On the representation of (weak) nilpotent minimum algebras

Umberto Rivieccio
DIMAp
UFRN
Natal, Brazil
ORCID 0000-0003-1364-5003
urivieccio@dimap.ufrn.br

Tommaso Flaminio
IIIA
CSIC
Bellaterra, Spain
ORCID 0000-0002-9180-7808
tommaso@iiia.csic.es

Thiago Nascimento
PPGSC - DIMAp
UFRN
Natal, Brazil
ORCID 0000-0002-9288-8929
thiagnascsilva@gmail.com

***Abstract*—We take a glimpse at the relation between WNM-algebras (algebraic models of the well-known *Weak Nilpotent Minimum* logic) and quasi-Nelson algebras, a non-involutive generalisation of Nelson algebras (models of *Nelson's constructive logic with strong negation*) that was introduced in a recent paper. We show that the two varieties can be related via the *twist-structure construction*, obtaining a new representation for a subvariety of WNM-algebras that includes the involutive ones (i.e. NM-algebras). Our results imply, in particular, that every pre-linear quasi-Nelson algebra is a WNM-algebra; we thus generalize the known result that the class of pre-linear Nelson algebras coincides with that of NM-algebras (models of *Nilpotent Minimum* logic).**

***Index Terms*—weak nilpotent minimum, quasi-Nelson, monoidal t-norm, twist representation**

I. Introduction

Monoidal t-norm logic (MTL), the logic of left-continuous t-norms, is among the most prominent systems in the mathematical fuzzy logic literature. MTL was introduced in [12], and the same paper [12, Sec. 3] considers certain axiomatic extensions of MTL that result from imposing stronger requirements on the negation connective. Among these, a *weak negation* function on the real interval $[0,1]$ determines *weak nilpotent minimum* logics, and a strong (involutive) negation defines *nilpotent minimum logic* (NML). Algebraic models of MTL as well as those of the above-mentioned extensions (called respectively *MTL-algebras*, *WNM-algebras* and *NM-algebras*) have been studied extensively, and several representation results are known.

Nelson algebras are the algebraic models of *Nelson's constructive logic with strong negation* [19], a system obtained by adding a new involutive negation to positive intuitionistic logic.

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Structurally, NM-algebras and Nelson algebras are closely related. Indeed, Busaniche and Cignoli [9] proved that if one adds the *pre-linearity* equation

$$(x \Rightarrow y) \vee (y \Rightarrow x) \approx 1$$

to Nelson algebras, one obtains precisely the variety of NM-algebras.

From a methodological viewpoint, this result is particularly interesting, for it entails that every NM-algebra (as a Nelson algebra) can be represented as a special binary power (called a *twist-structure*) of a (pre-linear) Heyting algebra (i.e. a *Gödel algebra*), and also as a *(dis)connected rotation* of a Gödel algebra (see [8], [18]).

(Dis)connected rotations have been generalized in [10] to account for some non-involutive structures (see also [2]). With a similar purpose, the twist-structure construction has been recently extended to a non-involutive setting [23], [24]. The twist construction, when applied to pairs of Heyting algebras, determines the class of *quasi Nelson-algebras*.

In the present paper we focus on the interplay between the pre-linearity equation and the non-involutive twist-structure construction. In particular, we prove that pre-linear quasi-Nelson algebras correspond precisely to the class of (non-involutive) twist-structures over pairs of Gödel algebras. As a class of abstract algebras, the latter is a proper subvariety of WNM-algebras.

II. NM, WNM and (quasi-)Nelson

We assume familiarity with basic results of universal algebra [7], residuated lattices [14] and fuzzy logics [16]. Although we shall be dealing exclusively with algebras, it is important to keep in mind that every variety considered here is the algebraic counterpart (in the strong sense of [6]) of some substructural/fuzzy logic. Thus, virtually all algebraic results stated in the next sections have a straightforward logical counterpart.

A *commutative integral bounded residuated lattice* (CIBRL) is an algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ of type $\langle 2,2,2,2,0,0\rangle$ such that:

(i) $\langle A; *, 1\rangle$ is a commutative monoid,
(ii) $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded lattice (with order $\leq$),

(iii) $a*b \leq c$ iff $a \leq b \Rightarrow c$ for all $a, b, c \in A$ (*residuation*).
The unary *negation* operation is defined by $\neg a := a \Rightarrow 0$ for all $a \in A$. Notice that the negation of every CIBRL $\mathbf{A}$ satisfies the properties postulated in [12] for a *weak negation*, that is, $\neg$ is order-reversing, $\neg 1 = 0$ and $a \leq \neg\neg a$ for all $a \in A$.

An *MTL-algebra* is a CIBRL that additionally satisfies the pre-linearity equation introduced in the preceding Section. MTL-algebras are thus said to be *pre-linear*, suggesting the well-known result that every MTL-algebra is isomorphic to a subdirect product of linearly ordered ones. The same applies to the subvarietes of MTL-algebras introduced below, and entails in particular that the lattice reduct of every such algebra is distributive. *WNM-algebras* are the subvariety of MTL-algebras defined by the *weak nilpotent minimum* equation:

$$\neg(x * y) \vee ((x \wedge y) \Rightarrow (x * y)) \approx 1.$$

In turn, *NM-algebras* are obtained from WNM by adding the involutive equation $\neg\neg x \approx x$ (or, equivalently, just $\neg\neg x \leq x$). Thus NM $\subseteq$ WNM $\subseteq$ MTL $\subseteq$ CIBRL (all inclusions being proper).

Moving to the realm of Nelson logics, a *quasi-Nelson algebra* (QN-algebra) is defined as a CIBRL that satisfies the *Nelson* equation:

$$(x \Rightarrow (x \Rightarrow y)) \wedge (\neg y \Rightarrow (\neg y \Rightarrow \neg x)) \approx x \Rightarrow y.$$

QN-algebras have been introduced only recently [23], [24], whereas *Nelson algebras* (N-algebras) have been around for more than four decades. N-algebras are precisely the involutive QN-algebras (i.e. those that satisfy $\neg\neg x \approx x$, or equivalently $\neg\neg x \leq x$). Thus N $\subseteq$ QN $\subseteq$ CIBRL (all inclusions proper).

The relationship between NM-algebras and Nelson algebras has been investigated in previous papers, a standard reference being [9]. There it is proved that the variety of NM-algebras coincides with the subvariety PN of Nelson algebras satisfying the pre-linearity equation. In the non-involutive setting (moving from Nelson to quasi-Nelson algebras on the one side, and from NM to WNM-algebras on the other), it is not difficult to produce a (linearly ordered) WNM-algebra that does *not* satisfy the Nelson equation (see Example 4.6 below). Denoting by PQN the variety of QN-algebras satisfying the pre-linearity equation, we thus have WNM $\not\subseteq$ PQN (a fortiori, WNM $\not\subseteq$ PN). This raises the following questions.

First: does the converse inclusion (PN $\subseteq$ WNM) hold?

Second: how can one describe the class WNM $\cap$ N (or, more generally, WNM $\cap$ QN)?

As we are going to see, the answer to the first question is that, indeed, one has PN $\subseteq$ WNM, and even PQN $\subseteq$ WNM. These observations (having seen that NM $\subseteq$ N) entail the above-mentioned result that NM $=$ PN.

The latter question brought to our attention an equational condition (only involving the $\wedge$ and $\neg$ operations: see Proposition 4.7) that, as far as we know, has never been singled out in the context of WNM-algebras.

We obtained both the above-mentioned results thanks to the twist representation of (quasi-)Nelson algebras. This perspective, which is new for WNM-algebras, led us to other interesting insights and questions, which we are going to recount in the next sections.

III. Twist-structures and rotations

The twist-structure is a method for constructing an N-algebra (extended in [23], [24] to QN-algebras) as a subalgebra of a special product of two Heyting algebras. In fact, twist-structures yield a representation theorem: every (quasi-)Nelson algebra arises in this way. We proceed to expound the details of the construction, restricting our attention (since we are in a pre-linear setting) to pre-linear Heyting algebras factors, known as *Gödel algebras* in the fuzzy literature (from now on G-algebras).

Let:

$$\mathbf{G}_+ = \langle G_+, \leq_+; \wedge_+, \vee_+, \rightarrow_+, \neg_+, 0_+, 1_+ \rangle$$
$$\mathbf{G}_- = \langle G_-, \leq_-; \wedge_-, \vee_-, \rightarrow_-, \neg_-, 0_-, 1_- \rangle$$

be both G-algebras, and $n\colon G_+ \to G_-$ and $p\colon G_- \to G_+$ be bounded lattice homomorphisms, additionally satisfying the following requirements: $n \cdot p = Id_{G_-}$ and $Id_{G_+} \leq_+ p \cdot n$. Define an algebra $\mathbf{G}_+ \bowtie \mathbf{G}_- = \langle G_+ \times G_-; \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$ as follows: for all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in G_+ \times G_-$,

$$1 = \langle 1_+, 0_- \rangle$$
$$0 = \langle 0_+, 1_- \rangle$$
$$\neg\langle a_+, a_- \rangle = \langle p(a_-), n(a_+) \rangle$$
$$\langle a_+, a_- \rangle \wedge \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, a_- \vee_- b_- \rangle$$
$$\langle a_+, a_- \rangle \vee \langle b_+, b_- \rangle = \langle a_+ \vee_+ b_+, a_- \wedge_- b_- \rangle$$
$$\langle a_+, a_- \rangle \rightarrow \langle b_+, b_- \rangle = \langle a_+ \rightarrow_+ b_+, n(a_+) \wedge_- b_-) \rangle.$$

The residuated operations are given by the following terms:

$$x \Rightarrow y = (x \rightarrow y) \wedge (\neg y \rightarrow \neg x)$$
$$x * y = x \wedge (y \wedge \neg(x \Rightarrow \neg y)).$$

Component-wise, these give us $\langle a_+, a_- \rangle \Rightarrow \langle b_+, b_- \rangle = \langle (a_+ \rightarrow_+ b_+) \wedge_+ (p(b_-) \rightarrow_+ p(a_-)), n(a_+) \wedge_- b_- \rangle$, and $\langle a_+, a_- \rangle * \langle b_+, b_- \rangle = \langle a_+ \wedge_+ b_+, (n(a_+) \rightarrow_- b_-) \wedge_- (n(b_+) \rightarrow_- a_-) \rangle$. Also observe that the lattice order $\leq$ on $\mathbf{G}_+ \bowtie \mathbf{G}_-$ is given, for all $\langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in G_+ \times G_-$, by $\langle a_+, a_- \rangle \leq \langle b_+, b_- \rangle$ iff ($a_+ \leq_+ b_+$ and $b_- \leq_- a_-$).

The algebra $\mathbf{G}_+ \bowtie \mathbf{G}_-$ may not even be a CIBRL, but we can obtain a quasi-Nelson algebra by considering the following subalgebra. Let:

$$D(G_+) = \{a_+ \in G_+ : \neg_+ a_+ = 0_+\}$$

be the set of *dense elements* of $\mathbf{G}_+$, and consider a lattice filter $\nabla \subseteq G_+$ such that $D(G_+) \subseteq \nabla$. One can then show that the set $Tw(G_+, G_-, n, p, \nabla) = \{\langle a_+, a_- \rangle \in G_+ \times G_- : a_+ \wedge_+ p(a_-) = 0_+, a_+ \vee_+ p(a_-) \in \nabla\}$ is the universe of a subalgebra of $\mathbf{G}_+ \bowtie \mathbf{G}_-$. We call the corresponding algebra $\mathbf{A} = \mathbf{Tw}\langle \mathbf{G}_+, \mathbf{G}_-, n, p, \nabla \rangle$ a *QN twist-structure*[1].

[1]A technical observation that will be useful in proofs: the requirement $a_+ \wedge_+ p(a_-) = 0_+$ entails that $n(a_+) \wedge_- a_- = n(a_+) \wedge_- np(a_-) = n(a_+ \wedge_+ p(a_-)) = n(0_+) = 0_-$.

The algebra $\mathbf{A}$ with the operations $\langle \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a CIBRL and a QN-algebra [24, Thm. 2]; moreover, every QN-algebra arises in this way.

To justify the above claim, let $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a pre-linear QN-algebra. Define the operation $\rightarrow$ by the term:

$$x \rightarrow y = x^2 \Rightarrow y$$

where $x^2 = x * x$. Define the relation $\equiv$, for all $a, b \in A$, by:

$$a \equiv b \quad \text{iff} \quad a \rightarrow b = b \rightarrow a = 1.$$

The connective $\rightarrow$ is known as the *weak implication* in the literature on Nelson logics (as opposed to the *strong* residuated one $\Rightarrow$), and it is the connective witnessing the deduction-detachment theorem for (quasi-)Nelson logic. One verifies that the relation $\equiv$ thus obtained is compatible with the operations $\langle \wedge, \vee, *, \rightarrow\rangle$, which gives us a quotient $\mathbf{A}_+ = \langle A/{\equiv}; \wedge, *, \vee, \rightarrow, 0, 1\rangle$. Also, the algebra $\mathbf{A}_+$ is a G-algebra (on which the operations $\wedge$ and $*$ coincide). Defining the set $F(A) = \{a \in A : \neg a \leq a\}$, we have that $\nabla_{\mathbf{A}} = F(A)/{\equiv}$ is a lattice filter of $\mathbf{A}_+$ and that $D(\mathbf{A}_+) \subseteq \nabla_+$. To obtain a second G-algebra factor, one considers the set $\neg A = \{\neg a : a \in A\}$ and lets $A_- = \neg A/\equiv$. Then A_- is the universe of a subalgebra of $\mathbf{A}_+$, which we denote by $\mathbf{A}_-$. Lastly, define maps $n_{\mathbf{A}} \colon A_+ \rightarrow A_-$ and $p_{\mathbf{A}} \colon A_- \rightarrow A_+$ as follows: $n_{\mathbf{A}}(a/\equiv) = \neg\neg a/\equiv$ and $p_{\mathbf{A}}(\neg a/\equiv) = \neg a/\equiv$. The tuple $\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$ satisfies the required properties for defining a QN twist-structure $\mathbf{Tw}\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$. The representation theorem proved in [24, Prop. 10] then states that $\mathbf{A} \cong \mathbf{Tw}\langle \mathbf{A}_+, \mathbf{A}_-, n_{\mathbf{A}}, p_{\mathbf{A}}, \nabla_{\mathbf{A}}\rangle$ through the map ι given by $\iota(a) = \langle a/{\equiv}, \neg a/{\equiv}\rangle$ for all $a \in A$.

Among QN-algebras, the involutive ones (i.e. N-algebras) are precisely those algebras $\mathbf{A}$ such that $A = \neg A$. Hence $\mathbf{A}_+ = \mathbf{A}_-$ and $n_{\mathbf{A}}, p_{\mathbf{A}}$ are both the identity map. Therefore, a (pre-linear) N-algebra is determined by just a pair $\langle \mathbf{G}, \nabla\rangle$. Another prominent subvariety of QN-algebras is the class of G-algebras itself (indeed, the Nelson equation is easily seen to be implied by $x \Rightarrow (x \Rightarrow y) \approx x \Rightarrow y$, which is valid on all G-algebras). In terms of the twist representation, G-algebras correspond precisely to those $\mathbf{A}$ such that $\mathbf{A}_+ \cong \mathbf{A}$.

On the other hand, NM-algebras (which coincide with pre-linear N-algebras) can be constructed from G-algebras by employing *connected* and *disconnected rotations*. Although the results of the present paper do not rely directly on these constructions, it will be useful, for further discussion, to recall the basic definitions. We begin with a special case. Let $\mathbf{G} = \langle G; \wedge, \vee, \rightarrow, \neg, 0, 1\rangle$ be a finite and directly indecomposable G-algebra. Define:

$$CR(\mathbf{G}) = \{\langle a, a'\rangle \in G \times G : a \wedge a' = 0\},$$

$$DR(\mathbf{G}) = \{\langle a, a'\rangle \in G \times G : (a \wedge a') \vee \neg(a \vee a') = 0\}.$$

For all $\langle a, a'\rangle, \langle b, b'\rangle \in CR(\mathbf{G})$ or $DR(\mathbf{G})$, let:

$$\begin{aligned}
\langle a, a'\rangle * \langle b, b'\rangle &= \langle (a' \vee b') \rightarrow (a \vee b), a' \wedge b'\rangle \\
\langle a, a'\rangle \Rightarrow \langle b, b'\rangle &= \langle a' \wedge b, (a' \vee b) \rightarrow (a \vee b')\rangle \\
\langle a, a'\rangle \wedge \langle b, b'\rangle &= \langle a \vee b, a' \wedge b'\rangle \\
\langle a, a'\rangle \vee \langle b, b'\rangle &= \langle a \wedge a', b \vee b'\rangle \\
\neg\langle a, a'\rangle &= \langle a', a\rangle.
\end{aligned}$$

The reader will have noticed a similarity between the above-defined operations and the operations of twist-structures introduced earlier; we shall return on this below.

Denote by $\mathbf{CR}(\mathbf{G})$ and $\mathbf{DR}(\mathbf{G})$ the algebras:

$$\mathbf{CR}(\mathbf{G}) = \langle CR(\mathbf{G}); *, \Rightarrow \wedge, \vee, \neg, \langle 1, 0\rangle, \langle 0, 1\rangle\rangle,$$

$$\mathbf{DR}(\mathbf{G}) = \langle DR(\mathbf{G}); *, \Rightarrow, \wedge, \vee, \neg, \langle 1, 0\rangle, \langle 0, 1\rangle\rangle.$$

The above constructions coincide, respectively, with the well-known *connected* and *disconnected* rotations of [8], [18].

For every finite directly indecomposable G-algebra $\mathbf{G}$, $\mathbf{CR}(\mathbf{G})$ is a finite directly indecomposable NM-algebra with negation fixpoint $\langle 0, 0\rangle$, and $\mathbf{DR}(\mathbf{G})$ is a finite directly indecomposable NM-algebra without any fixpoint (every NM-algebra $\mathbf{A}$ can have, at most, one negation fixpoint, i.e. an element $a \in A$ such that $a = \neg a$).

Conversely, given a directly indecomposable NM-algebra $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$, one defines:

$$G(\mathbf{A}) = \{a^2 : a \in A\}$$

where, as before, $a^2 = a * a$. Then the algebra $\mathbf{G}(\mathbf{A}) = \langle G(\mathbf{A}); \wedge, \vee, \Rightarrow^2, 0, 1\rangle$ is a directly indecomposable G-algebra (the operations $\wedge$ and $\vee$ are the restrictions of those of $\mathbf{A}$ and, for every $a, b \in G(\mathbf{B})$, one defines $a \Rightarrow^2 b = (a \Rightarrow b)^2$).

In general, let $\mathbf{G} = \prod_{i \in I} \mathbf{G}_i$ be a finite G-algebra represented as direct product of its directly indecomposable components $\mathbf{G}_i$ and let $\mathbf{f}$ be a (necessarily principal) filter of its *Boolean skeleton* (i.e. the largest Boolean subalgebra of $\mathbf{G}$, whose universe is the set of elements that have a Boolean complement). The generator a of $\mathbf{f}$ is hence a complemented element of $\mathbf{G}$, which can be written as a string of length $|I|$ whose components a_i are either 0's or 1's . We then define, for all $i \in I$, $\mathbf{A}_i = \mathbf{CR}(\mathbf{G}_i)$ if $a_i = 0$ and $\mathbf{A}_i = \mathbf{DR}(\mathbf{G}_i)$ if $a_i = 1$. Finally, let $\mathbf{A}$ the NM-algebra $\prod_{i \in I} \mathbf{A}_i$. As shown in [5], every finite NM-algebra is of this form, which entails that the finite NM-algebras are in one-to-one correspondence with pairs of the form $\langle \mathbf{G}, \mathbf{f}\rangle$.

The above construction lifts to the infinite case with no extra requirements, and it is proved in [5] that every NM-algebra corresponds to a unique pair $\langle \mathbf{G}, \mathbf{f}\rangle$ where $\mathbf{G}$ is a G-algebra and $\mathbf{f}$ is a filter of its Boolean skeleton[2].

From an abstract perspective, twist-structures and rotations are thus two different methods for associating (in a one-to-one fashion) a Nelson algebra (resp. an NM-algebra) $\mathbf{A}$ to a pair consisting of a G-algebra $\mathbf{G}$ and a filter of (the

[2]Besides recalling the representation of NM-algebras as pairs $\langle \mathbf{G}, \mathbf{f}\rangle$, we will not use any result from the unpublished paper [5].

Boolean skeleton of) $\mathbf{G}$. Furthermore, both methods yield representations that can be used to establish a categorical equivalence between the algebraic category of Nelson algebras (resp. of NM-algebras) and a category naturally associated to pairs of type $\langle \mathbf{G}, \mathbf{f} \rangle$.

It is therefore natural to ask whether this apparent parallelism is grounded on a structural relation between Nelson and NM-algebras. This is indeed the case, and the question can be addressed both on an abstract and on a concrete level: see Corollary 5.2 below and the subsequent observations.

IV. Quasi-Nelson and WNM-algebras

In the light of the twist representation result of the preceding section, from now on we shall, whenever convenient, assume that a QN-algebra is of the form $\mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$.

Lemma 4.1: Let $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$ be a QN-algebra. Then $\mathbf{A}$ is linearly ordered if and only if both $\mathbf{G}_+$ and $\mathbf{G}_-$ are linearly ordered.

Proof: if $\mathbf{A}$ is linearly ordered, then $\mathbf{G}_+$ and $\mathbf{G}_-$ are linearly ordered because both are isomorphic to quotients of the lattice reduct of $\mathbf{A}$. Conversely, assume $\mathbf{G}_+$ and $\mathbf{G}_-$ are linearly ordered. Then, for all $\langle a_+, a_- \rangle \in A$, by the requirement $a_+ \wedge_+ p(a_-) = 0_+$, we have either $a_+ = 0_+$ or $p(a_-) = 0_+$ (in the latter case, $np(a_-) = a_- = 0_-$). Thus all elements of $\mathbf{A}$ are of the form $\langle a_+, 0_- \rangle$ or $\langle 0_+, a_- \rangle$ for some $a_+ \in G_+$, $a_- \in G_-$. Note that $\langle 0_+, a_- \rangle \leq \langle a_+, 0_- \rangle$ for all $a_+ \in G_+$, $a_- \in G_-$. On the other hand, $\langle 0_+, a_- \rangle \leq \langle 0_+, b_- \rangle$ iff $b_- \leq_- a_-$ and $\langle a_+, 0_- \rangle \leq \langle b_+, 0_- \rangle$ iff $a_+ \leq_+ b_+$, for all $a_+, b_+ \in G_+$ and $a_-, b_- \in G_-$. Thus $\mathbf{A}$ is also linearly ordered (e.g.) as follows (assuming $b_- \leq_- a_-$ and $a_+ \leq_+ b_+$): $\ldots \leq \langle 0_+, a_- \rangle \leq \ldots \leq \langle 0_+, b_- \rangle \leq \ldots \leq \langle a_+, 0_- \rangle \leq \langle b_+, 0_- \rangle \leq \ldots$ ∎

Lemma 4.1 gives us the following useful characterisation of pre-linear QN-algebras.

Proposition 4.2: The following varieties of algebras coincide:

(i) Pre-linear quasi-Nelson algebras.
(ii) The class of all twist-structures of type $\mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$.

Proof: Taking our earlier considerations into account, we only need to prove that e.g. (ii) is a subclass of (i). To do so, we shall verify that every subdirectly irreducible algebra in (ii) is also in (i). Consider a subdirectly irreducible QN algebra $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$. By [24, Proposition 8], we have $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{G}_+)$. Hence the G-algebra $\mathbf{G}_+$ is also subdirectly irreducible. Thus $\mathbf{G}_+$ is linearly ordered [17, Lemma 3] and, by Lemma 4.1, $\mathbf{A}$ is also linearly ordered. Then $\mathbf{A}$ satisfies the pre-linearity equation, as required. ∎

Proposition 4.2 could be stated in a slightly more general form. As mentioned earlier, non-involutive twist-structures can be defined over pairs of Heyting algebras (rather than G-algebras, which are a special case). One can then observe that pre-linear quasi-Nelson algebras correspond to the class of twist-structures $\{\mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle : \mathbf{G}_+ \text{ is a G-algebra}\}$, simply because the Heyting algebra $\mathbf{G}_-$ must be pre-linear whenever $\mathbf{G}_+$ is.

For the reader familiar with Nelson algebras, we mention an easy but non-trivial consequence of Proposition 4.2: a QN-algebra $\mathbf{A}$ satisfies the equation $(x \Rightarrow y) \vee (y \Rightarrow x) \approx 1$ if and only if $\mathbf{A}$ satisfies the (seemingly weaker) equation $(x \to y) \vee (y \to x) \approx 1$, which employs the so-called weak Nelson implication given by $x \to y = x^2 \Rightarrow y$.

Lemma 4.3: Every linearly ordered QN-algebra $\mathbf{A}$ satisfies the WNM equation: $\neg(x * y) \vee ((x \wedge y) \Rightarrow (x * y)) \approx 1$.

Proof: Let $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$ and $a, b \in A$. We need to ensure that $\neg(a * b) \vee ((a \wedge b) \Rightarrow (a * b)) = 1$. We can assume, without loss of generality, that $a \leq b$. Then $\neg(a * b) \vee ((a \wedge b) \Rightarrow (a * b)) = \neg(a * b) \vee (a \Rightarrow (a * b))$. As observed in the proof of Lemma 4.1, all elements of $\mathbf{A}$ are of the form $\langle a_+, 0_- \rangle$ or $\langle 0_+, a_- \rangle$ for some $a_+ \in G_+$, $a_- \in G_-$. If $b = \langle 0_+, b_- \rangle$, then $a = \langle 0_+, a_- \rangle$ for some $a_- \in G_-$ such that $b_- \leq_- a_-$. Then $a * b = \langle 0_+, (n(0_+) \to_- b_-) \wedge (n(0_+) \to_- a_-) \rangle = \langle 0_+, (0_- \to_- b_-) \wedge (0_- \to_- a_-) \rangle = \langle 0_+, 1_- \rangle$. So $\neg(a * b) = 1$, and we are done. Thus, let us assume that $b = \langle b_+, 0_- \rangle$. If $a = \langle 0_+, a_- \rangle$, we calculate $\neg(\langle 0_+, a_- \rangle * \langle b_+, 0_- \rangle) \vee (\langle 0_+, a_- \rangle \Rightarrow (\langle 0_+, a_- \rangle * \langle b_+, 0_- \rangle)) = \langle p((n(0_+) \to_- 0_-) \wedge_- (n(b_+) \to_- a_-)), n(0_+ \wedge_+ b_+) \rangle \vee \langle (0_+ \to_+ (0_+ \wedge_+ b_+)) \wedge_+ p((n(0_+) \to_- 0_-) \wedge_- (n(b_+) \to_- a_-)) \to_+ p(a_-), n(0_+) \wedge_- (n(0_+) \to_- 0_-) \wedge_- (n(b_+) \to_- a_-) \rangle = \langle p((0_- \to_- 0_-) \wedge_- (n(b_+) \to_- a_-)), n(0_+) \rangle \vee \langle (0_+ \to_+ 0_+) \wedge_+ p((0_- \to_- 0_-) \wedge_- (n(b_+) \to_- a_-)) \to_+ p(a_-), 0_- \wedge_- (0_- \to_- 0_-) \wedge_- (n(b_+) \to_- a_-) \rangle = \langle p((n(b_+) \to_- a_-)), 0_- \rangle \vee \langle p((n(b_+) \to_- a_-)) \to_+ p(a_-), 0_- \rangle$. Thus, we need to check that $p((n(b_+) \to_- a_-)) \vee_+ p((n(b_+) \to_- a_-)) \to_+ p(a_-) = 1_+$. If $n(b_+) \leq_- a_-$, we are done. Thus (recalling that $\mathbf{G}_-$ is linearly ordered), assume $a_- <_- n(b_+)$. Then $n(b_+) \to_- a_- = a_-$ (this also holds on every linearly ordered G-algebra), and we have $p((n(b_+) \to_- a_-)) \vee_+ p((n(b_+) \to_- a_-)) \to_+ p(a_-) = p(a_-) \vee_+ (p(a_-) \to_+ p(a_-)) = p(a_-) \vee_+ 1_+ = 1_+$, as required. To conclude the proof, assume $a = \langle a_+, 0_- \rangle$, while $b = \langle b_+, 0_- \rangle$ as before and $a_+ \leq_+ b_+$. We claim that $a \Rightarrow (a * b) = 1$, which is clearly sufficient to obtain the required result. Let us compute $a \Rightarrow (a * b) = \langle a_+, 0_- \rangle \Rightarrow \langle a_+ \wedge_+ b_+, (n(a_+) \to_- 0_-) \wedge_- (n(b_+) \to_- 0_-) \rangle = \langle a_+, 0_- \rangle \Rightarrow \langle a_+, n(b_+) \to_- 0_- \rangle$. The last equality holds because from $a_+ \leq_+ b_+$ we have $n(a_+) \leq_- n(b_+)$ and from this $n(b_+) \to_- 0_- \leq_- n(a_+) \to_- 0_-$. We proceed and compute $\langle a_+, 0_- \rangle \Rightarrow \langle a_+, n(b_+) \to_- 0_- \rangle = \langle (a_+ \to_+ a_+) \wedge_+ (p(n(b_+) \to_- 0_-) \to_+ p(0_-)), n(a_+) \wedge_- (n(b_+) \to_- 0_-) \rangle = \langle p(n(b_+) \to_- 0_-) \to_+ 0_+, 0_- \rangle$. The second component of the last equality holds because from $a_+ \leq_+ b_+$ we have $n(a_+) \leq_- n(b_+)$, thus $n(a_+) \wedge_- (n(b_+) \to_- 0_-) \leq_- n(b_+) \wedge_- (n(b_+) \to_- 0_-) = n(b_+) \wedge_- 0_- = 0_-$. Hence, it remains to check that $p(n(b_+) \to_- 0_-) \to_+ 0_+ = 1_+$, that is $p(n(b_+) \to_- 0_-) = 0_+$. Observe that, since $0_+ <_+ b_+$ (we have considered case where $b_+ = 0_+$ earlier), we have $0_- <_- n(b_+)$. For otherwise, since $Id_{G_+} \leq_+ p \cdot n$, from $n(b_+) = 0_-$ we would obtain $b_+ \leq_+ pn(b_+) = p(0_-) = 0_+$, against our assumptions. Then $n(b_+) \to_- 0_- = 0_-$, which entails $p(n(b_+) \to_- 0_-) = 0_+$, as required. ∎

As shown in [24, Cor. 3], a QN-algebra $\mathbf{A}$ is subdirectly irreducible if and only if $\mathbf{A}$ has a unique co-atom. This observation allows one to prove that (similarly to MTL, WNM and NM-algebras) the variety of QN-algebras satisfying the pre-linearity equation is generated by its linearly ordered members. Thus, the result of Lemma 4.3 applies to all pre-linear QN-algebras.

Corollary 4.4: Every pre-linear quasi-Nelson algebra satisfies the WNM equation.

Thus PQN $\subseteq$ WNM, the inclusion being strict (as shown by Example 4.6 below).

Corollary 4.5: The following varieties coincide:

(i) Pre-linear QN-algebras.

(ii) Pre-linear QN-algebras satisfying the WNM equation.

(iii) The class of all twist-structures of type $\mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle$.

It is useful to recall that, on a linearly ordered WNM-algebra $\mathbf{A}$, the lattice structure together with the negation determine the other operations in the following way (see e.g. [3, p. 2]). For all $a, b \in A$, one has $a * b = a \wedge b$ if $a \leq \neg b$, and $a * b = 0$ otherwise; $a \Rightarrow b = 1$ if $a \leq b$, and $a \Rightarrow b = \neg a \vee b$ otherwise. We shall often use this observation in subsequent calculations, starting from the next Example.

Example 4.6: Let $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, \neg, 0, 1\rangle$ be an algebra with universe $A := \{0, a, b, 1\}$ such that the lattice $\langle A; \wedge, \vee, 0, 1\rangle$ is linearly ordered as follows: $0 < a < b < 1$. The negation $\neg$ is defined by: $\neg 0 = 1$, $\neg 1 = 0$, $\neg a = b = \neg b$. The operations $*$ and $\Rightarrow$ are then determined by the above prescriptions for WNM-chains. It is easy to check that $\mathbf{A}$ is a WNM-algebra–this is an application of a general method for producing WNM-chains: see [20, Definition 6.37]; in fact, $\mathbf{A}$ is a DP-algebra[3]. Now, $\mathbf{A}$ does not satisfy the Nelson equation, because

$$\begin{aligned}(b \Rightarrow (b \Rightarrow a)) \wedge (\neg a \Rightarrow (\neg a \Rightarrow \neg b)) &= (b \Rightarrow b) \wedge (\neg a \Rightarrow 1)\\ &= 1 \not\leq b = b \Rightarrow a.\end{aligned}$$

The following proposition shows that, as expected, the lattice (or even meet-semilattice) structure of a WNM-algebra $\mathbf{A}$ together with the negation determine whether $\mathbf{A}$ satisfies the Nelson equation or not.

Proposition 4.7: A WNM-algebra $\mathbf{A}$ is a (pre-linear) quasi-Nelson algebra if and only if $\mathbf{A}$ satisfies $\neg\neg x \wedge \neg x \leq x$.

Proof: It is shown in [22] and easy to check (using twist-structures) that every QN-algebra satisfies $\neg\neg x \wedge \neg x \leq x$. Conversely, relying on pre-linearity, we are going to show that every WNM-chain $\mathbf{C}$ that satisfies $\neg\neg x \wedge \neg x \leq x$. also satisfies the Nelson equation. Observe that, on a chain, $\neg\neg a \wedge \neg a \leq a$ implies $a = \neg\neg a$ or $\neg a \leq a$, for all $a \in C$. As mentioned earlier, on a WNM-chain, we have $a^2 = 0$ if $a \leq \neg a$ and $a^2 = a$ if $\neg a < a$. Thus, for all $a, b \in C$, if $\neg a < a$, then $(a^2 \Rightarrow b) \wedge ((\neg b)^2 \Rightarrow \neg a) = (a \Rightarrow b) \wedge ((\neg b)^2 \Rightarrow \neg a) \leq a \Rightarrow b$, as required. Thus, assume $a \leq \neg a$, which implies $(a^2 \Rightarrow b) = 0 \Rightarrow b = 1$. Thus $(a^2 \Rightarrow b) \wedge ((\neg b)^2 \Rightarrow \neg a) = 1 \wedge ((\neg b)^2 \Rightarrow \neg a) = (\neg b)^2 \Rightarrow \neg a$. If $a \leq b$, then $a \Rightarrow b = 1$, and we are done. Thus, assume $b < a \leq \neg a$. Then $a \Rightarrow b = \neg a \vee b = \neg a$. Thus, we need to show $(\neg b)^2 \Rightarrow \neg a \leq \neg a$. If $\neg b \leq \neg\neg b$, then $b < a \leq \neg a \leq \neg b \leq \neg\neg b$. Since $\neg\neg b \wedge \neg b \leq b$, we have either $b = \neg\neg b$ or $\neg b \leq b$: both are against our assumptions, for each of them implies $a \leq b$. Thus $\neg\neg b < \neg b$, which means $(\neg b)^2 \Rightarrow \neg a = \neg b \Rightarrow \neg a$. We thus need to show $\neg b \Rightarrow \neg a \leq \neg a$. If $\neg a < \neg b$, then $\neg b \Rightarrow \neg a = \neg\neg b \vee \neg a$. If $\neg\neg b \leq \neg a$, we are done. Thus, assume $\neg a < \neg\neg b$. Then $b < \neg a < \neg\neg b < \neg b$. Using $\neg\neg b \wedge \neg b \leq b$ again, we have either $b = \neg\neg b$ or $\neg b \leq b$: both against our assumptions. It thus remains to consider the case where $\neg b = \neg a$. Then $b < a \leq \neg a = \neg b$ and $\neg\neg b \wedge \neg b \leq b$ gives us $\neg\neg b = b$. This means that $\neg\neg a = \neg\neg b = b$. Since $a \leq \neg\neg a$, this would imply $a \leq b$, against our assumptions. This completes our proof. ∎

We summarise our findings below:

Corollary 4.8: The following varieties coincide:

(i) Pre-linear QN-algebras.

(ii) WNM-algebras satisfying $\neg\neg x \wedge \neg x \leq x$.

(iii) $\{\mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle : \mathbf{G}_+ \text{ is a G-algebra}\}$.

Corollary 4.8 thus identifies a subclass of WNM-algebras that are representable via the twist construction, and this is (as far as we are aware) the first result of this type for WNM-algebras. Given the parallel between twist-structures and rotations, Corollary 4.8 also suggests that a suitable modification of the rotation construction may allow us to give an alternative representation for this subclass of WNM-algebras.

Subdirectly irreducibles and directly indecomposables

We end the section with a sample application of the twist representation, which applies to (pre-linear) QN-algebras and therefore also to those WNM-algebras that satisfy the equation $\neg\neg x \wedge \neg x \leq x$. We shall use the following result from [24, Prop. 8].

Lemma 4.9: Let $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle$ be a pre-linear QN-algebra. The lattice $\mathrm{Con}(\mathbf{A})$ of congruences of $\mathbf{A}$ is isomorphic to the lattice $\mathrm{Con}(\mathbf{G}_+)$ of congruences of $\mathbf{G}_+$ via the maps $(.)_+$ and $(.)^{\bowtie}$ defined as follows:

(i) For $\theta \in \mathrm{Con}(\mathbf{A})$ and $a_+, b_+ \in G_+$, let $\langle a_+, b_+\rangle \in \theta_+$ if and only if there are $a_-, b_- \in G_-$ such that $\langle a_+ \to_+ b, a_-\rangle, \langle b_+ \to_+ a, b_-\rangle \in A$ and $\langle\langle a_+ \to_+ b, a_-\rangle, \langle 1_+, 0_-\rangle\rangle, \langle\langle b_+ \to_+ a, b_-\rangle, \langle 1_+, 0_-\rangle\rangle \in \theta$.

(ii) For $\eta \in \mathrm{Con}(\mathbf{G}_+)$ and $\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in A$, let $\langle\langle a_+, a_-\rangle, \langle b_+, b_-\rangle\rangle \in \eta^{\bowtie}$ if and only if $\langle a_+, b_+\rangle, \langle p(a_-), p(b_-)\rangle \in \eta$.

Proposition 4.10: Let $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle$ be a pre-linear QN-algebra. Then:

(i) $\mathbf{A}$ is subdirectly irreducible iff $\mathbf{G}_+$ is a subdirectly irreducible G-algebra.

(ii) $\mathbf{A}$ is directly indecomposable iff $\mathbf{G}_+$ is a directly indecomposable G-algebra.

[3]DP-algebras [?] are WNM-algebras satisfying $x \vee \neg x^2 \approx 1$. Using the twist representation, it is not difficult to show that the only DP-chains which satisfy the Nelson equation are the two-element and three-element one (isomorphic, respectively, to the two-element Boolean algebra and the three-element MV-algebra).

Proof: Item (i) is an application of Lemma 4.9. Regarding (ii), observe that $\mathbf{A}$ is not directly indecomposable iff there are non-trivial factor congruences $\theta, \theta' \in \mathrm{Con}(\mathbf{A})$. If this is the case, then $\theta_+, \theta'_+ \in \mathrm{Con}(\mathbf{G}_+)$ are non-trivial factor congruences of $\mathbf{G}_+$. Indeed, this follows from Lemma 4.9 together with the observation that $\mathbf{G}_+$, as a residuated lattice, is congruence-permutable [14, p. 94]. By the same token, $\mathbf{A}$ is congruence-permutable as well. Then, if $\eta_1, \eta_2 \in \mathrm{Con}(\mathbf{G}_+)$ are non-trivial factor congruences, then $\eta_1^{\bowtie}, \eta_2^{\bowtie} \in \mathrm{Con}(\mathbf{A})$ are non-trivial factor congruences. ■

Let $\mathbf{A}$ be a QN-algebra and $a \in A$. We say that a is a *splitting element* if, for all $b \in A$, either $a \leq b$ or $b < a$. We say that a is *idempotent* (with respect to the monoid operation) if $a^2 = a$.

Proposition 4.11: For every quasi-Nelson algebra $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle$, the following are equivalent:

(i) $\mathbf{G}_+$ (and therefore $\mathbf{G}_-$) has a unique atom.

(ii) $\mathbf{A}$ has a splitting idempotent element e such that $\neg e < e$, $a^2 = 0$ for all $a < e$, and $b^2 = b$ for all $e \leq b$.

Proof: Regarding (i), let us preliminary observe that, if e_+ is the unique atom of $\mathbf{G}_+$, then $n(e_+)$ is the unique atom of $\mathbf{G}_-$. Indeed, $n(e_+) \neq 0_-$, because $n(e_+) = 0_-$ would imply $e_+ \leq_+ pn(e_+) = p(0_-) = 0_+$, against the assumption that $e_+ \neq 0_+$. Further, for all $a_- \in G_-$ with $a_- \neq 0_-$, we have $p(a_-) \neq 0_+$. Indeed, $p(a_-) = 0_+$ would imply $a_- = np(a_-) = n(0_+) = 0_-$, against our assumptions. Then $e_+ \leq_+ p(a_-)$, which implies $n(e_+) \leq_- np(a_-) = a_-$, as claimed.

Now, assume (i) holds, and let $e_+ \in G_+$ be the unique atom of $\mathbf{G}_+$ (so $n(e_+)$ is the unique atom of $\mathbf{G}_-$). Take $e = \langle e_+, 0_-\rangle$, and observe that $\neg\langle e_+, 0_-\rangle = \langle p(0_-), n(e_+)\rangle = \langle 0_+, n(e_+)\rangle < \langle e_+, 0_-\rangle$, $(\neg\langle e_+, 0_-\rangle)^2 = \langle 0_+, n(0_+) \to_- n(e_+)\rangle = \langle 0_+, 1_-\rangle$, and that $e \in A$. The latter holds true because, on the one hand, $e_+ \wedge_+ p(0_-) = e_+ \wedge_+ 0_+ = 0_+$. On the other hand, since $e_+ \to_+ 0_+$ is the pseudo-complement of e_+, we have $e_+ \to_+ 0_+ = 0_+$ (hence also $a_+ \to_+ 0_+ \leq_+ e_+ \to_+ 0_+ = 0_+$ for every $a_+ \in G_+$ with $a_+ \neq 0_+$). So every non-zero element of $\mathbf{G}_+$ is dense, and $e_+ \in D(\mathbf{G}_+) \subseteq \nabla$ for any possible choice of ∇. Then $e_+ \vee_+ p(0_-) \in \nabla$, as required. Next, observe that every element of $\mathbf{A}$ is comparable with $\langle e_+, 0_-\rangle$. Indeed, for all $\langle a_+, a_-\rangle \in A$, the existence of a unique atom in $\mathbf{G}_+$ together with the requirement $a_+ \wedge_+ p(a_-) = 0_+$ entail that either $a_+ = 0_+$ or $p(a_-) = 0_+$ (in which case $a_- = np(a_-) = n(0_+) = 0_-$). Thus every element of $\mathbf{A}$ has the form $\langle a_+, 0_-\rangle$ or $\langle 0_+, a_-\rangle$ for some $a_+ \in G_+$ and $a_- \in G_-$. Obviously $\langle 0_+, a_-\rangle \leq \langle e_+, 0_-\rangle$ for all $a_- \in G_-$, and observe that $\langle 0_+, a_-\rangle^2 = \langle 0_+, n(0_+) \to_- a_-\rangle = \langle 0_+, 0_- \to_- a_-\rangle = \langle 0_+, 1_-\rangle$, as claimed in (ii). On the other hand, for all $a_+ \neq 0_+$, we have $\langle e_+, 0_-\rangle \leq \langle a_+, 0_-\rangle$. So every element of $\mathbf{A}$ is comparable with $\langle e_+, 0_-\rangle$, as required. Let us verify that $\langle e_+, 0_-\rangle$ is an idempotent. Since $n(e_+) \neq 0_-$, we have $n(e_+) \to_- 0_- = 0_-$. Then $\langle e_+, 0_-\rangle^2 = \langle e_+, n(e_+) \to_- 0_-\rangle = \langle e_+, 0_-\rangle$, as required. Lastly, since $n(e_+) \leq_+ n(a_+)$, we have $n(a_+) \neq 0_-$ for all $a_+ \in G_+$, so $\langle a_+, 0_-\rangle^2 = \langle e_+, n(a_+) \to_- 0_-\rangle = \langle a_+, 0_-\rangle$, as claimed.

Conversely, assume (ii) holds, and let $e = \langle e_+, e_-\rangle$ be the splitting element of $\mathbf{A}$. Let $a_+ \in G_+$ be such that $a_+ \neq 0_+$. Then there is $a_- \in G_-$ such that $\langle a_+, a_-\rangle \in A$. Moreover, $a_+ \neq 0_+$ entails $\langle a_+, a_-\rangle^2 = \langle a_+, n(a_+) \to_- a_-\rangle \neq \langle 0_+, 1_-\rangle$. Thus, it cannot be the case that $\langle a_+, a_-\rangle < e$. Hence (since e is a splitting element), $e \leq \langle a_+, a_-\rangle$, which entails $e_+ \leq_+ a_+$. This shows that e_+ is the unique atom of $\mathbf{G}_+$. As observed earlier, it follows that $n(e_+)$ is the unique atom of $\mathbf{G}_-$. ■

It may be worth mentioning that both Propositions 4.10 and 4.11 still hold true if we drop the pre-linearity hypothesis, replacing the G-algebras $\mathbf{G}_+, \mathbf{G}_-$ with Heyting algebras $\mathbf{H}_+, \mathbf{H}_-$ (Proposition 4.12 below, on the other hand, is specific to G-algebras).

Taking into account Proposition 4.10.ii, it is clear that (either of) the conditions in Proposition 4.11 entail that $\mathbf{A}$ is directly indecomposable. This implication becomes an equivalence in the case of finite pre-linear quasi-Nelson algebras, as the following proposition shows.

Proposition 4.12: For every finite pre-linear QN-algebra $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla\rangle$, the following are equivalent:

(i) The G-algebra algebra $\mathbf{G}_+$ (and therefore also $\mathbf{G}_-$) has a unique atom.

(ii) $\mathbf{A}$ has a splitting idempotent element e such that $a^2 = 0$ for all $a < e$ and $b^2 = b$ for all $e \leq b$.

(iii) $\mathbf{A}$ is directly indecomposable.

Proof: We seen in Proposition 4.11 the equivalence of (i) and (ii), together with the observation that $\mathbf{G}_-$ has a unique atom when $\mathbf{G}_+$ has a unique atom. We proceed to show that (i) and (iii) are equivalent. We have seen in Proposition 4.10.ii that $\mathbf{A}$ is directly indecomposable iff $\mathbf{G}_+$ is. To complete our proof, it is sufficient to recall that directly indecomposable finite G-algebras are precisely those having a unique atom (see e.g. [11, p. 56-57]). ■

V. Nelson and NM-algebras

In this section we show that things are different in the involutive setting: indeed, pre-linear Nelson algebras coincide with NM-algebras. This result is known since at least [9], but we present here a shorter proof that takes advantage of the recent insight on involutive CIBRLs gained in [26].

Lemma 5.1: Every WNM-algebra satisfies the equation:

$$x \approx x^2 \vee (x \wedge \neg x).$$

Proof: Relying on pre-linearity, we verify that the equation is satisfied by every WNM-chain $\mathbf{C}$. As mentioned earlier, on a WNM-chain we have $a^2 = 0$ if $a \leq \neg a$ and $a^2 = a$ if $\neg a < a$, for all $a \in C$. In the former case, we have $a^2 \vee (a \wedge \neg a) = 0 \vee a = a$. In the latter, $a^2 \vee (a \wedge \neg a) = a \vee \neg a = a$. ■

Corollary 5.2: Every NM-algebra is a pre-linear Nelson algebra. Hence Lemma 4.3 entails NM = PN.

Proof: Let $\mathbf{A}$ be a NM-algebra. Then $\mathbf{A}$ is involutive and, by Lemma 5.1, $\mathbf{A}$ satisfies the equation $x \approx x^2 \vee (x \wedge \neg x)$. It is shown in [26, Theorem 6.1] that, for an involutive CIBRL, this is equivalent to being a Nelson algebra. ■

Corollary 5.2, together with the structural results recalled in Section III, entails that NM-algebras are (as rotations) in a one-to-one correspondence with pairs $\langle \mathbf{G}, \mathbf{f} \rangle$ where $\mathbf{G}$ is a G-algebra and $\mathbf{f}$ a filter of the Boolean skeleton of $\mathbf{G}$, and (as twist-structures) are also in a one-to-one correspondence with pairs $\langle \mathbf{G}, \nabla \rangle$ where $\mathbf{G}$ is a G-algebra and ∇ a dense filter of $\mathbf{G}$. To see that the two perspectives indeed match, it is sufficient to observe that, on every G-algebra $\mathbf{G}$, the filters of the Boolean skeleton are in one-to-one correspondence with the dense filters[4]. On account of space limitations, a detailed analysis of this result will be deferred to a future publication.

It may be worth mentioning that Example 4.6, together with Lemma 5.1, provides an answer to a problem that was left open in [24]: namely, whether the equation $x \approx x^2 \vee (x \wedge \neg x)$ may be proven to be equivalent, in a non-involutive setting, to the Nelson equation. (The answer is, of course, negative.)

Another open problem mentioned in [24] can be recast (and resolved) in the present context. As observed earlier, every G-algebra is a (pre-linear) QN-algebra, and (by Corollary 5.2) every NM-algebra is also a (pre-linear) QN-algebra. Thus $\mathrm{G} \cup \mathrm{NM} \subseteq \mathrm{PQN}$. Indeed, in a fuzzy setting, one could motivate the introduction pre-linear QN-algebras as 'a common generalisation of Gödel and NM-algebras'. One might further enquire whether this is a 'minimal' generalisation, in the sense, for instance, that the variety $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$ generated by $\mathrm{G} \cup \mathrm{NM}$ is precisely PQN. Also in this case the answer is negative.

Let us begin by observing that, since $\mathrm{PQN} = \mathrm{V}(\mathrm{PQN})$, we have $\mathrm{V}(\mathrm{G} \cup \mathrm{NM}) \subseteq \mathrm{PQN}$. Thus $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$ is also a subvariety of WNM, i.e. a variety of algebras of fuzzy logic. Let us further note that the equation $(x \Rightarrow x^2) \vee (\neg\neg y \Rightarrow y) \approx 1$ is clearly satisfied by every algebra in $\mathrm{G} \cup \mathrm{NM}$, and therefore in $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$. However, there are algebras in PQN that do not satisfy it, as the following Example shows (see also [3, Def. 11]).

Example 5.3: Consider the (uniquely determined) three- and the two-element G-chains:

$$\mathbf{G}_+ = \langle G_+ = \{0_+, a_+, 1_+\}; \wedge_+, \vee_+, \rightarrow_+, 0_+, 1_+ \rangle$$
$$\mathbf{G}_- = \langle G_- = \{0_-, 1_-\}; \wedge_-, \vee_-, \rightarrow_-, 0_-, 1_- \rangle.$$

Let $\nabla = G_+$. Define $n \colon G_+ \to G_-$ by $n(a_+) = n(1_+) = 1_-$ and $n(0_+) = 0_-$, and $p \colon G_- \to G_+$ in the obvious way, i.e. $p(0_-) = 0_+$ and $p(1_-) = 1_+$. These determine a QN twist-structure $\mathbf{A} = \mathbf{Tw}\langle G_+, G_-, n, p, \nabla \rangle$. Observe that $\langle 0_+, 0_- \rangle, \langle a_+, 0_- \rangle \in A$. We have $\langle 0_+, 0_- \rangle^2 = \langle 0_+, 1_- \rangle$ and $\neg\neg\langle a_+, 0_- \rangle = \langle 1_+, 0_- \rangle$, which give us the following: $(\langle 0_+, 0_- \rangle \Rightarrow \langle 0_+, 0_- \rangle^2) \vee (\neg\neg\langle a_+, 0_- \rangle \Rightarrow \langle a_+, 0_- \rangle) = (\langle 0_+, 0_- \rangle \Rightarrow \langle 0_+, 1_- \rangle) \vee (\langle 1_+, 0_- \rangle \Rightarrow \langle a_+, 0_- \rangle) = \langle 0_+, 0_- \rangle \vee \langle a_+, 0_- \rangle = \langle a_+, 0_- \rangle \neq \langle 1_+, 0_- \rangle$.

The following lemma entails that $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$ is axiomatised precisely by adding $(x \Rightarrow x^2) \vee (\neg\neg y \Rightarrow y) \approx 1$ to the equational presentation of PQN.

Lemma 5.4: Let $\mathbf{A}$ be a linearly orered pre-linear QN-algebra. The following are equivalent:

(i) $\mathbf{A}$ satisfies $(x \Rightarrow x^2) \vee (\neg\neg y \Rightarrow y) \approx 1$.
(ii) $\mathbf{A}$ is either a Gödel algebra or a Nelson algebra.

Proof: The non-trivial implication is from (i) to (ii). Let then $\mathbf{A}$ be a subdirectly irreducible algebra in PQN. Suppose $\mathbf{A}$ is neither Gödel nor Nelson. Then there are elements $a, b \in A$ such that $a \neq a^2$ (thus $a^2 < a$) and $b \neq \neg\neg b$ (thus $b < \neg\neg b$). This means that $a \Rightarrow a^2 < 1$ and $\neg\neg b \Rightarrow b < 1$. Since $\mathbf{A}$ is linearly ordered, the preceding considerations imply $(a \Rightarrow a^2) \vee (\neg\neg b \Rightarrow b) \neq 1$. ■

Lemma 5.4 applies, in particular, to subdirectly irreducible QN-algebras (which are linearly ordered, by Proposition 4.10.i). The following result is thus an immediate consequence (as well as an instance of [14, Lemma 5.25]).

Corollary 5.5: $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$ is the subvariety of PQN axiomatised by:

$$(x \Rightarrow x^2) \vee (\neg\neg y \Rightarrow y) \approx 1.$$

Proof: Observe that the subvariety of PQN axiomatised by $(x \Rightarrow x^2) \vee (\neg\neg y \Rightarrow y) \approx 1$ and $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$ have the same subdirectly irreductible members; therefore, they must coincide [7, II, Cor. 9.7]. ■

The next (and last) corollary entails *standard completeness* (see item 3. in Section VI) of the logic associated to the class $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$. Denote by $[0,1]_{\mathbf{G}}$ and by $[0,1]_{\mathbf{NM}}$, respectively, the G-algebra and NM-algebra having as universe the real interval $[0,1]$; both algebras are unique up to isomorphism. It is well known that $\mathrm{V}([0,1]_{\mathbf{G}})$ is the variety of G-algebras and $\mathrm{V}([0,1]_{\mathbf{NM}})$ is the variety of NM-algebras.

Corollary 5.6: $\mathrm{V}(\mathrm{G} \cup \mathrm{NM}) = \mathrm{V}(\{[0,1]_{\mathbf{G}}, [0,1]_{\mathbf{NM}}\})$.

Proof: Our previous considerations entail that $\mathrm{V}(\{[0,1]_{\mathbf{G}}, [0,1]_{\mathbf{NM}}\}) \subseteq \mathrm{V}(\mathrm{G} \cup \mathrm{NM})$. For the converse inclusion we proceed as in Corollary 5.5. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathrm{V}(\mathrm{G} \cup \mathrm{NM})$. Then, by Lemma 5.4, $\mathbf{A}$ is either a G-algebra or an NM-algebra. Thus either $\mathbf{A} \in \mathrm{V}([0,1]_{\mathbf{G}})$ or $\mathbf{A} \in \mathrm{V}([0,1]_{\mathbf{NM}})$. In both cases we have $\mathbf{A} \in \mathrm{V}(\{[0,1]_{\mathbf{G}}, [0,1]_{\mathbf{NM}}\})$, as claimed. ■

VI. FUTURE WORK

As mentioned in the Introduction, this paper has been a first attempt at establishing a connection between the theory of quasi-Nelson algebras/logics and fuzzy systems extending MTL. Future research may take several directions, among which we mention a few below.

1. Is it possible to extend (some form of) twist-structure representation to the whole class of WNM-algebras? Recent results [22] show that twist-structures can be used to represent algebras in the implication-free language $\langle \wedge, \vee, \neg \rangle$, including De Morgan and Kleene lattices [21], as well as more general sub(quasi)varieties of semi-De Morgan algebras [25]. The algebras representable in this way have been dubbed *semi-Kleene lattices* in [22]. It is easy to check that the $\langle *, \Rightarrow \rangle$-free reduct of every WNM-algebra is a semi-Kleene lattice; as we have seen, it is this reduct that determines (on WNM-chains) the behaviour of the remaining operations. These considerations suggest that representing WNM-algebras via twist-structure may indeed be a feasible project.

[4]Interestingly, this correspondence does not generalise to Heyting algebras: which is perhaps suggesting that 'non-pre-linear NM-algebras' may not be representable as rotations (whereas they are, as twist-structures).

2. The above-mentioned question suggests another structural relation that may be worthwhile exploring: namely the one between WNM-algebras (and thus PQN) and *Sugihara monoids*. These structures, that are algebraic models of relevance logic, have also been studied from a twist representation point of view; interestingly, the twist representation proposed (e.g.) in [15] also decomposes a Sugihara monoid as a special binary power of a Gödel algebra. To make this even more intriguing, we may further observe that every Sugihara monoid (indeed, even every *generalised Sugihara monoid* in the sense of [15]) also has a semi-Kleene lattice reduct…

3. From a fuzzy logic point of view on the logic of pre-linear QN-algebras (i.e. the extension of WNM-logic by the axiom $(\neg\neg\varphi \wedge \neg\varphi) \Rightarrow \varphi$), an obvious (and open) question concerns so-called *standard completeness*. This property means for a logic to be complete not only with respect to its linearly ordered algebraic models (which we know to be true, by pre-linearity), but also with respect to a class of (possibly non-isomorphic) algebras defined over the real interval $[0, 1]$. Standard completeness is known to hold for WNM-logic [12, Thm. 3], but it is also well known that the property need not be preserved by axiomatic extensions. An even stronger form of standard completeness is *single-chain real completeness*, that is, completeness with respect to a unique algebraic model over the real unit interval. This property, which holds for Gödel and NM-logic, is entailed by the observation that the G-algebra (resp. NM-algebra) over $[0, 1]$ is unique up to isomorphism. Since every G-algebra and every NM-algebra is a pre-linear QN-algebra, we have over $[0, 1]$ at least two non-isomorphic QN-algebras (in fact, Example 5.3 suggests that there may be more). This however, does not destroy all hope of proving single-chain real completeness for the logic of pre-linear QN-algebras. Therefore, both standard and single-chain real completeness are currently open problems, which we intend to address in future work.

4. Lastly, let us mention a possible extension of our approach outside the setting of integral residuated lattices. Alongside constructive logic with strong negation, a later paper by David Nelson [1] introduced a paraconsistent weakening of N-logic that is nowadays known as N4. The algebraic models of N4 (called *N4-lattices*) are residuated structures related to algebras of relevance logics, which can also be represented as twist-structures by a straightforward generalisation of the construction presented in this paper. Indeed, the twist construction for N-algebras (though not the one for QN) may be seen as a special case of that for N4-lattices; abstractly, N-lattices correspond precisely to the subvariety of N4-lattices defined by the equation $x \Rightarrow x \approx y \Rightarrow y$. N4-lattices that are twist-structures over Gödel algebras have already been studied in the paper [4], but the algebras considered there are not pre-linear in the usual sense. We speculate that a more thorough investigation of 'pre-linear N4-lattices' (along the lines of the present paper, as well as extending the results of [4]) may turn out to be an intriguing project for future research.

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