Atanassov's intuitionistic fuzzy measure based on the Sugeno integral induced by (α, β)-cut

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***Abstract*—Atanassov's intuitionistic fuzzy sets (A-IFSs) are used to deal with that information, which is incomplete as well as imprecise. In this paper, we defined a similarity measure by using Sugeno integral and technique of (α, β)-cut. Hwang et al. [Hwang, Chao-Ming, et al. "A similarity measure of intuitionistic fuzzy sets based on the Sugeno integral with its application to pattern recognition." Information Sciences 189 (2012): 93-109.] defined Sugeno integral based similarity measure for the first time. But, in Hwang et al.'s similarity measure, only α-cut is utilized that neglected the contribution of non-membership function. The non-membership function plays an equal role in the A-IFS theory. Therefore, we proposed the Sugeno integral based similarity measure concerning both the (α, β)-cuts. We added one artificial constructed example to show that our proposal is different than to similarity measure defined by Hwang et al. Moreover, we added some more benchmark examples to show the efficacy of the proposed similarity measure.**

I. Introduction

Zadeh in [1], introduced the basic structure of fuzzy sets (FSs) in 1965 to deal with that information, which involves uncertainty. Henceforth, a variety of higher-order of FSs, for example, Interval-valued fuzzy sets [5], Type-2 fuzzy sets [1], Atanassov intuitionistic fuzzy sets [2], Vague sets [6] etc. have been proposed by the different researcher to generalized the conventional FSs. Atanassov intuitionistic fuzzy sets (A-IFSs); initially has been extended by Atanassov [2] in 1986, which is a popular extension of the FS theory, used to describe objective reality more realistically and precisely.

(α)-cut helps to make a crisp set from the universe of discourse X in such a way that each element of the crisp set has membership value less than or equal to α (see Fig. 1). Thus, (α)-cut has been used by the researchers to relate the FS and the classical sets. Every A-IFSs depends upon two independent components membership and non-membership. Therefore, to carry foreword this cut technique from FS to its more generalized and accurate extension A-IFS, Li and Zou [7], [8] discussed the concept of upper cut and lower cut or (α, β)-cuts of A-IFSs and discussed the decomposition and representation theorem concerning (α, β)-cut (see definition-2.4, 2.5 and Fig. 2). Omer and Bayeq [28], applied (α, β)-cut technique to solve initial value problem in differential equation. Moreover, by using (α, β)-cut Liu et al. [29], proposed a model to make decision in A-IFS. (α, β)-cut also have been used by the researchers to construct similarity

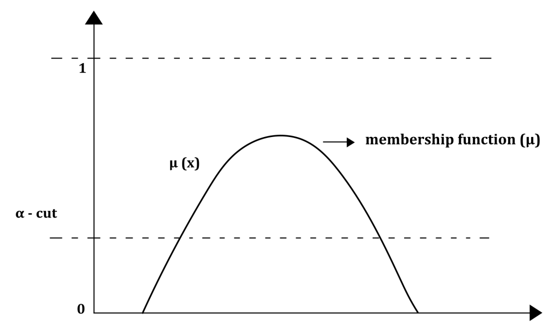

Fig. 1. Geometrical of α-cut.

measures [4] for solving real world problems.

Eventually, as a successor of the proposal of A-IFSs, many researchers studied it in different aspects practically and theoretically that transformed it into a burning area of research. In the development of A-IFS, similarity/distance measure is one of the essential theories studied under A-IFS. Similarity/distance measure, also known as information measure, is an explicit function used to differentiate two objects based on the given information. The similarity and distance measure both are interrelated concepts under the A-IFS theory, which is used to characterize the information by evaluating the degree of belongingness. Due to mathematical properties such as reflexive, symmetry and transitivity; similarity measure has high potential to separate information, and therefore, it received a huge amount of attention of the research community and used it in various applications, e.g., Decision making ([21],[22]), Pattern recognition, ([9],[10], [19], [18]), Cluster analysis ([11], [20], [16],[17]), Biology ([14]) and so on. For the space constraint, we refer to our previous papers [20] in which we did a detailed survey about the systematic development of the similarity/distance measure (please see the introduction section and table I in [20]).

In 1974, Sugeno integral [23] was first proposed by M. Sugeno in his doctoral thesis. Sugeno integral uses the maximum and minimum operations that make it a handy tool to induce similarity/dissimilarity measures, which are utilized in statistics as well in other fields [15], [13]. Hwang et al. [4], in

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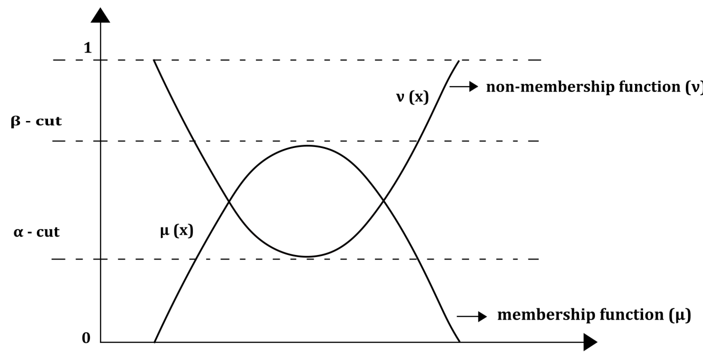

Fig. 2. Geometry of (α, β)-cut over A-IFS

their research, utilized the Sugeno integral to define similarity measures to measure the degree of similarity between two A-IFSs. As in A-IFSs, every element has two independent components. Thus, both the components have opposite nature's cut on them (see Fig. 2). Hwang et al., in [4], proposed their similarity measure between two A-IFSs by only utilizing the α-cut. However, the proper cut on any A-IFS is (α, β)-cut that uses both the independent components functions μ and ν. The main aim of this paper is to redefine Sugeno integral based intuitionistic fuzzy similarity measure by utilizing the cuts on both the independent components, membership, and non-membership.

Further paper is as follows; In section II, we discuss some basic definitions related to our study. In section III, by using (α, β)-cut, we proposed the Sugeno integral based similarity measure and studied its mathematical properties. In section IV, we compare the proposed similarity with the other well-known similarity measures. Finally, we conclude in section V.

II. Preliminaries

In this section, let us discuss some fundamental definitions associated with further study of this paper.

Definition 2.1: Atanassov intuitionistic fuzzy set (A-IFS) ([2]): Let X be an universe of discourse. Then, the Atanassov's intuitionistic fuzzy set (A-IFS) A is defined as an object of the form:

$$A = \{(x, \mu_A(x), \nu_A(x)) \ |x \in X\}$$

where, $\mu_A : X \to [0, 1]$ is membership function, $\nu_A : X \to [0, 1]$ is non-membership function, for each $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$. Moreover, the function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is degree of hesitancy.

Definition 2.2: Fuzzy Measure ([4]): A function m defined on sigma algebra Γ of X is called fuzzy measure if it satisfies the following conditions:

(1) $m(\phi) = 0$.
(2) $E, F \in \Gamma$ and $E \subset F$ imply $m(E \le m(F)$.
(3) $\{E_n\} \in \Gamma; E1 \subset E2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} E_n \in \Gamma$ imply $\lim_n m(E_n) = m(\bigcup_{n=1}^{\infty} E_n)$.
(4) $\{E_n\} \in \Gamma; E1 \supset E2 \supset \cdots, m(E_1) < \infty$ and $\bigcap_{n=1}^{\infty} E_n \in \Gamma$ imply $\lim_n m(E_n) = m(\bigcap_{n=1}^{\infty} E_n)$.

Definition 2.3: Sugeno integral ([4]): For a finite set X, we denote the value of function f at point x_i in X by f_i. If the values of the function f satisfy $f_1 < f_2 < ... < f_n$, then the sugeno integral of f with respect to fuzzy measure m is defined as:

$$\int f \, \mathrm{d}m = \vee_{i=1}^{n} (f_i \wedge m(A_i))$$

, where $A_i = x_i, x_{i+1}, ..., x_n, i = 1, ..., n$ and $A_{n+1} = \phi$.

Definition 2.4: (α, β)-cuts ([7], [8]): Let A be an A-IFS, then, its (α, β)-cuts is defined as:

$$A_{\alpha,\beta} = \{(x, \mu_A(x), \nu_A(x)) \ |x \in X, \mu_A(x) \ge \alpha, \nu_A(x) \le \beta\}$$

where, $\alpha, \beta \in [0, 1]$ with condition $\alpha + \beta \le 1$

Definition 2.5: Resolution identity ([7], [8]): Let A be an A-IFS with membership function ξ_A, non-membership function η_A and $A_{\alpha,\beta} = \{x : \xi_A(x) \ge \alpha, \eta_A \le \beta\}$. Then

$$\xi_A(x) = \sup_{0 \le \alpha \le 1} \alpha 1_{A_\alpha}(x); \eta_A(x) = \inf_{0 \le \beta \le 1} \beta \ 1_{A_\beta}(x)$$

where, A_α and A_β are the α and β level sets of the A-IFS corresponding to the membership and non-membership functions, respectively. Also, 1_{A_α} and 1_{A_β} are the called the Resolution identity and defined as follows:

$$1_{A_\alpha}(r) = \begin{cases} \alpha, & if \ r \in A_\alpha \\ 0, & if \ r \notin A_\alpha \end{cases}; \ 1_{A_\beta}(t) = \begin{cases} \beta, & if \ t \in A_\beta \\ 1, & if \ t \notin A_\beta \end{cases}$$

Definition 2.6: Similarity measure ([20]): For any $A_1, A_2, A_3 \in$ A-IFS, a similarity measure, S: A-IFS$\times$ A-IFS$\to [0, 1]$ is a mapping between A-IFSs with the following properties:

(1) $0 \le S(A_1, A_2) \le 1$.
(2) $S(A_1, A_2) = 0 \ \Leftrightarrow A_1 = A_2$.
(3) $S(A_1, A_2) = S(A_2, A_1)$.
(4) $A_1 \subseteq A_2 \subseteq A_3$ then $S(A_1, A_3) \le S(A_1, A_2)$ and $S(A_1, A_3) \le S(A_2, A_3)$.

The relationship between the distance measure and similarity measure is defined as $S(A_1, A_2) = 1 - D(A_1, A_2)$.

III. Sugeno integral as a similarity measure between A-IFSs

Sugeno integral is a useful operator to generate similarity measure. Similarity measures is utilized in many fields of applied mathematics and computer sciences. In this paper, we proposed a similarity measure for the intuitionistic fuzzy set by utilizing (α, β)-cuts. Moreover, as Sugeno integral uses maximum and minimum operation in their mathematical mechanism; thus, our proposed similarity measure between A-IFS will also enjoy those properties too. Let us assume that F be the class of all finite non-negative measurable functions with range $[0,1]$ defined in (X, Γ). For any given $f \in F$, we can write $f_\alpha = \{x : f(x \geq \alpha\}$, where $\alpha \in [0,1]$. f_α is called $\alpha - cut$ set (see Fig. 1). For a finite set X, we denote the value of function f at point x_i in X by f_i. Then, the Sugeno integral of f with respect to the fuzzy measure m is defined as:

$$\int f \,\mathrm{d}m = \vee_{i=1}^{n}(f_i \wedge m(A_i))$$

,

where $A_i = x_i, x_{i+1}, ..., x_n, i = 1, ..., n$ and $A_{n+1} = \phi$.

Moreover, as the membership function (μ) and non-membership function (ν) are of opposite nature. Thus, the only α-cut is not sufficient to deal with both the component functions simultaneously (see Fig. 2). Thus, by utilizing α-cut on μ and β-cut on ν we defined a novel similarity measure between two A-IFSs A and B.

For any two A-IFSs A and B, where;

$$A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$$

$$B = \{(x, \mu_B(x), \nu_B(x)); x \in X\}$$

Let us define the following sets;

$$C_\mu(A,B) = \{(x, |\mu_A(x) - \mu_B(x)|, x \in X\} \quad (1)$$

$$C_\nu(A,B) = \{(x, |\nu_A(x) - \nu_B(x)|, x \in X\} \quad (2)$$

Now, us define two Sugeno integral $E(C_\mu(A,B))$ and $E(C_\nu(A,B))$ with respect to (α, β)-cut as follows:

$$E(C_\mu(A,B)) = \frac{1}{n}\sum_{k_1=1}^{n} K_{\alpha_i} \int_{C_{\alpha_i \mu}(A,B)} C_\mu(A,B) dm \quad (3)$$

here, K_{α_i} is associated with the α-cut on the membership function μ (in discrete case, K_{α_i} is the number of $x \in C_\mu(A,B)$ such that $\mu_{C_\mu(A,B)(x)} = \alpha_i$).

and

$$E(C_\nu(A,B)) = \frac{1}{n}\sum_{k_2=1}^{n} K_{\beta_i} \int_{C_{\beta_i \nu}(A,B)} C_\nu(A,B) dm \quad (4)$$

here, K_{β_i} is associated with the β-cut on the non-membership function ν (in discrete case K_{β_i} is the number of $x \in C_\nu(A,B)$ such that $\nu_{C_\nu(A,B)(x)} = \beta_i$).

The Sugeno integral $E(C_\mu(A,B))$ can also be seen as expected total difference between A-IFSs A and B associated with α-cut based upon membership values μ_A and μ_B. And, Sugeno integral $E(C_\nu(A,B))$ seen as the expected total difference between A-IFSs A and B associated with β-cut based upon non-membership values ν_A and ν_B.

Now, on the basis of the two Sugeno integrals; $E(C_\mu(A,B))$ and $E(C_\nu(A,B))$, the new Sugeno integral based intuitionistic fuzzy similarity measure between two A-IFSs A and B induced by (α, β)-cuts, we define as follows:

$$S_{new}^{(\alpha,\beta)}(A,B) = exp\bigg(- \frac{E(C_\mu(A,B)) + E(C_\nu(A,B))}{2}\bigg) \quad (5)$$

here, α is associated with α-cut on μ and β is associated with β-cut on ν.

Theorem 3.1: $S_{new}^{(\alpha,\beta)}$ is an intuitionistic fuzzy similarity measure between two A-IFSs induced by (α, β)-cuts.

Proof 3.1: To proof $S_{new}^{(\alpha,\beta)}$ is a similarity measure, we have to prove all (1)-(4) conditions of definition-2.6 as follows:

(1) $0 \leq S_{new}^{(\alpha,\beta)} \leq 1$ (obvious property by the definition).

(2) $S_{new}^{(\alpha,\beta)}(A,B) = 0 \;\Leftrightarrow A = B$.
Now, let
$S_{new}^{(\alpha,\beta)}(A,B) = 0 \Leftrightarrow E(C_\mu(A,B)), E(C_\nu(A,B)) = 0$

$\Leftrightarrow$ each element in $E(C_\mu(A,B))$ and $E(C_\nu(A,B))$ are zero
$\Leftrightarrow$ $|\mu_A(x) - \mu_B(x)| = |\nu_A(x) - \nu_B(x)| = 0$ for each $x \in X \Leftrightarrow \quad A = B$.

(3) $S_{new}^{(\alpha,\beta)}(A,B) = S_{new}^{(\alpha,\beta)}(B,A)$ (obvious property by the definition).

(4) $A \subseteq B \subseteq C$, then $S_{new}^{(\alpha,\beta)}(A,C) \leq S_{new}^{(\alpha,\beta)}(A,B)$, $S_{new}^{(\alpha,\beta)}(A,C) \leq S_{new}^{(\alpha,\beta)}(B,C)$.

Since, $A \subseteq B \subseteq C$. Then, for each $x \in X$ and for each (α, β)-cut, we have;

$\mu_A(x) \leq \mu_B(x) \leq \mu_C(x)$ and $\nu_A(x) \geq \nu_B(x) \geq \nu_C(x)$.

$$\Rightarrow \quad |\mu_A(x) - \mu_B(x)| \leq |\mu_A(x) - \mu_C(x)|$$

and

$$|\nu_A(x) - \nu_B(x)| \leq |\nu_A(x) - \nu_C(x)|$$

$$\Rightarrow C_\mu(A,B) \subseteq C_\mu(A,C), \quad \alpha \in [0,1]$$

$$C_\nu(A,B) \subseteq C_\nu(A,C), \quad \beta \in [0,1]$$

$\Rightarrow$

$$E(C_\mu(A,B)) = \tfrac{1}{n}\sum_{k_1=1}^{n} K_{\alpha_i} \int_{C_{\alpha_i \mu}(A,B)} C_\mu(A,B) dm$$

by using definition of Sugeno integral, we have;

$$= \frac{1}{n}\sum_{k_1=1}^{n} K_{\alpha_i} \sup_{0\le\delta\le1}\Big(\delta \wedge m(C_{\alpha_i\mu}(A,B) \cap C_{\delta\mu}(A,B))\Big)$$

(here δ is the α cut on the membership function μ such that the quantity $\delta \wedge m(C_{\alpha_i\mu}(A,B) \cap C_{\delta\mu}(A,B)$ attended their suprima)

$$\le \frac{1}{n}\sum_{k_1=1}^{n} K_{\alpha_i} \sup_{0\le\delta\le1}\Big(\delta \wedge m(C_{\alpha_i\mu}(A,C) \cap C_{\delta\mu}(A,C))\Big)$$
$$= \frac{1}{n}\sum_{k_1=1}^{n} K_{\alpha_i}\int_{C_{\alpha_i\mu}(A,C)} C_\mu(A,C)dm$$
$$= E(C_\mu(A,C))$$

$$\Rightarrow E(C_\mu(A,B)) \le E(C_\mu(A,C))$$

In similar fashion,

$$E(C_\nu(A,B)) = \frac{1}{n}\sum_{k_1=1}^{n} K_{\beta_i}\int_{C_{\beta_i\nu}(A,B)} C_\nu(A,B)dm$$

Again by using definition of Sugeno integral, we have;

$$= \frac{1}{n}\sum_{k_1=1}^{n} K_{\beta_i} \inf_{0\le\delta\le1}\Big(\delta \wedge m(C_{\beta_i\nu}(A,B) \cap C_{\delta\nu}(A,B))\Big)$$

(here in above case δ is associated with β-cut on the non-membership function ν. Moreover, as the nature of the non-membership function ν is concave, so the quantity $\delta \wedge m(C_{\beta_i\nu}(A,B) \cap C_{\delta\nu}(A,B)$ will attend their suprima at the lowest level of δ (see figure I)

$$\le \frac{1}{n}\sum_{k_1=1}^{n} K_{\beta_i} \inf_{0\le\delta\le1}\Big(\delta \wedge m(C_{\beta_i\nu}(A,C) \cap C_{\delta\nu}(A,C))\Big)$$
$$= \frac{1}{n}\sum_{k_1=1}^{n} K_{\beta_i}\int_{C_{\beta_i\nu}(A,B)} C_\nu(A,C)dm$$
$$= E(C_\nu(A,C))$$

similarly for each (α,β)-cuts, we can easily prove $E(C_\mu(B,C)) \le E(C_\mu(A,C))$, $E(C_\nu(A,B)) \le E(C_\nu(A,C))$ and $E(C_\nu(B,C)) \le E(C_\nu(A,C))$

Therefore, by using Euq. 5, for $A \subseteq B \subseteq C$, we have;

$S_{new}^{(\alpha,\beta)}(A,C) \le S_{new}^{(\alpha,\beta)}(A,B)$ and $S_{new}^{(\alpha,\beta)}(A,C) \le S_{new}^{(\alpha,\beta)}(B,C)$.

Thus, $S_{new}^{(\alpha,\beta)}$ is an intuitionistic fuzzy similarity measure between two A-IFSs A and B induced by (α,β)-cuts.

Algorithm 1 Steps of $S_{new}^{(\alpha,\beta)}$ similarity measure

Input: Given two AIFSs

$$A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$$
$$B = \{(x, \mu_B(x), \nu_B(x)); x \in X\}$$

Computation of the set C_μ and C_ν:

$$C_\mu(A,B) = \{(x, |\mu_A(x) - \mu_B(x)|, x \in X\}$$
$$C_\nu(A,B) = \{(x, |\nu_A(x) - \nu_B(x)|, x \in X\}$$

Expected total sum of membership by using α-cut:

$$E(C_\mu(A,B))$$

Expected total sum of non-membership by using β-cut:

$$E(C_\nu(A,B))$$

Computation of $S_{new}^{(\alpha,\beta)}$ similarity:

$$S_{new}^{(\alpha,\beta)}(A,B) = exp\Big(-\frac{E(C_\mu(A,B)) + E(C_\nu(A,B))}{2}\Big)$$

Example: Let us include one artificially constructed example to understand the working of the proposed similarity measure which as follows:

Let $X = \{x_1, x_2, x_3\}$ be the universe of discourse and A and B are the three A-IFSs given as follows:

$$A = \{(0.1, 0.4), (0.2, 0.5), (0.4, 0.5)\}$$

$$B = \{(0.2, 0.3), (0.4, 0.4), (0.3, 0.1)\}$$

Then, we calculated $C_\mu(A,B)$ and $C_\nu(A,B)$ as follows:

$$C_\mu(A,B) = \{0.1, 0.2, 0.1\}$$

$$C_\nu(A,B) = \{0.1, 0.1, 0.4\}$$

Now, by applying Algorithm 1, we have;

$$S_{new}^{(\alpha,\beta)}(A,B) = exp\Big(-\frac{0.2 + 0.1}{2}\Big) = exp(-0.3/2) = 0.861$$

On the other hand, we also calculated similarity degree between A and B by using similarity measure S_{new} proposed by Hwang et al. [4] by using only α-cut on both the component function μ and ν given as;

$$S_{new}(A,B) = 0.741$$

This implies, $S_{new}^{(\alpha,\beta)}(A,B) \neq S_{new}(A,B)$ this validates that our proposed measure is new and different than to similarity measure proposed by Hwang et al. [4].

TABLE I
A COMPARATIVE OF THE PATTERN RECOGNITION PROBLEM DISCUSSED IN EXAMPLE 4.1 WITH $S_{new}^{(\alpha,\beta)}$ AND THE OTHER WELL KNOWN MEASURES.

SM	$S(P_1,Q)$	$S(P_2,Q)$	$S(P_3,Q)$	Output
S_{DC}	1.000	1.000	1.000	**Not Classify**
S_{LS1}	1.000	0.933	0.800	P_1
S_{LS2}	1.000	0.967	0.900	P_1
S_{LS3}	1.000	0.956	0.867	P_1
S_M	1.000	0.933	0.800	P_1
S_{HY1}^1	1.000	0.933	0.800	P_1
S_{HY2}^1	1.000	0.898	0.713	P_1
S_{HY3}^1	1.000	0.875	0.667	P_1
S_{new}	1.000	0.936	0.741	P_1
$S_{new}^{(\alpha,\beta)}$	1.000	0.906	0.8187	P_1

Bold entities means unreasonable results .

TABLE II
A COMPARATIVE OF THE PATTERN RECOGNITION PROBLEM DISCUSSED IN EXAMPLE 4.2 WITH $S_{new}^{(\alpha,\beta)}$ AND THE OTHER WELL KNOWN MEASURES.

SM	$S(P_1,Q)$	$S(P_2,Q)$	$S(P_3,Q)$	Output
S_{DC}	1.000	1.000	0.600	**Not Classify**
S_{LS1}	0.833	0.933	0.600	P_2
S_{LS2}	0.917	0.967	0.600	P_2
S_{LS3}	0.899	0.956	0.722	P_2
S_M	0.833	0.933	0.600	P_2
S_{HY1}^1	0.833	0.933	0.567	P_2
S_{HY2}^1	0.757	0.898	0.444	P_2
S_{HY3}^1	0.714	0.875	0.395	P_2
S_{new}	0.741	0.936	0.632	P_2
$S_{new}^{(\alpha,\beta)}$	0.8187	0.936	0.7047	P_2

Bold entities means unreasonable results .

IV. VERIFICATION AND COMPARISON OF $S_{new}^{(\alpha,\beta)}$ SIMILARITY MEASURE ON SOME BENCH MARK EXAMPLES

In this section, for the unbiased and proper justification of the proposed similarity measure $S_{new}^{(\alpha,\beta)}$, we have selected two more examples of pattern recognition. These examples have been chosen as a bench mark data by the researchers to examine the similarity measures. Simultaneously, for each examples, we compared the outcomes of our similarity measure with the other well known similarity measures S_{DC} ([24]), S_{LS1} ([25]), S_{LS2} ([25]), S_{LS3} ([25]), S_M ([26]), S_{HY1}^1 ([27]), S_{HY2}^1 ([27]), S_{HY3}^1 ([27]), S_{new}([4]) (due to space constrain, for the detail explanation and functional representation of these similarity measures please visit [20]), and shown that the proposed similarity measure is genuine.

Example 4.1: In this example, borrowed from [4] we have three given patterns P_1, P_2 and P_3 in terms of A-IFSs in the universe of discourse $X = \{x_1, x_2, x_3, x_4\}$ as follows:

$$P_1 = \{(x_1, 0.3, 0.3), (x_2, 0.2, 0.2), (x_3, 0.1, 0.1)\}$$

$$P_2 = \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\}$$

$$P_3 = \{(x_1, 0.4, 0.4), (x_2, 0.4, 0.4), (x_3, 0.4, 0.4)\}$$

On the basis of these patterns P_1, P_2 and P_3, we need to identify the unknown A-IFS Q in $X = \{x_1, x_2, x_3, x_4\}$ where:

$$Q = \{(x_1, 0.3, 0.3), (x_2, 0.2, 0.2), (x_3, 0.1, 0.1)\}$$

In order to solve this example, we evaluate the similarities between AIFSs (P_1, P_2, P_3) and Q by using our proposed similarity measure $S_{new}^{(\alpha,\beta)}$ and by other well known similarity measures. The similarity degree with respect to each measure is given in Table-I. By seeing the data, it is clear that $P_1 = Q$. The proposed similarity measure $S_{new}^{(\alpha,\beta)}$ along with S_{LS1}, S_{LS2}, S_{LS3}, S_M, S_{HY1}^1, S_{HY2}^1, S_{HY3}^1, S_{new} claiming the same output. But, S_{DC} measure failed to identify the patterns. Hence, the unknown pattern Q is classified into the pattern P_1 as the values of $S_{new}^{(\alpha,\beta)}(P_1, Q)$ are the largest.

Example 4.2: In this example, as given in [4], we have three given patterns P_1, P_2 and P_3 in $X = \{x_1, x_2, x_3, x_4\}$ as follows:

$$P_1 = \{(x_1, 0.1, 0.1), (x_2, 0.5, 0.1), (x_3, 0.1, 0.9)\}$$

$$P_2 = \{(x_1, 0.5, 0.5), (x_2, 0.7, 0.3), (x_3, 0.0, 0.8)\}$$

$$P_3 = \{(x_1, 0.7, 0.2), (x_2, 0.1, 0.8), (x_3, 0.4, 0.4)\}$$

On the basis of these provided patterns P_1, P_2 and P_3, we need to classify the unknown A-IFS Q in $X = \{x_1, x_2, x_3, x_4\}$ where Q is given as follows:

$$Q = \{(x_1, 0.4, 0.4), (x_2, 0.6, 0.2), (x_3, 0.0, 0.8)\}$$

To solve this example, we calculated the similarity of Q with the AIFSs (P_1, P_2, P_3) by using the our proposed similarity measure $S_{new}^{(\alpha,\beta)}$ and by other similarity measures. The outcomes of the all respective similarity degrees with respect to each measure is given in Table-II. Now, by usual observation, it is clear that the unknown pattern Q is more similar to P_2 than to other known patterns. The other measures S_{LS1}, S_{LS2}, S_{LS3}, S_M, S_{HY1}^1, S_{HY2}^1, S_{HY3}^1, S_{new} claiming the same output results. But, the similarity measure S_{DC} fails to recognize the patterns. Hence, the unknown pattern Q is classified into the pattern P_2 as the values of $S_{new}^{(\alpha,\beta)}(P_1, Q)$ is the largest.

Example 4.3: In This example, as given in [4], we have two given patterns P_1 and P_2 in the universe of discourse $X = \{x_1, x_2, x_3, x_4\}$ as follows:

$$P_1 = \{(x_1, 0.1, 0.4), (x_2, 0.4, 0.3), (x_3, 0.2, 0.1)\}$$

$$P_2 = \{(x_1, 0.3, 0.4), (x_2, 0.3, 0.4), (x_3, 0.1, 0.1)\}$$

On the basis of the patterns P_1 and P_2, we have to classify the unknown A-IFS pattern Q in $X = \{x_1, x_2, x_3, x_4\}$ where Q is given as follows:

$$Q = \{(x_1, 0.2, 0.2), (x_2, 0.2, 0.2), (x_3, 0.2, 0.2)\}$$

Now, by using Equ. 1 and 2, we have;

$$C_{1\mu} = \{0.1, 0.2, 0.1\}, \; C_{1\nu} = \{0.2, 0.1, 0.1\}$$

$$C_{2\mu} = \{0.1, 0.1, 0.1\}, \; C_{2\nu} = \{0.2, 0.2, 0.1\}$$

TABLE III
A COMPARATIVE OF THE PATTERN RECOGNITION PROBLEM DISCUSSED IN EXAMPLE 4.3 WITH $S_{new}^{(\alpha,\beta)}$ AND THE OTHER WELL KNOWN MEASURES.

SM	$S(P_1, Q)$	$S(P_2, Q)$	Output
S_{DC}	0.758	0.888	**Not Classify**
S_{LS1}	0.867	0.867	**Not Classify**
S_{LS2}	0.875	0.911	P_2
S_{LS3}	0.744	0.845	P_2
S_M	0.833	0.933	P_2
S^1_{HY1}	0.833	0.833	**Not Classify**
S^1_{HY2}	0.653	0.653	**Not Classify**
S^1_{HY3}	0.641	0.641	P_1
S_{new}	0.818	0.861	P_2
$S_{new}^{(\alpha,\beta)}$	0.985	0.990	P_2

Bold entities means unreasonable results .

A comparative analysis of example 4.3 with the proposed (α, β)-cuts based similarity measure $S_{new}^{(\alpha,\beta)}$ is given in Table-III. In Table-III, the measure $S_{new}^{(\alpha,\beta)}$ along with the similarity measures S_{LS2}, S_{LS3}, S_M, S_{new} classifies the pattern Q in to the pattern P_2. But, the similarity measures S_{DC}, S_{LS1}, S^1_{HY1}, S^1_{HY2}, S^1_{HY3} are either fails to identify or incorrectly identify the unknown pattern Q.

V. CONCLUSION

Atanassov's intuitionistic fuzzy set (A-IFS) is the general form of Zadeh's fuzzy set, which is used to deal with uncertainty more precisely. The similarity measure is a tool to differentiate two objects based on the given information. The Sugeno integral based similarity measure is highly appreciated in applied mathematics, statistics, computer science, etc. Therefore in this paper, we proposed a novel similarity measure by using Sugeno integral and (α, β)-cut and studied its mathematical properties. Moreover, some benchmark examples we borrowed from already published papers for verification and comparison purposes. The outcomes of the proposed measure $S_{new}^{(\alpha,\beta)}$ is either equivalent to existing similarity measure or provided better results.

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