Process of Inversion in Fuzzy Interpolation Model using Fuzzy Geometry

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***Abstract*—Fuzzy rule interpolation (FRI) predicts an accountable outcome of a possible course of action in sparse fuzzy rule base system (FRBS). However, in real life, we encounter some situations where the antecedent has to be predicted to obtain a desired consequent of FRBS. In this situation, inverse fuzzy rule interpolation (IFRI) or backward fuzzy rule interpolation (BFRI) is used to get the desired outcome. Here a geometry based inverse fuzzy rule base interpolation (GIFRI) is suggested. The mathematical detail of the proposed method is elaborated and its geometrical interpretation is given with the help of fuzzy geometry. It is to be noted that the proposed method ensures that the inverse of the inverse is the original one.**

***Index Terms*—Fuzzy rule base interpolation, Inverse rule base interpolation, backward rule base interpolation, Transformation of fuzzy point, Multi-dimensional rule base interpolation**

I. Introduction

Fuzzy Rule Interpolation (FRI) in a sparse fuzzy rule base system was introduced by K,ócy an Hirota (KH) [1] in 1993. In the subsequent years, the KH method has been improved and generalized by several researchers in [2]–[5]. The existing methodologies on FRI are mainly divided in two groups. The methodologies which belong to the first group [6]–[12], deduce the conclusion directly from the given rule base whereas the methodologies of second group obtain the conclusion in two steps. In the first step, an auxiliary rule is obtained from the given rule base and the conclusion is drawn using that auxiliary rule.

Solid cutting method [15], [16], fixed point law (FPL), fixed value law (FVL) [17]–[19], least square method (LS) [13] and polar cut method (PC) [14] are among few of the methodologies which belong to second group.

In literature, only two type of inverse fuzzy rule interpolation (IFRI) or backward fuzzy rule interpolation (BFRI) methods exist. Baranyi et al. [22] were first to propose a two steps inverse interpolation process. In the first step, an inverse rule base (IRB) is constructed from the given rule base. In the second step, required missing antecedent is obtained by using the driven IRB. Backward fuzzy rule interpolation (BFRI) has been proposed by Jin et al. [20], [21] based on scale and move transformation.

To describe the method proposed by Baranyi et al. [22], let us consider a rule base with two antecedents (x_1, x_2) and one consequent y which is given in form of Table I. We have to predict the missing antecedent x_1 to obtain a desired consequent $y = b^*$ for given $x_2 = a_2^*$. In the first step, an auxiliary inverse rule base (AIRB) of the given rule base (Table I) is obtained as Table II with the assumption that x_1 is the missing antecedent. This AIRB consists of eight rules with four unknown antecedents a_{m1}, a_{m2}, a_{m3} and a_{m4}. The antecedents a_{m1}, a_{m2} can obtained by interpolating the inverse model but a_{m3} and a_{m4} have to obtained by extrapolation. From the geometry of the AIRB it can be concluded that the AIRB can be effectively reduced to inverse rule base (IRB) which contain only four rules (see Table III) where $b_{min} = min\{b_1, b_2, b_3, b_4\}$ and $b_{max} = max\{b_1, b_2, b_3, b_4\}$. In the second step, the required antecedent $x_1 = a_1^*$ for the desired $y = b^*$ and given $x_2 = a_2^*$ can obtained from the reduced inverse rule base presented in Table III by using any interpolation technique. The same approach is extended to obtain the inverse rule base for multi-antecedents case also. The method illustrated above has considered that each of the

TABLE I
Forward Rule Base

x_2	$x_1 = a_{11}$	$x_1 = a_{12}$
a_{21}	b_1	b_2
a_{22}	b_3	b_4

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TABLE II
INVERSE RULE BASE

x_2	$y = b_1$	$y = b_2$	$y = b_3$	$y = b_4$
a_{21}	a_{11}	a_{12}	a_{m1}	a_{m2}
a_{22}	a_{m3}	a_{m4}	a_{11}	a_{12}

TABLE III
REDUCED INVERSE RULE BASE

x_2	$y = b_{min}$	$y = b_{max}$
a_{21}	d_1	d_2
a_{22}	d_3	d_4

antecedents x_i have only two possible values a_{i1} and a_{i2}. But, when the antecedent x_i has n number of possible values then the given rule base has to divided into $(n-1)$ sub-rule bases and for each sub-rule base we have find the inverse rule base. In this case, this method faces a very high computational difficulties.

The method proposed by Jin et al. [20], [21] is based on scale and move transformations of fuzzy numbers. In this method, the antecedents and consequent are represented by their generalized representative value and spread lengths. This is also a two stage technique. In the first step, an auxiliary rule is obtained from the given rule base and the observation. In the second step, the required conclusion is obtained from the auxiliary rule which goes through necessary scale and move transformation.

Suppose the rule base $\mathbf{R} = \{\widetilde{R}_i\}$ and observation $\widetilde{O}$ are given as follows:

$$\widetilde{R}_i : \text{If } x_k \text{ is } \widetilde{A}_k^i,\ k \in \{1, 2, \dots M\} \text{ then } y \text{ is } \widetilde{B}^i \tag{1}$$

$$\widetilde{O} : x_1 = \widetilde{A}_1^*, \dots x_l = ?, \dots x_M = \widetilde{A}_M^* \text{ then } y \text{ is } \widetilde{B}^* \tag{2}$$

For simplicity let us consider that each $\widetilde{A}_k$ is triangular fuzzy set $\widetilde{A}_k(a_{k0}, a_{k1}, a_{k2})$. After obtaining the auxiliary rule $\widetilde{R}(\widetilde{A}_1, \widetilde{A}_2, ...\widetilde{A}_M, \widetilde{B})$, the measure of scale transformation between $\widetilde{A}_k$ and $\widetilde{A}_k^*$ is given as:

$$s_{A_k} = \frac{a_{k2}^* - a_{k0}^*}{a'_{k2} - a'_{k0}} \tag{3}$$

The unknown parameter s_{A_l} is then obtained from the relation given as follows:

$$s_{A_l} = M \times s_B - \sum_{k=1, k\neq l}^{M} s_{A_k} \tag{4}$$

Using these scale measures s_k, we obtain other rule $\widetilde{R}^\dagger(\widetilde{A}_1^\dagger, \widetilde{A}_2^\dagger, ...\widetilde{A}_M^\dagger, \widetilde{B}^\dagger)$ from which we obtain the required $\widetilde{A}_l^*$ using move transformation as follows. Here $\widetilde{A}_k^\dagger$ is given as

$$\begin{cases} a_{k0}^\dagger = \frac{a'_{k0}(1+2s_{A_k}) + a'_{k0}(1-s_{A_k}) + a'_{k0}(1-s_{A_k})}{3} \\ a_{k1}^\dagger = \frac{a'_{k0}(1-s_{A_k}) + a'_{k0}(1+2s_{A_k}) + a'_{k0}(1-s_{A_k})}{3} \\ a_{k2}^\dagger = \frac{a'_{k0}(1-s_{A_k}) + a'_{k0}(1-s_{A_k}) + a'_{k0}(1+2s_{A_k})}{3} \end{cases} \tag{5}$$

The move transformation parameters m_{A_k} between $\widetilde{A}_k$ and $\widetilde{A}_k^*$ is given as follows

$$m_{A_k} = \begin{cases} \frac{3(a_{k0}^* - a_{k0}^\dagger)}{a_{k1}^\dagger - a_{k0}^\dagger} & \text{if } a_{k0}^* \geq a_{k0}^\dagger \\ \frac{3(a_{k0}^* - a_{k0}^\dagger)}{a_{k2}^\dagger - a_{k1}^\dagger} & \text{otherwise} \end{cases} \tag{6}$$

The unknown parameter m_{A_l} is then obtained from the relation given as follows

$$m_{A_l} = M \times m_B - \sum_{k=1, k\neq l}^{M} m_{A_k} \tag{7}$$

Then the required antecedent $\widetilde{A}_l^*$ is obtained through move transformation of $\widetilde{A}_l^\dagger$ with parameter m_{A_l}.

We can see that in Equations (4) and (7) the factor M is multiplied in the consequent dimension y which gives a biased emphasis on the consequent dimension without any valid geometrical significance.

Geometry based linear fuzzy rule base interpolation (GLFRI) is proposed by Das et al. [24] which associates the FRI technique to classical interpolation technique with a complete geometrical interpretation. In the present study, GLFRI is generalized for multi-antecedent case which is named as generalized geometry based linear fuzzy rule base interpolation (GGLFRI). Also, a technique for inverse-GGLFRI (IGGLFRI) is proposed. The geometrical and analytical interpretation of proposed inverse-GGLFRI (IGGLFRI) is in the same line of GGLFRI. In IGGLFRI, the resultant rule $\widetilde{R}$ is projected to the unknown antecedent axis x_m (say) to obtain a resultant prediction $\widetilde{A}_m$ of the required antecedent x_m. Then required antecedent A_m^* obtained from $\widetilde{A}_m$ through some geometrical transformations [23].

There are a few advantages in using the proposed GGLFRI and IGGLFRI. The methods are complement of each other in the sense that the conclusion $\widetilde{B}^*$ obtained from GGLFRI can also be used as a desired output in IGGLFRI. That is if $\widetilde{B}^*$ is output of GGLFRI for the observation $(\widetilde{A}_1^*, \widetilde{A}_2^*, ...\widetilde{A}_N^*)$ then $\widetilde{A}_m^*$ can also be obtained as output of IGGLFRI corresponding to the input $(\widetilde{A}_1^*, \widetilde{A}_2^*, ...\widetilde{A}_{m-1}^*, ?, \widetilde{A}_{m+1}^*, ...\widetilde{A}_N^*, \widetilde{B}^*)$. Also, it is to be noted that the inverse interpolation technique generates unique antecedent and there is no need to obtain auxiliary rule base in the process.

In the next section, a few basic definitions on fuzzy points which are related to the proposed method are given. The proposed forward and inverse methods are described simultaneously in two different stages. In the first stage, the proposed method is described for single input and single output rule base system which is given in section III. The proposed method is generalized for multiple inputs and single output case in section IV. Finally, section V concludes our work.

II. Preliminaries

Definition 2.1: **(Fuzzy Points [23]):** A fuzzy set $\widetilde{P}(a,b)$ at $(a,b) \in \mathbb{R}^2$ is called a fuzzy point (FP) if its membership function $\mu((x,y) \mid \widetilde{A})$ follows the properties:

1) $\mu((x,y) \mid \widetilde{P})$ is upper semi-continuous,
2) $\mu((x,y) \mid \widetilde{P} = 1$ if and only if $(x,y) = (a,b)$, and
3) the *alpha*-cut $\widetilde{P}(\alpha)$ is a compact and convex subset of $\mathbb{R}^2$, for all $\alpha \in [0,1]$.

Note 1: Support $\widetilde{P}(0)$ of the FP $\widetilde{P}(a,b)$ can be represented as: $\widetilde{P}(0) = \cup_{\substack{\alpha\in[0,1],\\ \theta\in[0,2\pi]}} \{P^{\alpha,\,\theta}\}$ where $\mu(P^{\alpha,\,\theta} \mid \widetilde{P}) = \alpha$ and θ is the angle between the lines through (a,b) parallel to x-axis and the line joining $P^{\alpha,\,\theta}$ and $P(a,\ b)$.

Definition 2.2: **(Same and Inverse Points of FP [23]):** Let $\widetilde{P}_1(a,b)$, $\widetilde{P}_2(c,d)$ be two fuzzy points at $\widetilde{P}_1(a,b)$ and $\widetilde{P}_2(c,d)$ respectively. Then $P_1^{\alpha,\,\theta}$ and $P_2^{\alpha,\,\theta}$ are same points of $\widetilde{P}_1$ and $\widetilde{P}_2$ where $\widetilde{P}_1(0) = \cup_{\substack{\alpha\in[0,1],\\ \theta\in[0,2\pi]}} \{P_1^{\alpha,\,\theta}\}$ and $\widetilde{P}_2(0) = \cup_{\substack{\alpha\in[0,1],\\ \theta\in[0,2\pi]}} \{P_2^{\alpha,\,\theta}\}$.

Also, $P_1^{\alpha,\,\theta}$ and $P_2^{\alpha,\,\theta+\pi}$ are inverse points of $\widetilde{P}_1$ and $\widetilde{P}_2$.

Definition 2.3: **(Fuzzy Line Segment [23]):** Fuzzy line segment (FLS) $\widetilde{L}_{P_1P_2}$ joining two fuzzy points $\widetilde{P}_1$ and $\widetilde{P}_2$ can be defined by its membership function as:

$$\begin{aligned}\mu((x,y) \mid \widetilde{L}_{P_1P_2}) = sup\{\alpha :\ &\text{where } (x,y) \text{ lies on the line}\\ &\text{joining same points } u \in \widetilde{P}_1(0) \text{ and } v \in \widetilde{P}_2(0)\\ &\text{where } \mu(u \mid \widetilde{P}_1) = \mu(v \mid \widetilde{P}_2) = \alpha\ \}\end{aligned} \tag{8}$$

In other words we can describe $\widetilde{L}_{P_1P_2}$ as:

$$\begin{aligned}\widetilde{L}_{P_1P_2} = \cup_{\substack{\alpha\in[0,1],\\ \theta\in[0,2\pi]}} \{l^{\alpha,\,\theta} :\ &\text{where } l^{\alpha,\,\theta} \text{ is the line joining}\\ &P_1^{\alpha,\,\theta} \text{ and } P_2^{\alpha,\,\theta} \text{ with } \mu(l^{\alpha,\,\theta} \mid \widetilde{L}_{P_1P_2}) = \alpha\ \}\end{aligned} \tag{9}$$

Expansion and contraction of fuzzy point is defined in the following. The expression $\mu((x,y) \mid \widetilde{Q}) = f(x-a, y-b)$ of the membership function $\mu((x,y) \mid \widetilde{Q})$ of a FP $\widetilde{Q}(a,b)$ is considered to define expansion/contraction of FP.

Definition 2.4: **Expansion/Contraction of FP [24]:** Expansion/Contraction of $\widetilde{Q}(a,b)$ with membership function $\mu((x,y) \mid \widetilde{Q}) = f(x-a, y-b)$ by a set parameters $t = \{t_1, t_2 ... t_m\}$ and $s = \{s_1, s_2 ... s_m\}$, where $t_i, s_i \geq 0$ in m different regions $D = \{D_1, D_2 ... D_m\}$ is defined in the following way:

$$((x,y) \mid \widetilde{Q}') = \begin{cases} f(\frac{x-a}{t_i}, \frac{y-b}{s_i}), \text{if } (x,y) \in D_i \text{ and } t_i, s_i > 0\\ 1, \text{if } (x,y) = (a,b) \in D_i,\ t_i = 0 \text{ and/or } s_i = 0\\ 0, \text{if } (a,b) \neq (x,y) \in D_i,\ t_i = 0 \text{ and/or } s_i = 0\\ f(x-a, y-b),\ \text{elsewhere}\end{cases} \tag{10}$$

Example 2.1: Let us consider the fuzzy point $\widetilde{Q}(5,5)$ with membership function defined as follows:

$$\mu((x,y) \mid \widetilde{Q}) = \begin{cases} 1 - \sqrt{(\frac{x-5}{2})^2 + (\frac{y-5}{2})^2}, \text{if } (\frac{x-5}{2})^2 + (\frac{y-5}{2})^2 \leq 1\\ 0,\ \text{elsewhere}\end{cases}$$

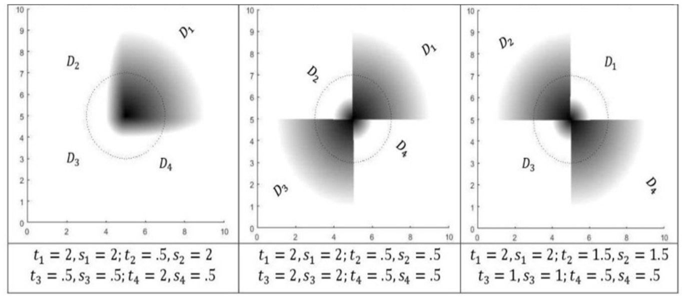

Fig. 1. Expansion/Contraction of Fuzzy Point

Then, the expansion/contraction of $\widetilde{Q}(5,5)$ in regions $D = \{D_1, D_2, D_3, D_4\}$ with set of parameters $t = \{t_1, t_2, t_3, t_4\}$ and $s = \{s_1, s_2, s_3, s_4\}$, where $D_1 = \{(x,y) \mid x \geq 5, y \geq 5\}$, $D_2 = \{(x,y) \mid x \leq 5, y \geq 5\}$, $D_3 = \{(x,y) \mid x \leq 5, y \leq 5\}$ and $D_4 = \{(x,y) \mid x \geq 5, y \leq 5\}$, is defined by its membership function as follows:

$$\mu((x,y) \mid \widetilde{Q}') = \begin{cases} 1 - \sqrt{(\frac{x-5}{2\times t_1})^2 + (\frac{y-5}{2\times s_1})^2},\\ \qquad \text{if } (\frac{x-5}{2\times t_1})^2 + (\frac{y-5}{2\times s_1})^2 \leq 1 \text{ and } (x,y) \in D_1\\ 1 - \sqrt{(\frac{x-5}{2\times t_2})^2 + (\frac{y-5}{2\times s_2})^2},\\ \qquad \text{if } (\frac{x-5}{2\times t_2})^2 + (\frac{y-5}{2\times s_2})^2 \leq 1 \text{ and } (x,y) \in D_2\\ 1 - \sqrt{(\frac{x-5}{2\times t_3})^2 + (\frac{y-5}{2\times s_3})^2},\\ \qquad \text{if } (\frac{x-5}{2\times t_3})^2 + (\frac{y-5}{2\times s_3})^2 \leq 1 \text{ and } (x,y) \in D_3\\ 1 - \sqrt{(\frac{x-5}{2\times t_4})^2 + (\frac{y-5}{2\times s_4})^2},\\ \qquad \text{if } (\frac{x-5}{2\times t_4})^2 + (\frac{y-5}{2\times s_4})^2 \leq 1 \text{ and } (x,y) \in D_4\\ 0, \qquad \text{elsewhere}\end{cases}$$

Few classes are explained through the above diagram (see Figure 1).

III. Proposed Method: Single Input-Single Output

Suppose the given knowledge base $\mathbf{R} = \{\widetilde{R}_i\}$ contains n rules. The rules $\widetilde{R}_i$ are of single input-single output type which are given in the following form:

$$\widetilde{R}_i :\ if\ x = A_i\ then\ y = B_i \tag{11}$$

A. Forward FRI System with Single Input-Single Output:

In GLFRI [24], each rule $\widetilde{R}_i$ is considered as an ordered pair of the antecedent $\widetilde{A}_i$ and consequent $\widetilde{B}_i$, i.e. $\widetilde{R}_i = (\widetilde{A}_i, \widetilde{B}_i)$. Different type of t-norms can be used in this purpose. In this study 'min'-norm (see Equation (12)) is used. Then the rule $\widetilde{R}_i = (\widetilde{A}_i, \widetilde{B}_i)$ represents a fuzzy point in the antecedent-consequent plane. The rules or fuzzy points $\widetilde{R}_i$, $\widetilde{R}_{i+1}$ are joined through their same points to obtain a collection of fuzzy line segments $\widetilde{L}_{i,i+1}$ (see Figure 2 and Equation (8)). Then a resultant rule $\widetilde{R}$ is obtained as convex combination $\widetilde{R} = \frac{q}{p+q}\widetilde{R}_i \oplus \frac{p}{p+q}\widetilde{R}_{i+1}$ of adjacent rules $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$ of the observation $x = \widetilde{A}^*$, corresponding to which we have to find the required consequent $\widetilde{B}^*$ (see Figure 2). Here, the vertical line passing through core

a^* of $\widetilde{A}^*$ intersects the line segment joining the cores of $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$ at ration $p:q$ internally. Next, the resultant rule $\widetilde{R}$ is decomposed in antecedent (x)- consequent(y) axes to obtain the resultant antecedent $\widetilde{A}$ and $\widetilde{B}$ (see Figure 3) respectively. Then the expansion/contraction parameters γ, δ are calculated between $\widetilde{A}$ and $\widetilde{A}^*$ and with the same parameters γ, δ resultant consequent $\widetilde{B}$ is transformed into required conclusion $\widetilde{B}^*$ (see Figure 4). In the following, the method is illustrated with an example.

Example 3.1: Suppose a rule base $\mathbf{R}$ is given by: $\mathbf{R} = \bigcup_{i=1}^{3}\{\widetilde{R}_i : \widetilde{A}_i \rightarrow \widetilde{B}_i\}$, where $\widetilde{A}_1(2,3,4)$, $\widetilde{A}_2(7,8,9)$, $\widetilde{A}_3(13,14,15)$, $\widetilde{B}_1(2,3,4)$, $\widetilde{B}_2(8,9.5,11)$ and $\widetilde{B}_3(13,14,15)$ are triangular fuzzy numbers respectively. Then the membership function of fuzzy rule $\widetilde{R}_i$ is given in following equation:

$$\mu((x,y) \mid \widetilde{R}_i) = min\{\mu(x \mid \widetilde{A}_i), \mu(y \mid \widetilde{B}_i)\},\ i = 1,2,3 \tag{12}$$

If a conclusion $\widetilde{B^*}$ corresponding to an observation $\widetilde{A}^*(9.5, 11, 12.5)$ has to be drawn from the given rule base $\mathbf{R}$, then the following steps are followed.

First a vertical line $l : x = 11$, through the core $a^* = 11$ of $\widetilde{A}^*$ is drawn which intersects the line segment $L_{R_2R_3}$ at $(11, 11.75)$. The point $(11, 11.75)$ divides the line segment $L_{R_2R_3}$ into $p:q = 1:1$ ratio internally.

So, the intermediate rule $\widetilde{R}$ is obtained as a convex combination of the rules $\widetilde{R}_2$ and $\widetilde{R}_3$ with the above mentioned ratio, i.e. $\widetilde{R} = \frac{1}{2}\widetilde{R}_2 \oplus \frac{1}{2}\widetilde{R}_3$ (see Figure 2).

The intermediate antecedent $\widetilde{A}$ obtained from intermediate rule $\widetilde{R}$ is $\widetilde{A}(10, 11, 12)$. But the observation $\widetilde{A}^*(9.5, 11, 12.5)$ is an expanded fuzzy set of $\widetilde{A}(10, 11, 12)$ with parameters $\gamma = \frac{11-9.5}{11-10} = 1.5$ and $\delta = \frac{12.5-11}{12-11} = 1.5$ (see Figure 3).

So, to obtain $\widetilde{R}^*$ the intermediate rule $\widetilde{R}$ is also expanded with set of parameters $t = \{t_1 = 1.5, t_2 = 1.5, t_3 = 1.5, t_4 = 1.5\}$ and $s = \{s_1 = 1.5, s_2 = 1.5, s_3 = 1.5, s_4 = 1.5\}$ in regions $D = \{D_1, D_2, D_3, D_4\}$ respectively, where $D_1 = \{(x,y) \mid x \geq 11, y \geq 11.75\}$, $D_2 = \{(x,y) \mid x \leq 11, y \geq 11.75\}$, $D_3 = \{(x,y) \mid x \leq 11, y \leq 11.75\}$ and $D_4 = \{(x,y) \mid x \geq 11, y \leq 11.75\}$.

Now, for $t_i = s_i,\ i = 1,2,3,4$, $\widetilde{R}^*$ is an uniformly expanded fuzzy rule of $\widetilde{R}$ (see Figure 4). Then the final conclusion $\widetilde{B}^*(9.875, 11.75, 13.625)$ is being drawn from $\widetilde{R}^*$ using following equation:

$$\mu(y \mid \widetilde{B}^*) = \sup_x\{\mu(x, y \mid \widetilde{R}^*)\}$$

B. Inverse FRI System with Single Input-Single Output:

Suppose we need to predict an antecedent $\widetilde{A}^*$ for a desired output $\widetilde{B}^*$ based on the given knowledge base $\mathbf{R}$. In this case also, we propose to apply the same geometry based concept for interpolation which is mentioned above. The advantage of the proposed method is that there is an exact matching of arguments through forward and reverse process of interpolation. The first two steps of inverse interpolation technique are same, *i.e.*, converting rule $\widetilde{R}_i$ to fuzzy point $\widetilde{R}_i$ and joining $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$ as a collection of lines $\widetilde{L}_{i,i+1}$ remain same. In the next step we have to find out in which ratio $p:q$ the core of $\widetilde{B}^*$ situated in between its adjacent rules $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$. Then we find out the resultant rule $\widetilde{R} = (\widetilde{A}, \widetilde{B})$ as a convex combination of the rules $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$ in the similar manner (see figure 5). The resultant $\widetilde{A}$ and $\widetilde{B}$ are obtained by projecting $\widetilde{R}$ in their respective dimension. It might happen that resultant $\widetilde{B}$ is an/a expanded/contracted variation of given $\widetilde{B}^*$ (see figure 6). In the next step, the transformation parameters γ, δ between desired consequent $\widetilde{B}^*$ and resultant consequent $\widetilde{B}$ are obtained. Finally, we obtain the required antecedent $\widetilde{A}^*$ by transforming $\widetilde{A}$ with parameters γ, δ. The same has been illustrated step by step in the following with an example.

Example 3.2: Suppose, we have to predict an antecedent $\widetilde{A}^*$ corresponding to a given output $\widetilde{B}^*(10, 11.5, 13)$ based on the knowledge base $\mathbf{R}$ given in Example 3.1.

Then for inversion of the desired consequent $\widetilde{B}^*(10, 11.5, 13)$ into corresponding antecedent, we first draw a vertical line $l_1 : y = 11.5$, through the core $b^* = 11.5$ of $\widetilde{B}^*$, which intersects the line segment $L_{R_2R_3}$ at $(10.67, 11.5)$.

The point $(10.67, 11.5)$ divides the line segment $L_{R_2R_3}$ into $p:q = 2:2.5$ ratio internally. So, the resultant rule $\widetilde{R}$ is obtained as a convex combination of $\widetilde{R}_2$ and $\widetilde{R}_3$ with the above ratio, i.e. $\widetilde{R} = \frac{2.5}{4.5}\widetilde{R}_2 \oplus \frac{2}{4.5}\widetilde{R}_3$. The intermediate consequent $\widetilde{B}$ is obtained from intermediate rule $\widetilde{R}$ as $\widetilde{B}(10.22, 11.5, 12.78)$.

But the conclusion $\widetilde{B}^*(10, 11.5, 13)$ is a contracted fuzzy set of $\widetilde{B}(10.22, 11.5, 12.78)$ with parameters $\gamma = \frac{11.5-10.22}{11.5-10} = 0.85$ and $\delta = \frac{12.78-11.5}{13-11.5} = 0.85$.

So, to obtain final $\widetilde{R}^*$, the intermediate rule $\widetilde{R}$ is also contracted with the set of parameters $t = \{t_1 = 0.85, t_2 = 0.85, t_3 = 0.85, t_4 = 0.85\}$ and $s = \{s_1 = 0.85, s_2 = 0.85, s_3 = 0.85, s_4 = 0.85\}$ in regions $D = \{D_1, D_2, D_3, D_4\}$ respectively, where $D_1 = \{(x,y) \mid x \geq 10.67, y \geq 11.5\}$, $D_2 = \{(x,y) \mid x \leq 10.67, y \geq 11.5\}$, $D_3 = \{(x,y) \mid x \leq 10.67, y \leq 11.5\}$ and $D_4 = \{(x,y) \mid x \geq 10.67, y \leq 11.5\}$.

Then the final conclusion $\widetilde{A}^*(9.82, 10.67, 11.52)$ is being drawn from $\widetilde{R}^*$ using following equation:

$$\mu(y \mid \widetilde{A}^*) = \sup_y\{\mu(x, y \mid \widetilde{R}^*)\}$$

Note 2: If we consider $\widetilde{B}^* = \widetilde{11.75}$ in the inverse FRI model as given conclusion then we will get the same antecedent $\widetilde{A}^* = \widetilde{11}$ as a result which implies that the above mentioned forward and inverse methods are complementary to each other.

IV. Proposed Method: FRI System with Multiple Inputs and Single Output

Suppose the given knowledge base $\mathbf{R}$ consists of n number of rules. Each rule $\widetilde{R}_i$ have N inputs and single

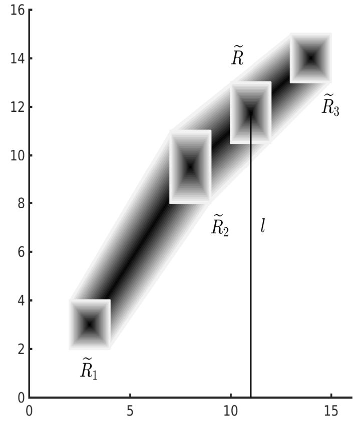

Fig. 2. Resultant rule in FRI

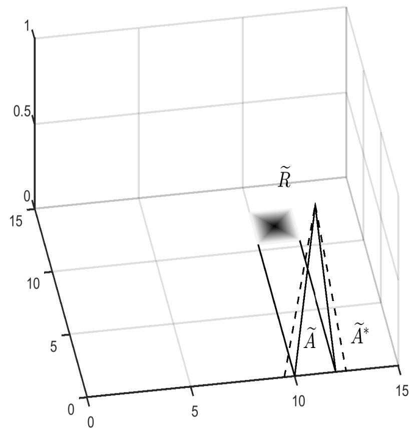

Fig. 3. Observation and resultant rule

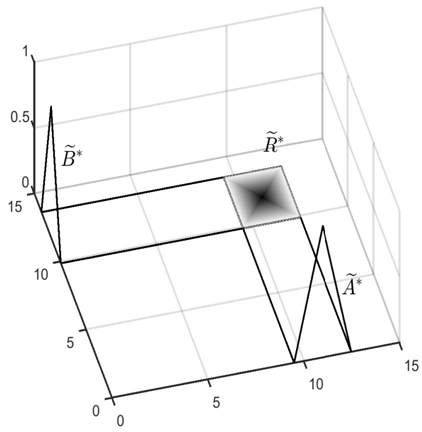

Fig. 4. Obtaining final conclusion in FRI

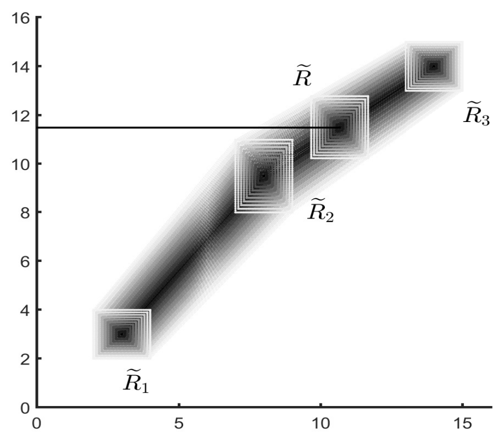

Fig. 5. Intermediate rule for inverse interpolation

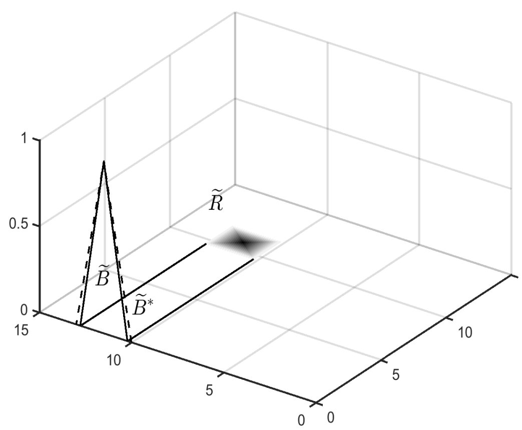

Fig. 6. Observation and resultant rule in inverse interpolation

output which are given as follows:

$$\widetilde{R}_i:\ if\ x_1=\widetilde{A}_{i1},\ x_2=\widetilde{A}_{i2},\dots x_N=\widetilde{A}_{iN}\ then\ y=\widetilde{B}_i \tag{13}$$

The above method GLRFI is generalized for the case of N antecedents and single consequent. We propose to visualize fuzzy rules as ordered pair $\widetilde{R}_i(\widetilde{A}_{i1},\widetilde{A}_{i2},\dots\widetilde{A}_{iN},\widetilde{B}_i)$ of antecedents and consequent. Geometrically, each antecedent and consequent may be considered as individual fuzzy set in Euclidean space along respective antecedent and consequent axes. Then the fuzzy rules can considered as fuzzy points in the antecedents-consequent geometrical space. Different type of $t-norms$ can be taken to construct the fuzzy point. 'min' t-norm is used in this study to perform the cross product of the fuzzy sets for its simplicity and linearity.

A. Forward FRI System with Multiple inputs-Single Output

Suppose we have to find the conclusion $\widetilde{B}^*$ based on the given knowledge base $\mathbf{R}$ corresponding to an observation $\widetilde{A}^*=(\widetilde{A}^*_1,\widetilde{A}^*_2,\dots\widetilde{A}^*_N)$. In general, it is not possible to find two rules $\widetilde{R}_k$ and $\widetilde{R}_{k+1}$ such that the core $(a^*_1,a^*_2,\dots a^*_N)$ of the given observation $\widetilde{A}^*=(\widetilde{A}^*_1,\widetilde{A}^*_2,\dots\widetilde{A}^*_N)$ lies in between the cores $(a_{1k},a_{2k},\dots a_{Nk})$ and $(a_{1k+1},a_{2k+1},\dots a_{Nk+1})$ of $\widetilde{R}_k$ and $\widetilde{R}_{k+1}$ respectively, i.e. $(a_{1k},a_{2k},\dots a_{Nk+1})\le(a^*_1,a^*_2)\le(a_{1k+1},a_{2k+1},\dots a_{Nk+1})$. But it is always possible to find $2N$ number of covering rules $\widetilde{R}_{t_1}$, $\widetilde{R}_{s_1}$; $\widetilde{R}_{t_2}$, $\widetilde{R}_{s_2}\dots\widetilde{R}_{r_N},\widetilde{R}_{s_N}$ such that $a_{1t_1}\le a^*_1\le a_{1s_1}$, $a_{2t_2}\le a^*_2\le a_{2s_2}\dots a_{Nt_N}\le a^*_N\le a_{Ns_N}$.

Geometrically, $\widetilde{R}_{t_1}$, $\widetilde{R}_{s_1}$; $\widetilde{R}_{t_2}$, $\widetilde{R}_{s_2}$;...$\widetilde{R}_{t_N}$, $\widetilde{R}_{s_N}$ are adjacent rules in the $x_1,x_2,\dots x_N$ antecedent dimensions respectively. If we assume that $p_1:q_1$, $p_2:q_2$,...$p_N:q_N$ are the ratios with which the cores a^*_1 and a^*_2,...a^*_N lie in between a_{1t_1},a_{1s_1}; a_{2t_2},a_{2s_2},...a_{Nt_N},a_{Ns_N} respectively, i.e. $\frac{q_i}{p_i+q_i}a_{it_i}+\frac{p_i}{p_i+q_i}a_{is_i}=a^*_i$ for $i=1,2,\dots N$. Then N intermediate rules $\widetilde{R}^*_i$ are obtained as convex combination of the rules $\widetilde{R}_{t_i}$, $\widetilde{R}_{s_i}$ respectively, i.e. $\widetilde{R}^*_i=\frac{q_i}{p_i+q_i}\widetilde{R}_{t_i}\oplus\frac{p_i}{p_i+q_i}\widetilde{R}_{s_i}$, for $i=1,2,\dots N$.

Now it is to be noted that the antecedents $(\widetilde{A}^*_{i1},\widetilde{A}^*_{i2},\dots\widetilde{A}^*_{iN})$ of $\widetilde{R}^*_i$ do not coincide with the given observation $(\widetilde{A}_1^*, \widetilde{A}_2^*, \dots \widetilde{A}_N^*)$. But we need to find the conclusion corresponding to the observation $(\widetilde{A}_1^*, \widetilde{A}_2^*, \dots \widetilde{A}_N^*)$. To achieve this requirement we first find the constants $\lambda_1, \lambda_2, \dots \lambda_N$ such that the core of $(\widetilde{A}_1^*, \widetilde{A}_2^*, \dots \widetilde{A}_N^*)$ coincides with the core of $\sum_{i=1}^N \lambda_i(\widetilde{A}_{i1}^*, \widetilde{A}_{i2}^*, \dots \widetilde{A}_{iN})$, i.e. $(a_1^*, a_2^*, \dots a_N^*) = \sum_{i=1}^N \lambda_i(a_{i1}^*, a_{i2}^*, \dots a_{iN})$.

Then the intermediate rule $\widetilde{R}$ is calculated as $\widetilde{R} = \sum\limits_{i=1}^{N} \lambda_i \widetilde{R}_i^*$.

The intermediate antecedent $\widetilde{A} = (\widetilde{A}_1, \widetilde{A}_2, \dots \widetilde{A}_N)$ and consequent $\widetilde{B}$ are obtained from $\widetilde{R}$ by taking the projection of $\widetilde{R}$ in the axes $x_1, x_2, \dots x_N$ and y respectively. Now it may be seen that A_i^* are expanded/contracted fuzzy numbers of $\widetilde{A}_i$ respectively. The corresponding parameters γ_i, δ_i of expansion/contraction are then calculated. Thus the final conclusion $\widetilde{B}^*$ is obtained by expanding/contracting $\widetilde{B}$ with parameters λ, δ where λ, δ are the weighted convex combination of the parameters γ_i, δ_i with weights λ_i respectively, i.e. $\gamma = \dfrac{\sum_{i=1}^N \lambda_i \gamma_i}{\sum_{i=1}^N \lambda_i}$ and $\delta = \dfrac{\sum\limits_{i=1}^{N} \lambda_i \delta_i}{\sum_{i=1}^N \lambda_i}$.

The proposed method is furthermore explained with the following Algorithm 1 and Example 4.1.

Example 4.1: Suppose a rule base $\mathbf{R}$ is given by: $\mathbf{R} = \bigcup\limits_{i=1}^{8} \{\widetilde{R}_i : \ if \ x_1 = \widetilde{A}_{1i}, \ \ x_2 = \widetilde{A}_{2i}, \ \ x_3 = \widetilde{A}_{3i}, \ \ x_4 = \widetilde{A}_{4i} \ \ then \ \ \widetilde{B}_i\}$, where $\widetilde{R}_1(\widetilde{2}, \widetilde{7}, \widetilde{5}, \widetilde{4}, \widetilde{3})$, $\widetilde{R}_2(\widetilde{6}, \widetilde{10}, \widetilde{8}, \widetilde{6}, \widetilde{6})$, $\widetilde{R}_3(\widetilde{10}, \widetilde{4}, \widetilde{6}, \widetilde{2}, \widetilde{5})$, $\widetilde{R}_4(\widetilde{12}, \widetilde{8}, \widetilde{10}, \widetilde{6}, \widetilde{10})$, $\widetilde{R}_5(\widetilde{7}, \widetilde{8}, \widetilde{7}, \widetilde{5}, \widetilde{6})$, $\widetilde{R}_6(\widetilde{14}, \widetilde{10}, \widetilde{6}, \widetilde{8}, \widetilde{8})$, $\widetilde{R}_7(\widetilde{7}, \widetilde{5}, \widetilde{8}, \widetilde{8}, \widetilde{7})$ and $\widetilde{R}_8(\widetilde{15}, \widetilde{12}, \widetilde{9}, \widetilde{10}, \widetilde{11})$ are cross products of triangular fuzzy numbers with one unit length in each spreads.

Suppose a conclusion corresponding to an observation $\widetilde{A}^*(\widetilde{5}, \widetilde{9.5}, \widetilde{8.5}, \widetilde{5.5})$ has to be drawn from the given rule base $\mathbf{R}$.

The core 5 of $\widetilde{A}_1^* = \widetilde{5}$ lies between the cores 2 and 6 of the first antecedents $\widetilde{A}_{11} = \widetilde{2}$ and $\widetilde{A}_{21} = \widetilde{6}$. So, the adjacent rules of the observation $\widetilde{A}^*$ corresponding first antecedent $\widetilde{A}_1^*$ are $\widetilde{R}_1$ and $\widetilde{R}_2$. Similarly, the adjacent rules of the observation $\widetilde{A}^*$ corresponding to $\widetilde{A}_2^*$,$\widetilde{A}_3^*$ and $\widetilde{A}_4^*$ are $\widetilde{R}_2$, $\widetilde{R}_4$; $\widetilde{R}_7$, $\widetilde{R}_8$ and $\widetilde{R}_4$, $\widetilde{R}_5$ respectively. The ratios at which the core 5 of $\widetilde{A}_1^* = \widetilde{5}$ lies between the cores 2 and 6 of $\widetilde{A}_{11} = \widetilde{2}$ and $\widetilde{A}_{21} = \widetilde{6}$ is $p_1 : q_1 = 3 : 1$. Similarly, the ratios corresponding to the other antecedents are calculated as $p_2 : q_2 = 1 : 3$, $p_3 : q_3 = 1 : 1$ and $p_4 : q_4 = 1 : 1$.

So, the intermediate rules $\widetilde{R}_1^*$, $\widetilde{R}_2^*$, $\widetilde{R}_3^*$ and $\widetilde{R}_4^*$ are calculated as $\widetilde{R}_1^*(\widetilde{5}, \widetilde{9.25}, \widetilde{7.25}, \widetilde{5.5}, \widetilde{5.25})$, $\widetilde{R}_2^*(\widetilde{7.5}, \widetilde{9.5}, \widetilde{8.5}, \widetilde{6}, \widetilde{7})$, $\widetilde{R}_3^*(\widetilde{11}, \widetilde{8.5}, \widetilde{8.5}, \widetilde{9}, \widetilde{9})$ and $\widetilde{R}_4^*(\widetilde{6.5}, \widetilde{9}, \widetilde{7.5}, \widetilde{5.5}, \widetilde{6})$.

The constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by solving the equation given in line 10 of Algorithm 1 is obtained as $\lambda_1 = 1.7, \lambda_2 = 2.7, \lambda_3 = -0.15, \lambda_4 = -3.4$.

Algorithm 1 Algorithm of GGLFRI

Require: Given rule base $\mathbf{R}$ with n rules and observation $\widetilde{A}^*(a_1^*, \ a_2^*, \dots a_N^*)$.

1: **for** $i = 1$ to N **do**
2:     **for** $j = 1$ to n **do**
3:         Find t_i^k and s_i^k so that $a_{it_i^k} \le a_i^* \le a_{is_i^k}$, $a_{it_i^k} = \max\limits_l \{a_{il} : a_{il} \le a_i^*\}$, $a_{is_i^k} = \min\limits_l \{a_{il} : a_{il} \ge a_i^*\}$.
4:         Find t_i so that $D(a^*, a_{t_i}) = \min\limits_k \{D(a^*, a_{t_i^k})\}$.
5:         Find s_i so that $D(a^*, a_{s_i}) = \min\limits_k \{D(a^*, a_{s_i^k})\}$.
6:     **end for**
7:     Find ratio $p_i : q_i$ so that $a_i^* = \frac{q_i}{p_i+q_i} a_{it_i} + \frac{p_i}{p_i+q_i} a_{is_i}$.
8:     Find intermediate rule $\widetilde{R}_i^* = \frac{q_i}{p_i+q_i} \widetilde{R}_{t_i} \oplus \frac{p_i}{p_i+q_i} \widetilde{R}_{s_i}$.
9: **end for**
10: Find the constants λ_i $(i = 1, 2, \dots N)$ from $(a_1^*, a_2^*, \dots a_N^*) = \sum\limits_{i=1}^{N} \lambda_i (a_{1i}^*, a_{2i}^*, \dots a_{Ni}^*)$.
11: Find the final intermediate rule $\widetilde{R} = \sum\limits_{i=1}^{N} \lambda_i \widetilde{R}_i^*$.
12: **for** $i = 1$ to N **do**
13:     Find the parameters $\gamma_i = \frac{a_i^* - \underline{a}_i^*}{a_i - \underline{a}_i}$ and $\delta_i = \frac{\bar{a}_i^* - a_i^*}{\bar{a}_i - a_i}$.
14: **end for**
15: Find the parameters γ and δ between $\widetilde{B}$ and $\widetilde{B}^*$ from $\gamma = \dfrac{\sum\limits_{i=1}^{N} \lambda_i \gamma_i}{\sum_{i=1}^N \lambda_i}$ and $\delta = \dfrac{\sum\limits_{i=1}^{N} \lambda_i \delta_i}{\sum_{i=1}^N \lambda_i}$.
16: Obtain final conclusion $\widetilde{B}^*$ by expanding/contracting $\widetilde{B}$ with parameters γ and δ.

Thus the final conclusion is obtained as $1.7 \times \widetilde{5.25} \oplus 2.7 \times \widetilde{7} \ominus 0.15 \times \widetilde{9} \ominus 3.4 \times \widetilde{6} = \widetilde{6.075}$.

Each fuzzy number involved in the given rule base $\mathbf{R}$ have same spread lengths with the fuzzy numbers involved in the observation $\widetilde{A}^*$. So, the any changes in spread lengths which incur at line 11 of Algorithm 1, adjusted in lines 15 and 16 of Algorithm 1 and the final conclusion obtained as $\widetilde{B}^* = \widetilde{6.075}$ with spread lengths one unit in each spread.

B. Inverse FRI System with Multiple inputs-Single Output

Let us consider that a desired output $\widetilde{B}^*$ is required based on the given rule base $\mathbf{R}$ and observation $\widetilde{A}^* = (\widetilde{A}_1^*, \widetilde{A}_2^*, \dots \widetilde{A}_{m-1}^*, ?, \widetilde{A}_{m+1}^*, \dots \widetilde{A}_N^*)$ where $\widetilde{A}_m^*$ is the missing antecedent. We propose to visualize this situation similar to GGLFRI where the missing antecedent x_m is considered as effective consequent and proceed in a similar manner. So, $2N$ number of covering rules $\widetilde{R}_{t_i}$, $\widetilde{R}_{s_i}$ for $i = 1, 2, \dots N+1$; $i \neq m$ will be obtained such that $a_{1t_1} \le a_1^* \le a_{1s_1}$, $a_{2t_2} \le a_2^* \le a_{2s_2} \dots a_{Nt_N} \le a_N^* \le a_{Ns_N}$ and $b_{t_{N+1}} \le b^* \le b_{s_{N+1}}$.

Then we have to find out the ratios $p_i : q_i$ such that $\frac{q_i}{p_i+q_i} a_{it_i} + \frac{p_i}{p_i+q_i} a_{is_i} = a_i^*$ for $i = 1, 2, \dots N$; $i \neq m$ and $\frac{q_{N+1}}{p_{N+1}+q_{N+1}} b_{t_{N+1}} + \frac{p_{N+1}}{p_{N+1}+q_{N+1}} b_{s_{N+1}} = b^*$. Then N intermediate rules $\widetilde{R}_i^*$ are obtained as convex combination of the rules $\widetilde{R}_{t_i}$, $\widetilde{R}_{s_i}$ respectively, i.e. $\widetilde{R}_i^* = \frac{q_i}{p_i+q_i}\widetilde{R}_{t_i} \oplus \frac{p_i}{p_i+q_i}\widetilde{R}_{s_i}$, $for\ i = 1, 2, \ldots N;\ i \neq m$ and $\widetilde{R}_{N+1}^* = \frac{q_{N+1}}{p_{N+1}+q_{N+1}}\widetilde{R}_{t_{N+1}} \oplus \frac{p_{N+1}}{p_{N+1}+q_{N+1}}\widetilde{R}_{s_{N+1}}$.

The constants λ_i are obtained from relation $(a_1^*, a_2^*, \ldots a_{m-1}^*, a_{m+1}^*, \ldots a_N^*, b^*) = \sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i(a_{i1}^*, a_{i2}, \ldots a_{im-1}^*, a_{im+1}^*, \ldots a_{iN}^*, b_i^*)$. Then the intermediate rule $\widetilde{R}$ is calculated as $\widetilde{R} = \sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i \widetilde{R}_i^*$.

Thus the final conclusion $\widetilde{B}^*$ is then obtained by expanding/contracting $\widetilde{B}$ with parameters λ, δ where λ, δ are the weighted convex combination of the parameters γ_i, δ_i with weights λ_i respectively,

i.e. $\gamma = \frac{\sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i\gamma_i}{\sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i}$ and $\delta = \frac{\sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i\delta_i}{\sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i}$.

The proposed method is furthermore explained with the following Algorithm 2 and Example 4.2.

Example 4.2: Suppose that we have to predict the possible antecedent $\widetilde{A}_1^*$ for a desired consequent $\widetilde{B}^* = \widetilde{6.075}$ and given antecedents $\widetilde{A}_2^* = \widetilde{9.5}, \widetilde{A}_3^* = \widetilde{8.5}$ and $\widetilde{A}_4^* = \widetilde{5.5}$ from the rule base $\mathbf{R}$ given in Example 4.1.

The adjacent rules of the corresponding to the $\widetilde{A}_2^*$, $\widetilde{A}_3^*$, $\widetilde{A}_4^*$ and desired consequent $\widetilde{B}^*$ are $\widetilde{R}_2$, $\widetilde{R}_4$; $\widetilde{R}_7$, $\widetilde{R}_8$; $\widetilde{R}_4$, $\widetilde{R}_5$ and $\widetilde{R}_2$, $\widetilde{R}_7$ respectively. The ratios corresponding to the antecedents and consequent are calculated as $p_2 : q_2 = 1 : 3$; $p_3 : q_3 = 1 : 1$; $p_4 : q_4 = 1 : 1$ and $p_5 : q_5 = 0.075 : 0.925$ respectively.

So, the intermediate rules $\widetilde{R}_2^*$, $\widetilde{R}_3^*$, $\widetilde{R}_4^*$ and $\widetilde{R}_5^*$ are calculated as $\widetilde{R}_2^*(\widetilde{7.5}, \widetilde{9.5}, \widetilde{8.5}, \widetilde{6}, \widetilde{7})$, $\widetilde{R}_3^*(\widetilde{11}, \widetilde{8.5}, \widetilde{8.5}, \widetilde{9}, \widetilde{9})$, $\widetilde{R}_4^*(\widetilde{9.5}, \widetilde{8}, \widetilde{8.5}, \widetilde{5.5}, \widetilde{8})$ and $\widetilde{R}_5^*(\widetilde{6.075}, \widetilde{9.625}, \widetilde{8}, \widetilde{6.15}, \widetilde{6.075})$.

The constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ by solving the equation given in line 17 of Algorithm 2 is obtained as $\lambda_2 = -0.729, \lambda_3 = -0.339, \lambda_4 = 0.57, \lambda_5 = 1.59$.

Thus the required antecedent $\widetilde{A}_1^*$ is obtained as $-0.729 \times \widetilde{7.5} \ominus 0.339 \times \widetilde{11} \oplus 0.57 \times \widetilde{9.5} \oplus 1.59 \times \widetilde{6.075} = \widetilde{5.88}$.

Each fuzzy number involved in the given rule base $\mathbf{R}$ have same spread lengths with the fuzzy numbers involved in the observation A^*. So, the any changes in spread lengths which incur at line 18 of Algorithm 2, adjusted in lines 23 and 24 of Algorithm 2 and the final conclusion obtained as $\widetilde{A}_1^* = \widetilde{5.88}$ with one unit spread lengths.

Algorithm 2 Algorithm of Inverse GGLFRI

Require: Rule base $\mathbf{R}$ containing n rules. Observation $x_i = \widetilde{A}_i^*$ $(i = 1, 2, \ldots N; i \neq m)$ and desired consequent $y = \widetilde{B}^*$. We have to predict the m^{th} antecedents $\widetilde{A}_m^*$.

1: **for** $i = 1$ to N and $i \neq m$ **do**
2:   **for** $j = 1$ to n **do**
3:     Find t_i^k and s_i^k so that $a_{it_i^k} \leq a_i^* \leq a_{is_i^k}$ and $a_{it_i^k} = \max_l\{a_{il} : a_{il} \leq a_i^*\}$, $a_{is_i^k} = \min_l\{a_{il} : a_{il} \geq a_i^*\}$.
4:     Find t_i so that $D(a'^*, a'_{t_i}) = \min_k\{D(a'^*, a'_{t_i^k})\}$, where $a' = (a_1, a_2, ..a_{m-1}, a_{m+1}, ...a_N, b)$.
5:     Find s_i so that $D(a'^*, a'_{s_i}) = \min_k\{D(a'^*, a'_{s_i^k})\}$.
6:   **end for**
7:   Find ratio $p_i : q_i$ so that $a_i^* = \frac{q_i}{p_i+q_i}a_{it_i} + \frac{p_i}{p_i+q_i}a_{is_i}$.
8:   Find intermediate rule $\widetilde{R}_i^* = \frac{q_i}{p_i+q_i}\widetilde{R}_{t_i} \oplus \frac{p_i}{p_i+q_i}\widetilde{R}_{s_i}$.
9: **end for**
10: **for** $j = 1$ to n **do**
11:   Find t_{N+1}^k, s_{N+1}^k so that $b_{t_{N+1}^k} \leq b^* \leq b_{s_{N+1}^k}$ and $b_{t_{N+1}^k} = \max_l\{b_l : b_l \leq b^*\}$, $b_{s_{N+1}^k} = \min_l\{b_l : b_l \geq b^*\}$.
12:   Find t_{N+1} from $D(b^*, b_{t_{N+1}}) = \min_k\{D(b^*, b_{t_{N+1}^k})\}$.
13:   Find s_{N+1} from $D(b^*, b_{s_{N+1}}) = \min_k\{D(b^*, b_{s_{N+1}^k})\}$.
14: **end for**
15: Find ratio $p_{N+1} : q_{N+1}$ so that $b^* = \frac{q_{N+1}}{p_{N+1}+q_{N+1}}b_{t_{N+1}} + \frac{p_{N+1}}{p_{N+1}+q_{N+1}}b_{s_{N+1}}$.
16: Find intermediate rule $\widetilde{R}_{N+1}^* = \frac{q_{N+1}}{p_{N+1}+q_{N+1}}\widetilde{R}_{t_{N+1}} \oplus \frac{p_{N+1}}{p_{N+1}+q_{N+1}}\widetilde{R}_{s_{N+1}}$.
17: Find the constants λ_i from $a_i^* = \sum_{\substack{j=1\\j\neq m}}^{N+1} \lambda_j a_{ji}$ $(i \neq m)$ and $b^* = \sum_{\substack{j=1\\j\neq m}}^{N+1} \lambda_j b_j$.
18: Find the final intermediate rule $\widetilde{R} = \sum_{\substack{i=1\\i\neq m}}^{N+1} \lambda_i \widetilde{R}_i^*$.
19: **for** $i = 1$ to N and $i \neq m$ **do**
20:   Find the parameters $\gamma_i = \frac{a_i^* - \underline{a}_i^*}{a_i - \underline{a}_i}$ and $\delta_i = \frac{\bar{a}_i^* - a_i^*}{\bar{a}_i - a_i}$.
21: **end for**
22: Find the parameters $\gamma_{N+1} = \frac{b^* - \underline{b}^*}{b - \underline{b}}$ and $\delta_{N+1} = \frac{\bar{b}^* - b^*}{\bar{b} - b}$.
23: Find the parameters γ and δ between $\widetilde{A}_m$ and $\widetilde{A}_m^*$ from $\gamma = \frac{\sum_{\substack{i=1\\j\neq m}}^{N+1} \lambda_i\gamma_i}{\sum_{i=1}^{N} \lambda_i}$ and $\delta = \frac{\sum_{\substack{i=1\\j\neq m}}^{N+1} \lambda_i\delta_i}{\sum_{i=1}^{N} \lambda_i}$.
24: Obtain final antecedent $\widetilde{A}_m^*$ by expanding/contracting $\widetilde{A}_m$ with parameters γ and δ.

V. Conclusion

The geometry based linear fuzzy rule base interpolation technique has been generalized for multi-antecedents. The same geometrical and analytical interpretations are used to obtain the inverse generalized geometry based linear fuzzy rule base interpolation. The proposed methods are equipped with complete geometrical visualization and analogous to classical interpolation. Moreover in the process of obtaining conclusion, the proposed method is able to capture the variations of the fuzziness of involved fuzzy sets both in knowledge base and observation. Thus the amount of uncertainty involved in given information is well captured to obtain the unknown parameter.

The proposed method can be extended for extrapolation. A detail comparison on the results obtained from different methodologies is due for future work. Also, the exactness of the proposed method for real life scenarios remain to check. Behaviour of the proposed methods can be analyzed for choice of different $t - norms$.

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