1}$$

II. Preliminary

A. Discrete T-S fuzzy system

Consider the uncertain nonlinear discrete-time system, described by

$$x(k+1) = f(z(k))x(k) + g(z(k))u(k), \tag{2}$$

where $f(\cdot) \in \mathbb{R}^{n_x \times n_x}$ and $g(\cdot) \in \mathbb{R}^{n_x \times n_u}$ are nonlinear functions, $x(k) = \begin{bmatrix} x_1(k) & \cdots & x_{n_x}(k) \end{bmatrix}^T \in \mathbb{R}^{n_x}$ is the state vector, $u(k) = \begin{bmatrix} u_1(k) & \cdots & u_{n_u}(k) \end{bmatrix}^T \in \mathbb{R}^{n_u}$ is the input vector, and $z(k) = \begin{bmatrix} z_1(k) & \cdots & z_{n_z}(k) \end{bmatrix}^T \in \mathbb{R}^{n_z}$ is a vector composed by the state vector $x(k)$ and a vector $v = \begin{bmatrix} v_1 & \cdots & v_{n_v} \end{bmatrix}^T \in \mathbb{R}^{n_v}$, whose entries v_ς, $\varsigma \in \mathbb{K}_{n_v}$, are uncertain bounded parameters of (2), that is,

$$z(k) = \begin{bmatrix} x_{a_1}(k) & \cdots & x_{a_q}(k) & v_1 & \cdots & v_{n_v} \end{bmatrix}^T \in \mathbb{R}^{n_z}, \tag{3}$$

where the set of indexes $\{a_1,a_2,\cdots,a_q\} \subseteq \{1,2,\cdots,n_x\}$, such that $q + n_v = n_z$.

Consider $\mathcal{V}$ a compact set defined by

$$\mathcal{V} := \{v \in \mathbb{R}^{n_v} :\ v_\varsigma \in [\underline{v_\varsigma},\ \overline{v_\varsigma}],\ \varsigma \in \mathbb{K}_{n_v}\}, \tag{4}$$

where for all $\varsigma \in \mathbb{K}_{n_v}$, $\overline{v_\varsigma}$ and $\underline{v_\varsigma}$ are known real numbers, and a operation region $\mathcal{L} \subset \mathbb{R}^{\overline{n_x}}$ in the state space defined as follows [10]:

$$\mathcal{L} := \{x(k) \in \mathbb{R}^{n_x} :\ x_{a_\eta}(k) \in [-\bar{x}_{a_\eta},\ \bar{x}_{a_\eta}],\ \eta \in \mathbb{K}_q\}, \tag{5}$$

where $q \le n_x$ and $\bar{x}_{a_\eta} > 0$ a known real number.

To represent the uncertain nonlinear system (2) by a fuzzy T-S model, it is necessary to calculate the bounds of the uncertain linear parameters and the system nonlinearities [12]–[14], [16], [17]. Thus, the fuzzy model T-S presents known local models and unknown normalized weights. Therefore, suppose that for all $v \in \mathcal{V}$ and $x(k) \in \mathcal{L}$, the uncertain nonlinear system (2) can be represented by a T-S fuzzy model, whose i-th rule can be described as

$$\begin{aligned} \underline{\text{Rule } i\text{:}}\ & \text{IF } z_1(k) \text{ is } M_1^i \text{ and } \ldots \text{ and } z_{n_z}(k) \text{ is } M_p^i, \\ & \text{THEN}\quad x(k+1) = A_i x(k) + B_i u(k) \end{aligned} \tag{6}$$

where $i \in \mathbb{K}_r$, M_j^i is a fuzzy set j of the rule i, $j \in \mathbb{K}_{n_z}$, $A_i \in \mathbb{R}^{n_x \times n_x}$, $B_i \in \mathbb{R}^{n_x \times n_u}$ are system matrices, $z_1(k)$, $\cdots$, $z_{n_z}(k)$ are the premise variable and r is the number of fuzzy rules.

From [24], the final mathematical model of the T-S fuzzy system, which represents the dynamics of the plant (2) is given by

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{r} h_i(z(k))\,(A_i x(k) + B_i u(k)) \\ &= A_{z(k)}x(k) + B_{z(k)}u(k) \end{aligned} \tag{7}$$

where $h_i(z(k))$, $i \in \mathbb{K}_r$, are differentiable functions that represent the normalized weight of each local model (A_i, B_i), defined in (6), for $i \in \mathbb{K}_r$, such that

$$\sum_{i=1}^{r} h_i(z(k)) = 1, \text{ and } h_i(z(k)) \ge 0,\ i \in \mathbb{K}_r. \tag{8}$$

B. Non-quadratic Lyapunov function and switched control law

As in [17], based on [7], the following non-quadratic Lyapunov function is employed as the Lyapunov function candidate:

$$V(x(k)) = x^T(k)G^{-T}\left(\sum_{i=1}^{r} h_i(z(k))P_i\right)G^{-1}x(k)$$
$$= x^T(k)G^{-T}P_{z(k)}G^{-1}x(k), \tag{9}$$

where $G \in \mathbb{R}^{n_x \times n_x}$ is a nonsingular matrice and P_i, $i \in \mathbb{K}_r$, are symmetric positive definite matrices.

Consider the switched control law, proposed in [17], that selects a state-feedback gain $K_{\sigma(k)} = F_{\sigma(k)}G^{-1}$, which belongs to the set of gains $\{K_l = F_l G^{-1} \in \mathbb{R}^{n_u \times n_x}, l \in \mathbb{K}_r\}$. The switching law $\sigma(k)$ uses symmetric matrices $G^{-T}Q_l G^{-1}$, $l \in \mathbb{K}_r$, to choose the state-feedback gain $K_{\sigma(k)}$. The switched control law is given by:

$$u(k) = -F_{\sigma(k)}G^{-1}x(k),$$
$$\sigma(k) = \arg^* \min_{l \in \mathbb{K}_r}\left\{x^T(k)G^{-T}Q_l G^{-1}x(k)\right\}, \tag{10}$$

where $\arg^* \min_{l \in \mathbb{K}_r}\left\{x^T(k)G^{-T}Q_l G^{-1}x(k)\right\}$ denotes the smallest index $\sigma(k) \in \mathbb{K}_r$, such that $x^T(k)G^{-T}Q_{\sigma(k)}G^{-1}x(k) = \min_{l \in \mathbb{K}_r}\left\{x^T(k)G^{-T}Q_l G^{-1}x(k)\right\}$. The symmetric matrices Q_l, $l \in \mathbb{K}_r$, are calculated using LMI criterion and the minimization of $x^T(k)G^{-T}Q_l G^{-1}x(k)$, for $l \in \mathbb{K}_r$ and $x(k) \neq 0$, causes reduction of the variation of Lyapunov function, $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$ [17], that is negative for $x(k) \neq 0$.

Therefore, from (1), the controlled system (7) and (10) can be represented as

$$x(k+1) = \sum_{i=1}^{r} h_i(z(k))\left(A_i - B_i F_{\sigma(k)}G^{-1}\right)x(k)$$
$$= \left(A_{z(k)} - B_{z(k)}F_{\sigma(k)}G^{-1}\right)x(k). \tag{11}$$

C. Preliminary result

The following lemmas will be used throughout this paper.

Lemma 1. *[17] Assume that there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_i \in \mathbb{R}^{n_x \times n_x}$, matrices $F_l \in \mathbb{R}^{n_u \times n_x}$ and $G \in \mathbb{R}^{n_x \times n_x}$, for all i, j, $l \in \mathbb{K}_r$, such that the condition holds:*

$$\begin{bmatrix} Z_i + Q_l & (A_i G - B_i F_l)^T \\ A_i G - B_i F_l & G + G^T - P_j \end{bmatrix} > 0. \tag{12}$$

Thus, considering (1) and the switched control law (10), the condition below also holds, for all $x(k) \neq 0$:

$$x^T(k)\Big\{\left(A_{z(k)} - B_{z(k)}F_{\sigma(k)}G^{-1}\right)^T\left(G^{-T}P_{z(k+1)}G^{-1}\right)$$
$$\times\left(A_{z(k)} - B_{z(k)}F_{\sigma(k)}G^{-1}\right)\Big\}x(k)$$
$$< x^T(k)\Big\{G^{-T}Z_{z(k)}G^{-1} + G^{-T}Q_{z(k)}G^{-1}\Big\}x(k). \tag{13}$$

Considering the non-quadratic Lyapunov function (9) and the switched control law (10), assuming $x(k) \in \mathcal{L}$ and $v \in \mathcal{V}$, then the next theorem, proposed in [17], offers conditions for the equilibrium point $x(k) = 0$ of the system (7) to be locally asymptotically stable.

Theorem 1. *[17] Assume that there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_i \in \mathbb{R}^{n_x \times n_x}$, matrices $F_l \in \mathbb{R}^{n_u \times n_x}$ and $G \in \mathbb{R}^{n_x \times n_x}$, for all i, j, $l \in \mathbb{K}_r$, such that the following LMIs are feasible:*

$$\begin{bmatrix} Z_i + Q_l & (A_i G - B_i F_l)^T \\ A_i G - B_i F_l & G + G^T - P_j \end{bmatrix} > 0, \tag{14}$$
$$Z_i + Q_i - P_i \leq 0. \tag{15}$$

Then, the switched control law (10), with the controller gains $F_l G^{-1}$, $l \in \mathbb{K}_r$, makes locally asymptotically stable the equilibrium point $x(k) = 0$ of the uncertain nonlinear system (7).

Remark 1. *An alternative to stabilize the uncertain T-S fuzzy system (7), without the need of using the membership functions, is to adopt a linear time-invariant controller, $u(k) = -Kx(k)$. In [17], is demonstrated that the procedure presented in the Theorem (1), presents less conservatism in the stabilization of the uncertain nonlinear system (7), than the procedure which uses a linear time-invariant controller.*

D. A estimation of the DOA

Suppose $\chi(k, x_0)$ the state trajectory of system (2), with an initial condition $x(0) = x_0$. Then the DOA of the origin is given by

$$\mathscr{R}_A = \left\{x_0 \in \mathbb{R}^{n_x} : \lim_{k \to \infty} \chi(k, x_0) = 0\right\}. \tag{16}$$

Consider the non-quadratic Lyapunov function candidate (9) and, for all $v \in \mathcal{V}$, the ellipsoidal set

$$\Omega(G^{-T}P_{z(k)}G^{-1}, \gamma) := \left\{x(k) \in \mathbb{R}^{n_x} : x^T(k)G^{-T}P_{z(k)}G^{-1}x(k) \leq \gamma\right\}, \tag{17}$$

with $\gamma > 0$.

The ellipsoidal set $\Omega(G^{-T}P_{z(k)}G^{-1}, \gamma)$ is said to be contractively invariant if $\Omega(G^{-T}P_{z(k)}G^{-1}, \gamma)\backslash\{0\} \subseteq \{x(k) \in \mathbb{R}^{n_x} : \Delta V(x(k)) < 0\}$. In this case, the ellipsoidal set defined in (17), is an estimate of the DOA (16), that is, $\Omega(G^{-T}P_{z(k)}G^{-1}, \gamma) \subseteq \mathscr{R}_A$ [25].

E. Control problem

To ensure that the system (2) can be represented by (7), it is necessary to guarantee that the system operates within the operation region $\mathcal{L}$. However, as noted in [11], it is necessary to ensure that for all $x(k) \in \mathcal{L}$ at the next instant $x(k+1)$ will remain in the region $\mathcal{L}$, because if this is not guaranteed, then for some $x(k) \in \mathcal{L}$, such that $x(k+1) \notin \mathcal{L}$, both $V(x(k+1))$ and $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$ cannot be defined. So, based on [10], [11], [21], consider the following set

$$\mathcal{R} := \left\{x(k) \in \mathcal{L} : \left(A_{z(k)} - B_{z(k)}F_{\sigma(k)}G^{-1}\right)x(k) \in \mathcal{L}\right\}. \tag{18}$$

Note that, from (11) and for all $x(k) \in \mathcal{R}$, the next instant, $x(k+1)$, will remain in the region $\mathcal{L}$. Assim, para provar que

Therefore, for the ellipsoidal set (17) to be an estimate for the attraction domain (16), of the equilibrium point of the uncertain nonlinear system (2); and for the representation (7) of the system (2) to be valid at every sample k, the following conditions must be satisfied:

1) $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{L}$ defined in (5), because this inclusion ensures that the system (2) can be represented by the T-S fuzzy model (7);
2) $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{R}$ described in (18), because the existence of $\Delta V(x(k))$ is ensured;
3) $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \setminus \{0\} \subseteq \{x(k) \in \mathcal{R} : \ \Delta V(x(k)) < 0\}$, for that $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ be an invariant subset of the DOA (16).

Remark 2. *Note that the condition* 2)*, from* (11) *and* (18)*, implies that* $x(k+1) \in \mathcal{L}$ *defined in* (5)*. This condition allows the calculation of* $\Delta V(x(k)) = V(x(k+1) - V(x(k))$*, because from* (9)*, if the condition* 2) *holds then* $x(k+1) \in \mathcal{L}$ *and* $h_i(z(k+1))$*,* $i \in K_r$*, is well defined because in this case the nonlinear plant* (2) *can be exactly represented by the T-S model* (7)*, replacing* k *by* $k+1$*.*

F. Auxiliary results

The Lemmas 2 and 3, presented in [10], ensure $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{L}$ and $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{R}$ respectively. The inclusion $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \setminus \{0\} \subseteq \{x(k) \in \mathcal{R} : \Delta V(x(k)) < 0\}$ stated in condition 3) above, will be shown later in the main result of this paper.

Lemma 2. *[10] Let the sets* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *and* $\mathcal{L}$ *given in* (17) *and* (5)*, respectively. The constraint* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{L}$ *is enforced if the following conditions hold*

$$\begin{bmatrix} P_i & G^T e_{a_\eta} \\ e_{a_\eta}^T G & \bar{x}_{a_\eta}^2 \gamma^{-1} \end{bmatrix} > 0, \tag{19}$$

for all $i \in \mathbb{K}_r$ *and* $\eta \in \mathbb{K}_q$*, where*

$$e_{a_\eta}^T := \begin{bmatrix} 0 & \cdots & 0 & \underbrace{1}_{a_\eta - th} & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_x}. \tag{20}$$

Lemma 3. *[10] Let the sets* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *and* $\mathcal{R}$ *given in* (17) *and* (18)*, respectively. The constraint* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right) \subset \mathcal{R}$ *is enforced if the following conditions hold*

$$\begin{bmatrix} P_i & (A_iG - B_iF_l)^T e_{a_\eta} \\ e_{a_\eta}^T (A_iG - B_iF_l) & \gamma^{-1}\bar{x}_{a_\eta}^2 \end{bmatrix} > 0, \tag{21}$$

for all $i,l \in \mathbb{K}_r$ *and* $\eta \in \mathbb{K}_q$*.*

III. Main Results

A. Maximization of the DOA

The ellipsoidal set $\Omega\left(G^{-T}P_zG^{-1},\gamma\right)$, defined in (17), can be a very conservative estimate of the DOA $\mathscr{R}_A$ given in (16) [20], [22]. Thus, in this section, new conditions will be presented in order to obtain a less conservative estimation of the DOA (17).

The problem of "enlargement" the estimation of the DOA (17), has been approached in some works in the literature.

- **Method A:**

In [10], for example, the following set is considered

$$\mathcal{Y} := \{x(k) \in \mathbb{R}^{n_x} : \ x(k)^T x(k) \leq 1/\delta\}, \tag{22}$$

where $\delta > 0$ is a real number. LMIs conditions are proposed to ensure that the set (22) is contained in the set $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ given in (17). Thus, the minimization of δ causes the maximization of the set defined in (22), that makes $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ to be enlarged.

Lemma 4. *[10] Let the sets* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *and* $\mathcal{Y}$ *given in* (17) *and* (22)*, respectively. The constraint* $\mathcal{Y} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *is enforced if, for all* $i \in \mathbb{K}_r$ *and* $m \in \mathbb{K}_{n_L}$*, the following conditions hold,*

$$\begin{bmatrix} \delta I_{n_x} & I_{n_x} \\ I_{n_x} & G^T + G - P_i \end{bmatrix} > 0. \tag{23}$$

- **Method B:**

On the other hand, in [25], [26], it is considered a convex polyhedral set, $\varpi\mathcal{W}$, defined by

$$\varpi\mathcal{W} = \varpi\text{co}\{w_1, \cdots, w_{n_L}\}, \tag{24}$$

where ϖ is a positive real number, $w_m \in \mathbb{R}^{n_x}$, $m \in \mathbb{K}_{n_L}$, are given vectors, and n_L is the number of polyhedron vertices.

To ensure that $\varpi\mathcal{W} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$, an immediate adaptation of the conditions LMIs presented in [25], for the case in which the Lyapunov function candidate (9) is considered, can be stated as follows:

Lemma 5. *[25] Let the sets* $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *and* $\varpi\mathcal{W}$ *given in* (17) *and* (24)*, respectively. The constraint* $\varpi\mathcal{W} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ *is enforced if, for all* $i \in \mathbb{K}_r$ *and* $m \in \mathbb{K}_{n_L}$*, the following conditions hold,*

$$\begin{bmatrix} \varpi^{-2}\gamma & w_m^T \\ w_m & G^T + G - P_i \end{bmatrix} > 0. \tag{25}$$

Note that, if the conditions of Lemma 5 hold, the maximization of the variable ϖ enlarge the size of the polyhedral set $\varpi\mathcal{W}$ and, consequently, provides a less conservative estimation of the DOA $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$, given in (17). In this case, $\varpi = \sup\left\{\tau : \ \tau\mathcal{W} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)\right\}$.

- **Method C (Proposed method):**

In order to develop a method that provides a less conservative estimation of the DOA, consider the following polyhedral set

$$\mathcal{B} = \text{co}\{\varpi_1 w_1, \cdots, \varpi_{n_L} w_{n_L}\}, \tag{26}$$

where ϖ_m, $m \in \mathbb{K}_{n_L}$, are positive scalars and $w_m \in \mathbb{R}^{n_x}$ are given vectors that define the directions in which the set

$\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ should be maximized. Thus, for all $\xi \in \mathcal{B}$, there exist $\theta_m \geq 0$, with $\sum_{m=1}^{n_L}\theta_m = 1$, such that

$$\xi = \sum_{m=1}^{n_L}\theta_m\varpi_m = (\varpi w)_\theta. \tag{27}$$

Based on [26], [27], the following lemma provides conditions for $\mathcal{B} \subset \Omega\left(G^{-T}P_zG^{-1},\gamma\right)$.

Lemma 6. *Let the sets $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ and $\mathcal{B}$ given in (17) and (26), respectively. The constraint $\mathcal{B} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ is enforced if the following conditions hold,*

$$\begin{bmatrix} \gamma & \varpi_m w_m^T \\ \varpi_m w_m & G^T+G-P_i \end{bmatrix} > 0, \tag{28}$$

for all $i \in \mathbb{K}_r$ and $m \in \mathbb{K}_{n_L}$.

Proof: Multiplying (28) by $h_i(z(k))$ and taking the sum from $i=1$ to $i=r$, multiplying the result by θ_m and taking the sum from $m=1$ to $m=n_L$, considering the notations given in (1), (26) and (27), one obtains

$$\begin{bmatrix} \gamma & (\varpi w)_\theta^T \\ (\varpi w)_\theta & G^T+G-P_{z(k)} \end{bmatrix} > 0. \tag{29}$$

Pre- and post-multiplying (29) by $\begin{bmatrix} 1 & -(\varpi w)_\theta^T G^{-T}\end{bmatrix}$ and its transposed, respectively, it follows that

$$((\varpi w)_\theta^T)G^{-T}P_{z(k)}G^{-1}((\varpi w)_\theta) < \gamma. \tag{30}$$

Note that $(\varpi w)_\theta \in \mathcal{B}$. Thus, from (30), one obtains $(\varpi w)_\theta \in \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$. Therefore, $\mathcal{B} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$. ∎

Note that, once the conditions of Lemma 6 hold, the maximization of $\sum_{m=1}^{n_L}\varpi_m$ enlarge the size of the polyhedral set $\mathcal{B}$, which provides a less conservative estimation of the DOA $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$.

The procedure presented in [25] provides a uniform expansion in all directions of the polyhedral set $\mathcal{W}$, while the proposed method in Lemma 6 provides the maximization of n_L variables ϖ_m, from the optimization of $\sum_{m=1}^{n_L}\varpi_m$. Therefore, different expansions are possible in each specific direction of the polyhedral set $\mathcal{B}$.

Lemma 7. *Assume that the conditions of Lemma 5 hold. Then, the conditions of Lemma 6 also hold.*

Proof: Suppose that the conditions (25) of Lemma 5 hold for all $i \in \mathbb{K}_r$ and $m \in \mathbb{K}_{n_L}$.

Pre- and post-multiplying (25) by $diag\{\varpi, I\}$ and its transposed, respectively, one obtains

$$\begin{bmatrix} \gamma & \varpi w_m^T \\ \varpi w_m & G^T+G-P_i \end{bmatrix} > 0. \tag{31}$$

Thus, as (31) hold for all $i \in \mathbb{K}_r$ e $m \in \mathbb{K}_{n_L}$, considering $\varpi_1 = \cdots = \varpi_{n_L}$, so that (28) hold for all $i \in \mathbb{K}_r$ and $m \in \mathbb{K}_{n_L}$. ∎

Remark 3. *In addition to the methods explored in this manuscript, there are other ways to maximize the set $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$. An alternative to this, is to maximize the ellipsoidal subsets $\Omega\left(G^{-T}P_iG^{-1},\gamma\right)$, defined by each local Lyapunov matrices P_i, $i \in \mathbb{K}_r$. As the volume of each ellipsoidal set $\Omega\left(G^{-T}P_iG^{-1},\gamma\right)$, $i \in \mathbb{K}_r$, is proportional to $\left(\det(\gamma^{-1}P_i^{-1})\right)^{\frac{1}{2}}$, the minimization of $\log(\det(\gamma P_i))$, $i \in \mathbb{K}_r$, cause the maximization of the set $\Omega\left(G^{-T}P_iG^{-1},\gamma\right)$, $i \in \mathbb{K}_r$ [3]. Furthermore, each eigenvalue of P_i, $i \in \mathbb{K}_r$, is associated to the length of one ellipsoid axis, then the minimization of $Tr(P_i)$, expands homogeneously the set $\Omega\left(G^{-T}P_iG^{-1},\gamma\right)$, $i \in \mathbb{K}_r$ [27].*

B. Local stabilization of uncertain nonlinear discrete-time system

Considering the Lyapunov function candidate (9), the switched control (10) and assuming that the uncertain parameters, v_ς, $\varsigma \in \mathbb{K}_{n_v}$, of the system (7), are bounded, that is, $v \in \mathcal{V}$ set defined in (4), the following theorem is proposed.

Theorem 2. *Consider the uncertain nonlinear system* (2) *described by T-S fuzzy model* (7) *in an operation region $\mathcal{L}$ given in* (5)*, where $\bar{x}_{a\eta} > 0$, $\eta \in \mathbb{K}_q$ are known, as well as the bounds of the uncertain parameters defined in* (4)*. Suppose that there exist symmetric positive definite matrices $P_i \in \mathbb{R}^{n_x \times n_x}$, symmetric matrices Z_i, $Q_i \in \mathbb{R}^{n_x \times n_x}$, matrices $F_l \in \mathbb{R}^{n_u \times n_x}$, $G \in \mathbb{R}^{n_x \times n_x}$, e real numbers $\varpi_m > 0$, with $\Pi = diag\{\varpi_1, \cdots \varpi_{n_L}\} \in \mathbb{R}^{n_L \times n_L}$, such that the following optimization problem*

$$\max_{P_i,Z_i,Q_l,F_l,G,H} Tr(\Pi) \text{ subject to}$$

$$\begin{bmatrix} P_i & G^T e_{a\eta} \\ e_{a\eta}^T G & \gamma^{-1}\bar{x}_{a\eta}^2 \end{bmatrix} > 0, \tag{32}$$

$$\begin{bmatrix} P_i & (A_iG-B_iF_l)^T e_{a\eta} \\ e_{a\eta}^T(A_iG-B_iF_l) & \gamma^{-1}\bar{x}_{a\eta}^2 \end{bmatrix} > 0, \tag{33}$$

$$\begin{bmatrix} Z_i+Q_l & (A_iG-B_iF_l)^T \\ A_iG-B_iF_l & G+G^T-P_j \end{bmatrix} > 0, \tag{34}$$

$$Z_i+Q_i-P_i \leq 0, \tag{35}$$

$$\begin{bmatrix} \gamma & \varpi_m w_m^T \\ \varpi_m w_m & G^T+G-P_i \end{bmatrix} > 0, \tag{36}$$

hold for all i, l, δ, e $j \in \mathbb{K}_r$, $\eta \in \mathbb{K}_q$, $s \in \mathbb{K}_{2^{n_u}}$ and $m \in \mathbb{K}_{n_L}$, where $e_{a\eta}$ defined in (20) *and $w_m \in \mathbb{R}^{n_x}$ given in* (26)*, are known. Then, the switched control law* (10) *with the controller gains $K_l = F_lG^{-1}$, $l \in \mathbb{K}_r$, makes locally asymptotically stable the equilibrium point $x(k) = 0$ of the uncertain nonlinear system* (2)*. Moreover, the ellipsoidal set $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ defined in* (17)*, is an invariant subset of the DOA* (16)*.*

Proof: According to Lemmas 2 and 3, the conditions (32) and (33), ensure that $\Omega\left(G^{-T}P_zG^{-1},\gamma\right) \subset \mathcal{L}$, $\Omega\left(G^{-T}P_zG^{-1},\gamma\right) \subset \mathcal{R}$. That is, for all $x(k) \in \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ the uncertain nonlinear systems (2) can be represented by a T-S fuzzy model (7) and the inequality $\Delta V(x(k)) < 0$ for $x(k) \neq 0$, can be verified.

Now, from the definition of $\mathcal{R}$, it follows that if $x(k) \in \mathcal{R}\backslash\{0\}$, then $x(k+1) \in \mathcal{L}$. Then, for $x(k) \in \mathcal{R}\backslash\{0\}$, the functions $h_i(z(k+1))$, $i \in \mathbb{K}_r$, described in (8), are well defined. Thus, for all $x(k) \in \mathcal{R}\backslash\{0\}$, using the notations defined in (1), multiplying (35) by $h_i(z(k))$ and taking the sum from $i=1$ to $i=r$, and pre- and post-multiplying the result by G^{-T} and G^{-1}, respectively, one obtains

$$x^T(k)\left\{G^{-T}Z_{z(k)}G^{-1}+G^{-T}Q_{z(k)}G^{-1}-G^{-T}P_{z(k)}G^{-1}\right\}x(k) \leq 0. \tag{37}$$

Considering the switched control law (10), from (34), Lemma 1 and (37), for all $x(k) \in \mathcal{R}\backslash\{0\}$, it follows that

$$\begin{aligned} x^T(k)\Big\{ &\left(A_{z(k)}-B_{z(k)}K_{\sigma(k)}\right)^T\left(G^{-T}P_{z(k+1)}G^{-1}\right) \\ &\times\left(A_{z(k)}-B_{z(k)}K_{\sigma(k)}\right)\Big\}x(k) \\ &< x^T(k)\left\{G^{-T}Z_{z(k)}G^{-1}+G^{-T}Q_{z(k)}G^{-1}\right\}x(k) \\ &\leq x^T(k)G^{-T}P_{z(k)}G^{-1}x(k). \end{aligned} \tag{38}$$

Thus, for all $x(k) \in \mathcal{R}\backslash\{0\}$,

$$\begin{aligned} x^T(k)\Big\{ &\left(A_{z(k)}-B_{z(k)}K_{\sigma(k)}\right)^T\left(G^{-T}P_{z(k+1)}G^{-1}\right) \\ &\times\left(A_{z(k)}-B_{z(k)}K_{\sigma(k)}\right)-G^{-T}P_{z(k)}G^{-1}\Big\}x(k) < 0. \end{aligned} \tag{39}$$

Considering the controlled system (11) and the Lyapunov function candidate, given in (9), the inequality (39), implies that $\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0$. Then,

$$\mathcal{R}\backslash\{0\} \subseteq \{x(k) \in \mathcal{L}:\ \Delta V(x(k)) < 0\}. \tag{40}$$

Thus, from (33) and (40), it follows that

$$\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)\backslash\{0\} \subseteq \{x(k) \in \mathcal{R}:\ \Delta V(x(k)) < 0\}. \tag{41}$$

Therefore, from (41), it follows that, the switched control law (10), with the controller gains $K_l = F_l G^{-1}$, $l \in \mathbb{K}_r$, makes the equilibrium point, $x(k)=0$, of the uncertain system (7), locally asymptotically stable and $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ is an invariant subset of the DOA.

Finally, note that LMI (36) is equivalent to LMI (28) of the Lemma 6. Thus from (36) and Lemma 6, it follows that $\mathcal{B} \subset \Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$, with $\mathcal{B}$ given in (26). Therefore, the maximization of the $Tr(\Pi)$ enlarges the size of the ellipsoidal set $\Omega\left(G^{-T}P_{z(k)}G^{-1},\gamma\right)$ that is, provides a less conservative estimation of the DOA. ∎

Remark 4. *In this work, the normalized membership functions are considered uncertain or unknown, which makes the calculation of* $V(x(k)) = x^T(k)G^{-T}\left(\sum_{i=1}^r h_i(z(k))P_i\right)G^{-1}x(k)$ *impossible. Thus, one method of obtaining a conservative estimate of the set* $\Omega\left(G^{-T}P_zG^{-1},\gamma\right)$*, consists of finding the intersection of the ellipsoidal sets* $\Omega\left(G^{-T}P_iG^{-1},\gamma\right)$*,* $i \in \mathbb{K}_r$*. In this case* $\Omega\left(G^{-T}P_zG^{-1},\gamma\right) \supseteq \bigcap_{i=1}^r \Omega\left(G^{-T}P_iG^{-1},\gamma\right)$*.*

IV. Numerical example

Example 1. *Consider the nonlinear chaotic Lorenz system discretized with forward Euler approximation [28]:*

$$\begin{cases} x_1(k+1) = x_1(k) + T_s\left(-\mu_1 x_1(k) + \mu_1 x_2(k)\right) \\ x_2(k+1) = x_2(k) + T_s\left(\mu_2 x_1(k) - x_2(k) - x_1(k)x_3(k)\right) \\ x_3(k+1) = x_3(k) + T_s\left(x_1(k)x_2(k) - \mu_3 x_3(k)\right), \end{cases} \tag{42}$$

where T_s is the fixed step of discretization and for the emergence of chaos, the nominal values $(\mu_1,\mu_2,\mu_3) = (10, 28, 8/3)$ are considered.

Suppose that the parameters μ_1 and μ_2 are uncertain but bounded within 50% *of their nominal values. Therefore, considering a vector $\mu = \begin{bmatrix}\mu_1 & \mu_2\end{bmatrix}^T \in \mathbb{R}^2$, whose entries are uncertain bounded parameters of* (42)*, the set $\mathcal{V}$ defined in* (4)*, is given by*

$$\mathcal{V} := \left\{\mu \in \mathbb{R}^2:\ \mu_1 \in [5,\ 15],\ \mu_2 \in [14,\ 42]\right\}. \tag{43}$$

Consider the operation region

$$\mathcal{L} := \left\{x(k) \in \mathbb{R}^3:\ x_1(k) \in [-\beta,\ \beta]\right\}. \tag{44}$$

Using the ranges of values presented in (43) *and* (44)*, for the uncertain parameters, μ_1 and μ_2, of the system* (42) *and for the state variable x_1, respectively, the following local model matrices of the T-S fuzzy system* (7)*, that represent the chaotic system* (42)*, are obtained:*

$$\begin{aligned} A_1 &= \begin{bmatrix} 1-5T_s & 5T_s & 0 \\ 14T_s & 1-T_s & \beta T_s \\ 0 & -\beta T_s & 1-\mu_3 T_s \end{bmatrix}, & A_2 &= \begin{bmatrix} 1-15T_s & 15T_s & 0 \\ 14T_s & 1-T_s & \beta T_s \\ 0 & -\beta T_s & 1-\mu_3 T_s \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 1-5T_s & 5T_s & 0 \\ 42T_s & 1-T_s & \beta T_s \\ 0 & -\beta T_s & 1-\mu_3 T_s \end{bmatrix}, & A_4 &= \begin{bmatrix} 1-15T_s & 15T_s & 0 \\ 42T_s & 1-T_s & \beta T_s \\ 0 & -\beta T_s & 1-\mu_3 T_s \end{bmatrix}, \\ A_5 &= \begin{bmatrix} 1-5T_s & 5T_s & 0 \\ 14T_s & 1-T_s & -\beta T_s \\ 0 & \beta T_s & 1-\mu_3 T_s \end{bmatrix}, & A_6 &= \begin{bmatrix} 1-15T_s & 15T_s & 0 \\ 14T_s & 1-T_s & -\beta T_s \\ 0 & \beta T_s & 1-\mu_3 T_s \end{bmatrix}, \\ A_7 &= \begin{bmatrix} 1-5T_s & 5T_s & 0 \\ 42T_s & 1-T_s & -\beta T_s \\ 0 & \beta T_s & 1-\mu_3 T_s \end{bmatrix}, & A_8 &= \begin{bmatrix} 1-15T_s & 15T_s & 0 \\ 42T_s & 1-T_s & -\beta T_s \\ 0 & \beta T_s & 1-\mu_3 T_s \end{bmatrix}. \end{aligned} \tag{45}$$

The input matrices B_1, B_2, B_3, B_4, B_5, B_6, B_7 and B_8 were arbitrarily chosen as

$$B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = B_8 = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T. \tag{46}$$

Then, with this procedure, one has known local models and unknown normalized weights.

The following solutions were obtained for the optimization problem given by the LMIs (32)-(36) *presented in Theorem 2, considering $T_s = 0.002$ s, $\gamma = 1$, $\bar{x}_1 = \beta = 50$, $w_1^T = \begin{bmatrix}1 & 0 & 0\end{bmatrix}$, $w_2 = -w_1$, $w_3^T = \begin{bmatrix}0 & 1 & 0\end{bmatrix}$, $w_4 = -w_3$, $w_5^T = \begin{bmatrix}0 & 0 & 1\end{bmatrix}$ and $w_6 = -w_5$:*

$$\begin{aligned} K_1 &= \begin{bmatrix}-0.9783 & -0.0201 & -1.5300\times10^{-6}\end{bmatrix}, \\ K_2 &= \begin{bmatrix}-0.9783 & -0.0201 & 1.5300\times10^{-6}\end{bmatrix}, \\ K_3 &= \begin{bmatrix}-0.9791 & -0.0201 & -2.1913\times10^{-7}\end{bmatrix}, \\ K_4 &= \begin{bmatrix}-0.9791 & -0.0201 & 2.1913\times10^{-7}\end{bmatrix}, \\ K_5 &= \begin{bmatrix}-0.9788 & -0.0201 & -1.7399\times10^{-6}\end{bmatrix}, \\ K_6 &= \begin{bmatrix}-0.9788 & -0.0201 & 1.7399\times10^{-6}\end{bmatrix}, \\ K_7 &= \begin{bmatrix}-0.9790 & -0.0201 & -4.1993\times10^{-7}\end{bmatrix}, \\ K_8 &= \begin{bmatrix}-0.9790 & -0.0201 & 4.1993\times10^{-7}\end{bmatrix}, \end{aligned} \tag{47}$$

$$P_1 = \begin{bmatrix} 2503.4145 & -82.0527 & -0.6179 \\ -82.0527 & 765515.4923 & 25.9001 \\ -0.6179 & 25.9001 & 777420.9898 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 2503.4145 & -82.0527 & 0.6179 \\ -82.0527 & 765515.4923 & -25.9001 \\ 0.6179 & -25.9001 & 777420.9898 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 2503.6194 & -88.2740 & -1.0224 \\ -88.2740 & 765415.1759 & 13.0573 \\ -1.0224 & 13.0573 & 777355.7218 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} 2503.6194 & -88.2740 & 1.0224 \\ -88.2740 & 765415.1759 & -13.0573 \\ 1.0224 & -13.0573 & 777355.7218 \end{bmatrix},$$

$$P_5 = \begin{bmatrix} 2503.4231 & -82.1080 & -0.5195 \\ -82.1080 & 765506.9255 & 55.0938 \\ -0.5195 & 55.0938 & 777395.6498 \end{bmatrix},$$

$$P_6 = \begin{bmatrix} 2503.4231 & -82.1080 & 0.5195 \\ -82.1080 & 765506.9255 & -55.0938 \\ 0.5195 & -55.0938 & 777395.6498 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 2503.6359 & -89.1073 & -1.2062 \\ -89.1073 & 765396.3768 & 33.3830 \\ -1.2062 & 33.3830 & 777354.6413 \end{bmatrix},$$

$$P_8 = \begin{bmatrix} 2503.6359 & -89.1073 & 1.2062 \\ -89.1073 & 765396.3768 & -33.3830 \\ 1.2062 & -33.3830 & 777354.6413 \end{bmatrix},$$

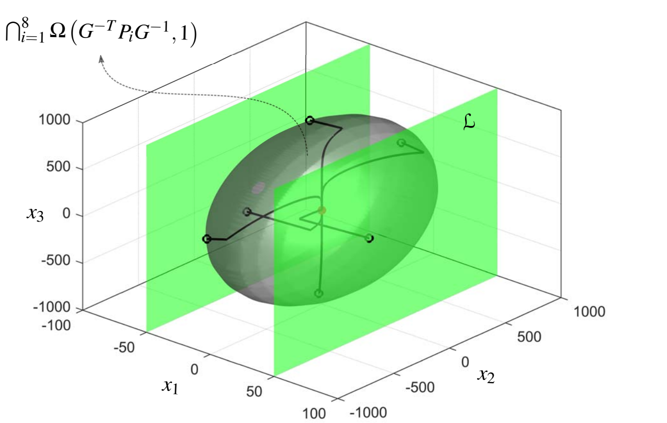
$$G = \begin{bmatrix} 2501.7024 & -84.0690 & 2.1461 \times -10 \\ -84.0585 & 765497.6411 & 2.6572 \times -8 \\ -4.7390 \times -10 & -1.1378 \times -9 & 777224.7235 \end{bmatrix},$$

$$\varpi_1 = \varpi_2 = 49.9819, \ \varpi_3 = \varpi_4 = 874.9082, \ \varpi_5 = \varpi_6 = 881.4864, \tag{48}$$

where $K_i = F_i G^{-1}$, $i \in \mathbb{K}_4$.

The estimate for the DOA obtained with the procedure presented in Theorem 2, using the solutions (47) *and* (48), *is plotted in Figures 1 and 2.*

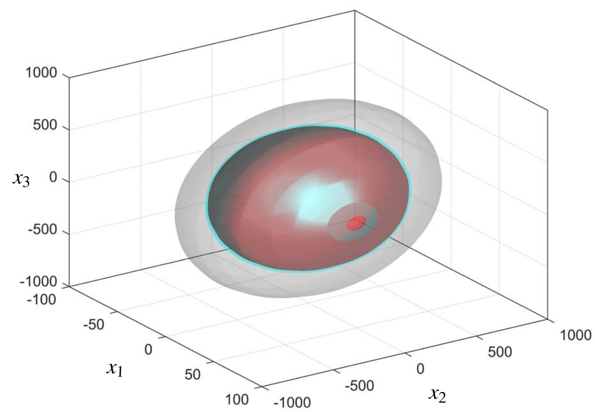
Fig. 1. Surface of set $\mathcal{L}$ and estimate of the DOA computed by using Theorem 2, with $\gamma = 1$ and $\bar{x}_1 = 50$, where the black solid line represent the state trajectories that start at "○" and converge to the origin "•".



Remark 5. *For the application of the switched control law* (10)*, the control design proposed in Theorem 2, does not use the membership functions or the uncertain parameters* μ_1 *and* μ_2*. Therefore, even though the premise variables and the membership functions are uncertain, with the proposed procedure in Theorem 2, it is possible to find an estimate of the DOA for the discrete-time T-S fuzzy systems. Figure 1 illustrates the validity of the proposed approach.*

Figure 2 shows the ellipsoidal sets obtained with different procedures to optimize the estimation of the DOA using Theorem 2. The ellipsoidal set in red was obtained using Theorem 2 modified with replacement of condition (36) *by LMI* (25) *and optimizing a single variable* ϖ. *The ellipsoidal set in cyan was obtained using Theorem 2 modified with replacement of condition* (36) *by LMI* (23) *and optimizing a single variable* δ. *Finally, the estimate of the DOA in gray was obtained with the procedure proposed in Theorem 2. Note that the ellipsoidal sets obtained using the methods of maximization proposed in [10] and [25] are inside of the domain obtained with the procedure proposed in Theorem 2.*

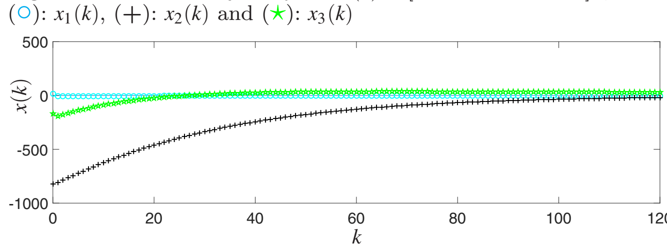
Fig. 2. Estimate of the DOA obtained by different maximization methods: In gray, obtained using Theorem 2, with $\gamma = 1$ and $\bar{x}_1 = 50$; In cyan obtained using the procedure proposed in [10]; In red obtained using the procedure proposed in [25].



Using the switched control law (10)*, with the set of gains given in* (47)*, it was perform a simulation of the closed-loop system* (7)*, considering* $\mu_1 = 15$, $\mu_2 = 14$ *and the initial condition* $x(0) = \begin{bmatrix} 13.78 & -820 & -170 \end{bmatrix}^T$*, belonging to the estimate of the DOA presented in Figure 1. Note that the initial condition does not belong to the ellipsoidal sets obtained using the methods proposed in [10] and [25]*

Figure 3 presents the trajectories of $x_1(k)$, $x_2(k)$ *and* $x_3(k)$*, symbolized by* ○, + *and* ★*, respectively.*

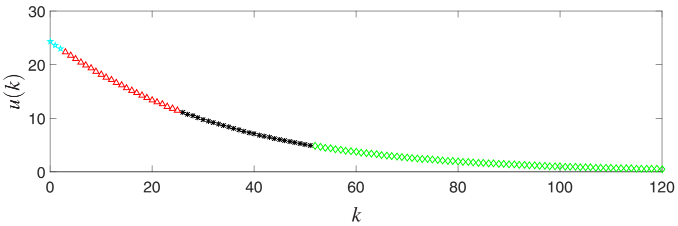
Fig. 3. The state trajectory for $x(0) = [13.78 \ -820 \ -170]^T$, where (○): $x_1(k)$, (+): $x_2(k)$ and (★): $x_3(k)$



The effort of the switched control law (10) *is shown in Figure 4, where the symbols* ×, □, ◇, ★, ○, ▽, ∗ *and* △

represent that the controller F_1G^{-1}, F_2G^{-1}, F_3G^{-1}, F_4G^{-1}, F_5G^{-1}, F_6G^{-1}, F_7G^{-1} or F_8G^{-1} is active, respectively, at the each instant k.

Fig. 4. The switched control law (10): control effort and $\sigma(k)$, where ($\times$): $\sigma(k) = 1$, ($\square$): $\sigma(k) = 2$, ($\diamond$): $\sigma(k) = 3$, ($\star$): $\sigma(k) = 4$, ($\circ$): $\sigma(k) = 5$, ($\bigtriangledown$): $\sigma(k) = 6$, ($\ast$): $\sigma(k) = 7$ and ($\triangle$): $\sigma(k) = 8$.



V. Conclusions

In this paper the local stability analysis problem for discrete-time uncertain nonlinear systems described by T-S fuzzy models has been investigated. By the use of the switched control and based on a non-quadratic Lyapunov function, in terms of a maximization problem of the trace of a matrix subject to LMI constraints have been proposed conditions to ensure the local stability and to obtain a less conservative estimative of the domain of attraction (DOA). This result generalizes that presented in [17], that did not consider the DOA. A numerical example has been given to illustrate the effectiveness of the proposed design method.

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