On the representation of (weak) nilpotent minimum algebras

Umberto Rivieccio  
DIMAp  
UFRN  
Natal, Brazil  
ORCID 0000-0003-1364-5003  
urivieccio@dimap.ufrn.br

Tommaso Flaminio  
IIA  
CSIC  
Bellaterra, Spain  
ORCID 0000-0002-9180-7808  
tommaso@iiia.csic.es

Thiago Nascimento  
PPGSC - DIMAp  
UFRN  
Natal, Brazil  
ORCID 0000-0002-9288-8929  
thiagnascsilva@gmail.com

Abstract—We take a glimpse at the relation between WNM-algebras (algebraic models of the well-known Weak Nilpotent Minimum logic) and quasi-Nelson algebras, a non-involutive generalisation of Nelson algebras (models of Nelson’s constructive logic with strong negation) that was introduced in a recent paper. We show that the two varieties can be related via the twist-structure construction, obtaining a new representation for a subvariety of WNM-algebras that includes the involutive ones (i.e. NM-algebras). Our results imply, in particular, that every pre-linear quasi-Nelson algebra is a WNM-algebra; we thus generalize the known result that the class of pre-linear Nelson algebras coincides with that of NM-algebras (models of Nilpotent Minimum logic).

Index Terms—weak nilpotent minimum, quasi-Nelson, monoidal t-norm, twist representation

I. INTRODUCTION

Monoidal t-norm logic (MTL), the logic of left-continuous t-norms, is among the most prominent systems in the mathematical fuzzy logic literature. MTL was introduced in [12], and the same paper [12, Sec. 3] considers certain axiomatic extensions of MTL that result from imposing stronger requirements on the negation connective. Among these, a weak negation function on the real interval [0,1] determines weak nilpotent minimum logics, and a strong (involutive) negation defines nilpotent minimum logic (NML). Algebraic models of MTL as well as those of the above-mentioned extensions (called respectively MTL-algebras, WNM-algebras and NM-algebras) have been studied extensively, and several representation results are known.

Nelson algebras are the algebraic models of Nelson’s constructive logic with strong negation [19], a system obtained by adding a new involutive negation to positive intuitionistic logic.

Structurally, NM-algebras and Nelson algebras are closely related. Indeed, Busaniche and Cignoli [9] proved that if one adds the pre-linearity equation

\[ (x \Rightarrow y) \lor (y \Rightarrow x) \approx 1 \]

to Nelson algebras, one obtains precisely the variety of NM-algebras.

From a methodological viewpoint, this result is particularly interesting, for it entails that every NM-algebra (as a Nelson algebra) can be represented as a special binary power (called a twist-structure) of a (pre-linear) Heyting algebra (i.e. a Gödel algebra), and also as a (dis)connected rotation of a Gödel algebra (see [8], [18]).

(Dis)connected rotations have been generalized in [10] to account for some non-involutive structures (see also [2]). With a similar purpose, the twist-structure construction has been recently extended to a non-involutive setting [23], [24]. The twist construction, when applied to pairs of Heyting algebras, determines the class of quasi Nelson-algebras.

In the present paper we focus on the interplay between the pre-linearity equation and the non-involutive twist-structure construction. In particular, we prove that pre-linear quasi-Nelson algebras correspond precisely to the class of (non-involutive) twist-structures over pairs of Gödel algebras. As a class of abstract algebras, the latter is a proper subvariety of WNM-algebras.

II. NM, WNM AND (QUASI-)NELSON

We assume familiarity with basic results of universal algebra [7], residuated lattices [14] and fuzzy logics [16]. Although we shall be dealing exclusively with algebras, it is important to keep in mind that every variety considered here is the algebraic counterpart (in the strong sense of [6]) of some substructural/fuzzy logic. Thus, virtually all algebraic results stated in the next sections have a straightforward logical counterpart.

A commutative integral bounded residuated lattice (CIBRL) is an algebra \( \mathbf{A} = \langle A; \land, \lor, *, \Rightarrow, 0, 1 \rangle \) of type \( (2, 2, 2, 2, 0, 0) \) such that:

(i) \( \langle A; *, 1 \rangle \) is a commutative monoid,

(ii) \( \langle A; \land, \lor, 0, 1 \rangle \) is a bounded lattice (with order \( \leq \)).
(iii) \(a \ast b \leq c\) iff \(a \leq b \Rightarrow c\) for all \(a, b, c \in A\) (residuation).

The unary negation operation is defined by \(\neg a := a \Rightarrow 0\) for all \(a \in A\). Notice that the negation of every CIBL \(A\) satisfies the properties postulated in [12] for a weak negation, that is, \(\neg\) is order-reversing, \(\neg 1 = 0\) and \(a \leq \neg\neg a\) for all \(a \in A\).

An MTL-algebra is a CIBL that additionally satisfies the pre-linearity equation introduced in the preceding Section. MTL-algebras are thus said to be pre-linear, suggesting the well-known result that every MTL-algebra is isomorphic to a subdirect product of linearly ordered ones. The same applies to the subvarieties of MTL-algebras introduced below, and entails in particular that the lattice reduct of every such algebra is distributive.

WNM-algebras are the subvariety of MTL-algebras defined by the weak nilpotent minimum equation:

\[\neg(x \ast y) \lor ((x \land y) \Rightarrow (x \ast y)) \approx 1.\]

In turn, NM-algebras are obtained from WNM by adding the involutive equation \(\neg\neg x \approx x\) (or, equivalently, just \(\neg x \leq x\)). Thus \(\text{NM} \subseteq \text{WNM} \subseteq \text{MTL} \subseteq \text{CIBL}\) (all inclusions being proper).

Moving to the realm of Nelson logics, a quasi-Nelson algebra (QN-algebra) is defined as a CIBL that satisfies the Nelson equation:

\[(x \Rightarrow (x \Rightarrow y)) \land (\neg y \Rightarrow (\neg y \Rightarrow \neg x)) \approx x \Rightarrow y.\]

QN-algebras have been introduced only recently [23], [24], whereas Nelson algebras (N-algebras) have been around for more than four decades. N-algebras are precisely the involutive QN-algebras (i.e., those that satisfy \(\neg\neg x \approx x\), or equivalently \(\neg x \leq x\)). Thus \(N \subseteq \text{QN} \subseteq \text{CIBL}\) (all inclusions proper).

The relationship between NM-algebras and Nelson algebras has been investigated in previous papers, a standard reference being [9]. There it is proved that the variety of NM-algebras coincides with the subvariety PN of Nelson algebras satisfying the pre-linearity equation. In the non-involutive setting (moving from Nelson to quasi-Nelson algebras on the one side, and from NM to WNM-algebras on the other), it is not difficult to produce a (linearly ordered) WNM-algebra that does not satisfy the Nelson equation (see Example 4.6 below).

Denoting by PQN the variety of QN-algebras satisfying the pre-linearity equation, we thus have \(\text{WNM} \not\subseteq \text{PQN}\) (for the former, WNM \not\subseteq PN). This raises the following questions.

First: does the converse inclusion (\(\text{PN} \subseteq \text{WNM}\)) hold?

Second: how can one describe the class \(\text{WNM} \cap \text{PN}\) (or, more generally, \(\text{WNM} \cap \text{QN}\))?

As we are going to see, the answer to the first question is that, indeed, one has \(\text{PN} \subseteq \text{WNM}\), and even \(\text{PQN} \subseteq \text{WNM}\). These observations (having seen that \(\text{NM} \subseteq \text{N}\)) entail the above-mentioned result that \(\text{NM} = \text{PN}\).

The latter question brought to our attention an equational condition (only involving the \(\land\) and \(\neg\) operations; see Proposition 4.7) that, as far as we know, has never been singled out in the context of WNM-algebras.

We obtained both the above-mentioned results thanks to the twist representation of (quasi-)Nelson algebras. This perspective, which is new for WNM-algebras, led us to other interesting insights and questions, which we are going to recount in the next sections.

III. Twist-structures and Rotations

The twist-structure is a method for constructing an N-algebra (extended in [23], [24] to QN-algebras) as a subalgebra of a special product of two Heyting algebras. In fact, twist-structures yield a representation theorem: every (quasi-)Nelson algebra arises in this way. We proceed to expand the details of the construction, restricting our attention (since we are in a pre-linear setting) to pre-linear Heyting algebras factors, known as Gödel algebras in the fuzzy literature (from now on G-algebras).

Let:

\[G_+ = \langle G_+, \leq_+; \land_+, \lor_+, \neg_+, 0_+, 1_+ \rangle\]

\[G_- = \langle G_-, \leq_-; \land_-, \lor_-, \neg_-, 0_-, 1_- \rangle\]

be both G-algebras, and \(n: G_+ \rightarrow G_-\) and \(p: G_- \rightarrow G_+\) be bounded lattice homomorphisms, additionally satisfying the following requirements: \(n \cdot p = 1d_G\) and \(1d_{G_+} \leq_+ p \cdot n\).

Define an algebra \(G_+ \bowtie G_- = \{G_+ \times G_-; \land, \lor, \neg, 0, 1\}\) as follows: for all \(\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in G_+ \times G_-\),

\[\begin{align*}
1 &= \langle 1_+, 0_- \rangle \\
0 &= \langle 0_+, 1_- \rangle \\
\neg\neg\langle a_+, a_-\rangle &= \langle \neg\neg a_+, \neg\neg a_- \rangle \\
\langle a_+, a_-\rangle \land \langle b_+, b_-\rangle &= \langle a_+ \land b_+, a_- \lor b_- \rangle \\
\langle a_+, a_-\rangle \lor \langle b_+, b_-\rangle &= \langle a_+ \lor b_+, a_- \land b_- \rangle \\
\langle a_+, a_-\rangle \rightarrow \langle b_+, b_-\rangle &= \langle a_+ \rightarrow b_+, n(a_+) \land b_- \rangle.
\end{align*}\]

The residuated operations are given by the following terms:

\[x \Rightarrow y = (x \rightarrow y) \land (\neg y \rightarrow \neg x)\]

\[x \ast y = x \land (y \lor (x \Rightarrow y)).\]

Component-wise, these give us \(\langle a_+, a_-\rangle \Rightarrow \langle b_+, b_-\rangle = \langle a_+ \rightarrow b_+, b_- \rangle \land \langle p(b_+) \rightarrow p(a_+), n(a_+) \land b_- \rangle\), and \(\langle a_+, a_-\rangle \ast \langle b_+, b_-\rangle = \langle a_+ \land b_+, n(a_+) \lor b_- \rangle \land \langle n(b_+) \rightarrow a_- \rangle\). Also observe that the lattice order \(\leq\) on \(G_+ \bowtie G_-\) is given, for all \(\langle a_+, a_-\rangle, \langle b_+, b_-\rangle \in G_+ \times G_-\), by \(\langle a_+, a_-\rangle \leq \langle b_+, b_-\rangle\) if \((a_+ \leq b_+\) and \(b_- \leq a_-\)).

The algebra \(G_+ \bowtie G_-\) may not even be a CIBL, but we can obtain a quasi-Nelson algebra by considering the following subalgebra. Let:

\[D(G_+) = \{a_+ \in G_+: \neg_+ a_+ = 0_+\}\]

be the set of dense elements of \(G_+\), and consider a lattice filter \(\nabla \subseteq G_+\) such that \(D(G_+) \subseteq \nabla\). One can then show that the set \(Tw(G_+, G_-, n, p, \nabla) = \{\langle a_+, a_-\rangle \in G_+ \times G_-: a_+ \land \neg_+ p(a_+) = 0_+\}\) is the universe of a subalgebra of \(G_+ \bowtie G_-\). We call the corresponding algebra \(A = Tw(G_+, G_-, n, p, \nabla)\) a QN twist-structure\(^1\).

\(^1\)A technical observation that will be useful in proofs: the requirement \(a_+ \land \neg_+ p(a_+) = 0_+\) entails that \(n(a_+) \land a_- = n(a_+) \land n(p(a_)) = n(a_+ \land \neg_+ p(a_+)) = n(0_+) = 0_+\).
The algebra $A$ with the operations $\langle \land, \lor, *, \Rightarrow, 0, 1 \rangle$ is a CIBRL and a QN-algebra [24, Thm. 2]; moreover, every QN-algebra arises in this way.

To justify the above claim, let $A = \langle A; \land, \lor, *, \Rightarrow, 0, 1 \rangle$ be a pre-linear QN-algebra. Define the operation $\rightarrow$ by the term:

$$x \rightarrow y = x^2 \Rightarrow y$$

where $x^2 = x \ast x$. Define the relation $\equiv$, for all $a, b \in A$, by:

$$a \equiv b \iff a \rightarrow b = b \rightarrow a = 1.$$

The connective $\rightarrow$ is known as the weak implication in the literature on Nelson logics (as opposed to the strong residuated one $\Rightarrow$), and it is the connective witnessing the deduction-detachment theorem for (quasi-)Nelson logic. One verifies that the relation $\equiv$ thus obtained is compatible with the operations $\langle \land, \lor, *, \Rightarrow \rangle$, which gives us a quotient $A_\equiv = \langle A/\equiv; \land, *, \lor, \Rightarrow, 0, 1 \rangle$. Also, the algebra $A_\equiv$ is a G-algebra (on which the operations $\land$ and $*$ coincide). Defining the set $F(A) = \{a \in A: \neg a \leq a\}$, we have that $A_\equiv = F(A)/\equiv$ is a lattice filter of $A_\equiv$, and that $D(A_\equiv) \subseteq \nabla_\equiv$. To obtain a second G-algebra factor, one considers the set $\neg A = \{a : a \in A\}$ and let $A_- = \neg A/\equiv$. Then $A_-$ is the universe of a subalgebra of $A_\equiv$, which we denote by $A_-$. Lastly, define maps $\nu_A: A_\equiv \rightarrow A_-$ and $\rho_A: A_- \rightarrow A_\equiv$ as follows: $\nu_A(a/\equiv) = \neg a/\equiv$ and $\rho_A(a/\equiv) = a/\equiv$. The tuple $\langle A_\equiv, A_-, \nu_A, \rho_A, \nabla_A \rangle$ satisfies the required properties for defining a QN twist-structure $\text{Tw}(A_\equiv, A_-, A_\equiv, \nabla_A)$. The representation theorem proved in [24, Prop. 10] then states that $A \cong \text{Tw}(A_\equiv, A_-, \nu_A, \rho_A, \nabla_A)$ through the map $\iota$ given by $\iota(a) = \langle a/\equiv, \neg a/\equiv \rangle$ for all $a \in A$.

Among QN-algebras, the involutive ones (i.e. N-algebras) are precisely those algebras $A$ such that $A = \neg A$. Hence $A_\equiv = A_-$ and $\nu_A, \rho_A$ are both the identity map. Therefore, a (pre-linear) N-algebra is determined by just a pair $\langle G, \nabla \rangle$. Another prominent subvariety of QN-algebras is the class of boolean skeleton $G$, and, the Nelson equation is easily seen to be implied by $x \Rightarrow (x \Rightarrow y) \approx x \Rightarrow y$, which is valid on all G-algebras. In terms of the twist representation, G-algebras correspond precisely to those $A$ such that $A_\equiv \cong A$.

On the other hand, NM-algebras (coinciding with pre-linear N-algebras) can be constructed from G-algebras by employing connected and disconnected rotations. Although the results of the present paper do not rely directly on these constructions, it will be useful, for further discussion, to recall the basic definitions. We begin with a special case. Let $G = \langle G; \land, \lor, \Rightarrow, \neg, 0, 1 \rangle$ be a finite and directly indecomposable G-algebra. Define:

$$CR(G) = \{\langle a, a' \rangle \in G \times G : a \land a' = 0\},$$

$$DR(G) = \{\langle a, a' \rangle \in G \times G : (a \land a') \lor (a \lor a') = 0\}.$$

The reader will have noticed a similarity between the above-defined operations and the operations of twist-structures introduced earlier; we shall return on this below.

Denote by $CR(G)$ and $DR(G)$ the algebras:

$$CR(G) = \langle CR(G); *, \Rightarrow, \land, \lor, \neg, (1, 0), (0, 1)\rangle,$$

$$DR(G) = \langle DR(G); *, \Rightarrow, \land, \lor, \neg, (1, 0), (0, 1)\rangle.$$
Boolean skeleton of) $G$. Furthermore, both methods yield representations that can be used to establish a categorical equivalence between the algebraic category of Nelson algebras (resp. of NM-algebras) and a category naturally associated to pairs of type $\langle G, f \rangle$.

It is therefore natural to ask whether this apparent parallelism is grounded on a structural relation between Nelson and NM-algebras. This is indeed the case, and the question can be addressed both on an abstract and on a concrete level: see Corollary 5.2 below and the subsequent observations.

IV. QUASI-NELSON AND WNM-ALGEBRAS

In the light of the twist representation result of the preceding section, from now on we shall, whenever convenient, assume that a QN-algebra is of the form $\text{Tw}(G_+, G_-, n, p, \nabla)$. Let $A = \langle G_+, G_-, n, p, \nabla \rangle$ be a QN-algebra. Then $A$ is linearly ordered if and only if both $G_+$ and $G_-$ are linearly ordered.

**Lemma 4.1** Let $A = \text{Tw}(G_+, G_-, n, p, \nabla)$ be a QN-algebra. Then $A$ is linearly ordered if and only if only if both $G_+$ and $G_-$ are linearly ordered.

**Proof:** if $A$ is linearly ordered, then $G_+$ and $G_-$ are linearly ordered because both are isomorphic to quotients of the lattice reducible of $A$. Conversely, assume $G_+$ and $G_-$ are linearly ordered. Then, for all $(a_+, a_-) \in A$, by the requirement $a_+ \land p(a_+) = 0$, we have either $a_+ = 0_+$ or $p(a_+) = 0_+$. (in the latter case, $np(a_-) = a_- = 0_-$). Thus all elements of $A$ are of the form $(a_+, 0_-)$ or $(0_+, a_-)$ for some $a_+ \in G_+, a_- \in G_-$. Note that $(0_+, a_-) \leq (a_+, 0_-)$ for all $a_+ \in G_+, a_- \in G_-$. On the other hand, $(0_+, a_-) \leq (0_+, b_-)$ if $b_- \leq a_- \land (a_+, 0_-) \leq (b_+, 0_-) \land (a_+ \land p(a_+) = 0) \land (b_+ \land p(b_+) = 0)$, for all $a_+, b_+ \in G_+$ and $a_-b_- \in G_-$. Thus $A$ is also linearly ordered (e.g.) as follows (assuming $b_- \leq a_- \land a_- \leq b_+$): $\ldots \leq (a_+, a_-) \leq \ldots \leq (0_+, b_-) \leq \ldots \leq (a_+, 0_-) \leq (b_+, 0_-) \ldots$

Lemma 4.1 gives us the following useful characterisation of pre-linear QN-algebras.

**Proposition 4.2** The following varieties of algebras coincide:

(i) Pre-linear quasi-Nelson algebras.

(ii) The class of all twist-structures of type $\text{Tw}(G_+, G_-, n, p, \nabla)$.

**Proof:** Taking our earlier considerations into account, we only need to prove that e.g. (ii) is a subclass of (i). To do so, we shall verify that every subdirectly irreducible algebra in (ii) is also in (i). Consider a subdirectly irreducible QN algebra $A = \text{Tw}(G_+, G_-, n, p, \nabla)$. By [24, Proposition 8], we have $\text{Con}(A) \cong \text{Con}(G_+)$. Hence the G-algebra $G_+$ is also subdirectly irreducible. Thus $G_+$ is linearly ordered [17, Lemma 3] and, by Lemma 4.1, $A$ is also linearly ordered. Then $A$ satisfies the pre-linearity equation, as required.

Proposition 4.2 could be stated in a slightly more general form. As mentioned earlier, non-involutive twist-structures can be defined over pairs of Heyting algebras (other than Nelson algebras, which are a special case). One can then observe that pre-linear quasi-Nelson algebras correspond to the class of twist-structures $\{\text{Tw}(G_+, G_-, n, p, \nabla) : G_+ \text{ is a G-algebra}\}$, simply because the Heyting algebra $G_-$ must be pre-linear whenever $G_+$ is.

For the reader familiar with Nelson algebras, we mention an easy but non-trivial consequence of Proposition 4.2: a QN-algebra $A$ satisfies the equation $(x \rightarrow y) \lor (y \rightarrow x) \approx 1$ if and only if $A$ satisfies the (seemingly weaker) equation $(x \rightarrow y) \lor (y \rightarrow x) \approx 1$, which employs the so-called weak Nelson implication given by $x \rightarrow y = x^2 \Rightarrow y$.

**Lemma 4.3** Every linearly ordered QN-algebra $A$ satisfies the WNM equation: $(x \land y) \lor ((x \land y) \rightarrow (x \land y)) \approx 1$.

**Proof:** Let $A = \text{Tw}(G_+, G_-, n, p, \nabla)$ and $a, b \in A$. We need to ensure that $a \land b \lor ((a \land b) \rightarrow (a \land b)) = 1$. We can assume, without loss of generality, that $a \leq b$. Then $a \land b \lor ((a \land b) \rightarrow (a \land b)) = a \land b \lor (a \land b) = 1$. Thus we use assumption that $b = (0_+, 0_-)$. If $a = (0_+, a_-)$, then $a = (0_+, a_-)$ for some $a_- \in G_-$. If $b = (0_+, b_-)$, then $b = (0_+, a_-)$ for some $a_- \in G_-$ such that $b_- \leq a_-$. Then $a \land b = (0_+, (n(b_+) \rightarrow b_-) \land (n(a_+) \rightarrow a_-)) \lor (0_+, (0_- \rightarrow b_-) \land (0_- \rightarrow a_-)) = (0_+, 1_-)$. So $a \land b = 1$, and $a \rightarrow b = 0$, and we are done. Thus, let us assume that $a = (0_+, a_-)$. If $a = (0_+, a_-)$, we calculate $\neg((0_+, a_-) \rightarrow (0_+, a_-)) \lor ((0_+, a_-) \rightarrow (0_+, a_-)) = (0_+,(n(a_+) \rightarrow 0_-) \land (n(b_+) \rightarrow 0_-)) \lor ((0_+,(0_- \rightarrow 0_-) \land (0_- \rightarrow a_-)) = (0_+,(n(a_+) \rightarrow 0_-) \land (n(b_+) \rightarrow 0_-)) \lor ((0_+,(0_- \rightarrow 0_-) \land (0_- \rightarrow a_-)) = (0_+,(n(a_+) \rightarrow 0_-) \land (n(b_+) \rightarrow 0_-)) \lor ((0_+,(0_- \rightarrow 0_-) \land (0_- \rightarrow a_-)) = (0_+,(n(a_+) \rightarrow 0_-) \land (n(b_+) \rightarrow 0_-)) \lor ((0_+,(0_- \rightarrow 0_-) \land (0_- \rightarrow a_-)) = (0_+,(n(a_+) \rightarrow 0_-) \land (n(b_+) \rightarrow 0_-)). Thus, we need to check that $(n(b_+) \rightarrow a_-) \lor ((n(b_+) \rightarrow a_-)) = p(a_-)$ and $p(a_-) = 1$. If $n(b_+) \leq a_-$, we are done. Thus (recalling that $G_-$ is linearly ordered), assume $a_- < n(b_+)$. Then $n(b_+) \rightarrow a_- = a_- = 0_-$ (this also holds on every linearly ordered G-algebra), and we have $p((n(b_+) \rightarrow a_-) \lor ((p(n(b_+) \rightarrow a_-)) = p(a_-) = 1$, which is clearly sufficient to obtain the required result. Let us compute $a \rightarrow (a \land b) = (a_\land b_\land (a_\land b_\rightarrow a_\land b_\rightarrow 0_-) \lor (a_\land b_\rightarrow 0_-) = (a_\land b_\rightarrow 0_-) \lor (a_\land b_\rightarrow 0_-) = (a_\land b_\rightarrow 0_-) \lor (a_\land b_\rightarrow 0_-)$. The last equality holds from $a_\land b_\rightarrow 0_- \lor (n(b_+) \rightarrow a_-) \lor (n(b_+) \rightarrow 0_-) \lor (n(a_+) \rightarrow 0_-) = (a_\land b_\rightarrow 0_-) \lor (a_\land b_\rightarrow 0_-) = (a_\land b_\rightarrow 0_-) \lor (a_\land b_\rightarrow 0_-) = (a_\land b_\rightarrow 0_-)$.

As shown in [24, Cor. 3], a QN-algebra $A$ is subdirectly
irreducible if and only if \( A \) has a unique co-atom. This observation allows one to prove that (similarly to MTL, WNM and NM-algebras) the variety of QN-algebras satisfying the pre-linearity equation is generated by its linearly ordered members. Thus, the result of Lemma 4.3 applies to all pre-linear QN-algebras.

**Corollary 4.4**: Every pre-linear quasi-Nelson algebra satisfies the WNM equation.

Thus PN \( \subseteq \) WNM, the inclusion being strict (as shown by Example 4.6 below).

**Corollary 4.5**: The following varieties coincide:

(i) Pre-linear QN-algebras.

(ii) Pre-linear QN-algebras satisfying the WNM equation.

(iii) The class of all twist-structures of type \( \text{Tw}(G_+, G_-, n, p, \nabla) \).

It is useful to recall that, on a linearly ordered WNM-algebra \( A \), the lattice structure together with the negation determine the other operations in the following way (see e.g. [3, p. 2]). For all \( a, b \in A \), one has \( a \ast b = a \lor b \) if \( a \leq b \), and \( a \ast b = 0 \) otherwise; \( a \Rightarrow b = 1 \) if \( a \leq b \), and \( a \Rightarrow b = 0 \) otherwise.

We shall often use this observation in subsequent calculations, starting from the next Example.

**Example 4.6**: Let \( A = \langle A ; \lor, \land, \ast, \Rightarrow, \neg, \leq, 0, 1 \rangle \) be an algebra with universe \( A \) := \{0, a, b, 1\} such that the lattice \( A; \lor, \land, \leq, 0, 1 \) is linearly ordered as follows: \( 0 < a < b < 1 \). The negation \( \neg \) is defined by: \( \neg 0 = 1 \), \( \neg 1 = 0 \), \( \neg a = b \). The operations \( \ast \) and \( \Rightarrow \) are then determined by the above prescriptions for WNM-chains. It is easy to check that \( A \) is a WNM-algebra—this is an application of a general method for producing WNM-chains: see [20, Definition 6.37]; in fact, \( A \) is a DP-algebra\(^3\). Now, \( A \) does not satisfy the Nelson equation, because

\[
(b \Rightarrow (b \Rightarrow a)) \land (\neg a \Rightarrow (\neg a \Rightarrow \neg b)) = (b \Rightarrow b) \land (\neg a \Rightarrow 1) = 1 \nless b \Rightarrow b \Rightarrow a.
\]

The following proposition shows that, as expected, the lattice (or even meet-semilattice) structure of a WNM-algebra \( A \) together with the negation determine whether \( A \) satisfies the Nelson equation or not.

**Proposition 4.7**: A WNM-algebra \( A \) is a (pre-linear) quasi-Nelson algebra if and only if \( A \) satisfies \( \neg \neg x \land \neg \neg x \leq x \).

**Proof**: It is shown in [22] and easy to check (using twist-structures) that every QN-algebra satisfies \( \neg \neg x \land \neg \neg x \leq x \). Conversely, relying on pre-linearity, we are going to show that every QN-chain \( C \) that satisfies \( \neg \neg x \land \neg \neg x \leq x \) also satisfies the Nelson equation. Observe that, on a chain, \( \neg \neg a \land \neg \neg a \leq a \) implies \( \neg \neg a \) or \( \neg \neg a \leq a \), for all \( a \in C \). As mentioned earlier, on a WNM-chain, we have \( a^2 = 0 \) if \( a \leq a \) and \( a^2 = a \) if \( a < a \). Thus, for all \( a, b \in C \), if \( \neg \neg a \leq a \), then \( (a^2 \Rightarrow b) \land ((\neg b)^2 \Rightarrow \neg a) = (a \Rightarrow b) \land ((\neg b)^2 \Rightarrow \neg a) \leq a \Rightarrow b \), as required. Thus, assume \( a \leq \neg a \), which implies \( (a^2 \Rightarrow b) = 0 \Rightarrow b = 1 \). Thus \( (a^2 \Rightarrow b) \land ((\neg b)^2 \Rightarrow \neg a) = 1 \land ((\neg b)^2 \Rightarrow \neg a) = (\neg b)^2 \Rightarrow \neg a \). If \( a \leq b \), then \( a \Rightarrow b = 1 \), and we are done. Thus, assume \( b \leq \neg a \). Then \( a \Rightarrow b = \neg a \lor b = \neg a \). Thus, we need to show \((\neg b)^2 \Rightarrow \neg a \leq \neg a \). If \( \neg b \leq \neg b \), then \( b \leq \neg a \leq \neg b < b \). Since \( \neg b \land \neg b \leq b \), we have either \( b = \neg b \) or \( \neg b \leq \neg b \); both are against our assumptions, for each of them implies \( a \leq b \). Thus \( \neg b < \neg b \), which means \((\neg b)^2 \Rightarrow \neg a = \neg b \Rightarrow \neg a \). We thus need to show \((\neg b)^2 \Rightarrow \neg a \leq \neg a \). If \( \neg a < \neg b \), then \( b \Rightarrow b = \neg a \Rightarrow \neg b \lor \neg a \). If \( \neg b \leq \neg a \), we are done. Thus, assume \( b \leq \neg b \). Then \( b < \neg a < \neg b < \neg b \). Using \( \neg b \land \neg b \leq b \), again, we have either \( b = \neg b \) or \( \neg b < \neg b \); both against our assumptions. It thus remains to consider the case where \( b = \neg a \). Then \( b < a \leq \neg a = \neg b \) and \( \neg b \land \neg b \leq b \) gives us \( \neg b = b \).

This means \( \neg \neg a = \neg \neg b = \neg \neg b \). Since \( a \leq \neg \neg a \), this would imply \( a \leq b \), against our assumptions. This completes our proof.

We summarise our findings below:

**Corollary 4.8**: The following varieties coincide:

(i) Pre-linear QN-algebras.

(ii) WNM-algebras satisfying \( \neg \neg x \land \neg \neg x \leq x \).

(iii) \{\text{Tw}(G_+, G_-, n, p, \nabla) : G_+ \text{ is a G-algebra}\}.

**Corollary 4.8** thus identifies a subclass of WNM-algebras that are representable via the twist construction, and this is (as far as we are aware) the first result of this type for WNM-algebras. Given the parallel between twist-structures and rotations, Corollary 4.8 also suggests that a suitable modification of the rotation construction may allow us to give an alternative representation for this subclass of WNM-algebras.

**Subdirectly irreducibles and directly indecomposables**

We end the section with a sample application of the twist representation, which applies to (pre-linear) QN-algebras and therefore also to those WNM-algebras that satisfy the equation \( \neg \neg x \land \neg \neg x \leq x \). We shall use the following result from [24, Prop. 8].

**Lemma 4.9**: Let \( A = \text{Tw}(G_+, G_-, n, p, \nabla) \) be a pre-linear QN-algebra. The lattice \( \text{Con}(A) \) of congruences of \( A \) is isomorphic to the lattice \( \text{Con}(G_+) \) of congruences of \( G_+ \) via the maps \( (\_\_\_\_\_\_\_\_\_) ) \) defined as follows:

(i) For \( \theta \in \text{Con}(A) \) and \( a_+, b_+ \in G_+ \), let \( \langle a_+, b_+ \rangle \in \theta \) if and only if there are \( a_-, b_- \in G_- \) such that \( \langle a_+ \rightarrow a_-, b_+ \rangle, \langle a_+ \rightarrow a, b_- \rangle \in A \) and \( \langle a_+ \rightarrow a, b_- \rangle, \langle a_+, b_- \rangle, \langle a_+, 0 \rangle, \langle b_+ \rightarrow a, b_- \rangle, \langle b_+, 0 \rangle \rangle \in \theta \).

(ii) For \( \eta \in \text{Con}(G_+) \) and \( a_+, b_+ \in G_+ \), let \( \langle a_+, a_- \rangle, \langle b_+, b_- \rangle \in \eta \) if and only if \( \langle a_+, b_+ \rangle, \langle p(a_-), p(b_-) \rangle \in \eta \).

**Proposition 4.10**: Let \( A = \text{Tw}(G_+, G_-, n, p, \nabla) \) be a pre-linear QN-algebra. Then:

(i) \( A \) is subdirectly irreducible iff \( G_+ \) is a subdirectly irreducible G-algebra.

(ii) \( A \) is directly indecomposable iff \( G_+ \) is a directly indecomposable G-algebra.
Proof: Item (i) is an application of Lemma 4.9. Regarding (ii), observe that $A$ is not directly indecomposable iff there are non-trivial factor congruences $\theta_1, \theta_2 \in \text{Con}(A)$. If this is the case, then $\theta_1, \theta_2 \in \text{Con}(G_+)$ are non-trivial factor congruences of $G_+$. Indeed, this follows from Lemma 4.9 together with the observation that $G_+$, as a residuated lattice, is congruence-permutable [14, p. 94]. By the same token, $A$ is congruence-permutable as well. Then, if $\eta_1, \eta_2 \in \text{Con}(G_+)$ are non-trivial factor congruences, then $\eta_1^n, \eta_2^n \in \text{Con}(A)$ are non-trivial factor congruences.

Let $A$ be a $\text{QN}$-algebra and $a \in A$. We say that $a$ is a splitting element if, for all $b \in A$, either $a \leq b$ or $b \leq a$. We say that $a$ is idempotent (with respect to the monoid operation) if $a^2 = a$.

Proposition 4.11: For every quasi-Nelson algebra $A = \text{Tw}(G, G_-, n, p, \nabla)$, the following are equivalent:

(i) $G_+$ (and therefore $G_-$) has a unique atom.

(ii) $A$ has a splitting idempotent element $e$ such that $a < e$, $a^2 = 0$ for all $a < e$, and $b^2 = b$ for all $e \leq b$.

Proof: Regarding (i), let us preliminary observe that, if $e_+$ is the unique atom of $G_+$, then $n(e_+)$ is the unique atom of $G_-$. Indeed, $n(e_+) \neq 0_-$, because $n(e_+) = 0_-$ would imply $e_+ \leq p_0 n(e_+) = p(0_-) = 0_-$, against the assumption that $e_+ \neq 0_+$. Further, for all $a_+ \in G_+$ with $a_+ \neq 0_+$, we have $p(a_+) = 0_-$, because $p(a_+) = 0_-$ would imply $a_+ = np(a_+) = n(0_+ + a_+) = 0_+$, against our assumption. Then $e_+ \leq p(a_+)$, which implies $n(e_+) \leq n(p(a_+)) = a_+$, as claimed.

Now, assume (i) holds, and let $e_+ \in G_+$ be the unique atom of $G_+$ (so $n(e_+)$ is the unique atom of $G_-$). Take $e \equiv \langle e_+, 0_+ \rangle$, and observe that $\langle e_+, 0_+ \rangle = \langle p(0_+), n(e_+) \rangle = \langle 0_+, n(e_+) \rangle$. Thus $e_+ \leq p(a_+)$, and $e_+ \neq 0_+$ for all $a_+ \in G_+$ with $a_+ \neq 0_+$. So every non-zero element of $G_+$ is dense, and $e_+ \in D(G_+) \subseteq \nabla$ for any possible choice of $\nabla$. Then $e_+ \vee \eta$, $p(0_+) \in \nabla$, as required. Next, observe that every element of $A$ is comparable with $\langle e_+, 0_+ \rangle$. Indeed, for all $a_+ \in A$, the existence of a unique atom in $G_+$ together with the requirement $a_+ \wedge p(a_-) = 0_+$ entail that either $a_+ = 0_+$ or $p(a_-) = 0_+$ (in which case $a_- = np(a_-) = n(0_+) = 0_+$). Thus every element of $A$ has the form $a_+ = 0_+ = 0_+ = 0_+$ or $\langle 0_+, a_+ \rangle$ for some $a_+ \in G_+$ and $a_+ \in G_+$, and $a_+ \in G_+$, and $\langle 0_+, a_+ \rangle \subseteq \langle 0_+, 0_+ \rangle$ for all $a_+ \in G_+$, and observe that $\langle 0_+, a_+ \rangle^2 = \langle 0_+, n(0_+) \rangle \rightarrow a_+ = \langle 0_+, 0_+ \rangle$, as claimed (in (ii)).

On the other hand, for all $a_+ \neq 0_+$ we have $\langle e_+, 0_+ \rangle \leq \langle a_+, 0_+ \rangle$, so every element of $A$ is comparable with $\langle e_+, 0_+ \rangle$, as required. Let us verify that $\langle e_+, 0_+ \rangle$ is an idempotent. Since $n(e_+) \neq 0_+$, we have $n(e_+) \rightarrow 0_+ = 0_+$. Then $\langle e_+, 0^2 \rangle = \langle e_+, n(e_+) \rightarrow 0_+ \rangle = \langle 0_+, 0_+ \rangle$, as required. Lastly, since $n(e_+) \leq n(a_+)$, we have $n(a_+)$ for all $a_+ \in G_+$, so $\langle a_+, 0^2 \rangle = \langle e_+, n(a_+) \rightarrow 0_+ \rangle = \langle a_+, 0_+ \rangle$, as claimed.

Conversely, assume (ii) holds, and let $e = \langle e_+, e_- \rangle$ be the splitting element of $A$. Let $a_+ \in G_+$ be such that $a_+ \neq 0_+$. Then $n_+ \leq 0_+$ such that $\langle a_+, a_- \rangle \in A$. Moreover, $a_+ \neq 0_+$ entails $\langle a_+, a_- \rangle^2 = \langle a_+, n(a_+) \rightarrow a_- \rangle \neq (0_+, 1_+)$. Thus, it cannot be the case that $\langle a_+, a_- \rangle < e$. Hence (since $e$ is a splitting element), $e \leq \langle a_+, a_- \rangle$, which entails $e_+ \leq a_+$. This shows that $e_+$ is the unique atom of $G_+$. As observed earlier, it follows that $n(e_+) = n(e)$ is the unique atom of $G_-$.

It may be worth mentioning that both Propositions 4.10 and 4.11 still hold true if we drop the pre-linearity hypothesis, replacing the $G$-algebras $G_+, G_-$ with Heyting algebras $H_+, H_-$ (Proposition 4.12 below, on the other hand, is specific to G-algebras).

Taking into account Proposition 4.10(ii), it is clear that (either of) the conditions in Proposition 4.11 entail that $A$ is directly indecomposable. This implication becomes an equivalence in the case of finite pre-linear quasi-Nelson algebras, as the following proposition shows.

Proposition 4.12: For every finite pre-linear $\text{QN}$-algebra $A = \text{Tw}(G, G_-, n, p, \nabla)$, the following are equivalent:

(i) The G-algebra $G_+$ (and therefore also $G_-$) has a unique atom.

(ii) $A$ has a splitting idempotent element $e$ such that $a^2 = 0$ for all $a < e$ and $b^2 = b$ for all $e \leq b$.

(iii) $A$ is directly indecomposable.

Proof: We seen in Proposition 4.11 the equivalence of (i) and (ii), together with the observation that $G_-$ has a unique atom when $G_+$ has a unique atom. We proceed to show that (i) and (iii) are equivalent. We have seen in Proposition 4.10(ii) that $A$ is directly indecomposable iff $G_+$ is. To complete our proof, it is sufficient to recall that directly indecomposable finite $G$-algebras are precisely those having a unique atom (see e.g. [11, p. 56-57]).

V. NELSON AND NM-ALGEBRAS

In this section we show that things are different in the involutive setting: indeed, pre-linear Nelson algebras coincide with NM-algebras. This result is known since at least [9], but we present here a shorter proof that takes advantage of the recent insight on involutive CIBRLs gained in [26].

Lemma 5.1: Every WNM-algebra satisfies the equation:

$$x \approx x^2 \lor (x \land \neg x).$$

Proof: Relying on pre-linearity, we verify that the equation is satisfied by every WNM-chain $C$. As mentioned earlier, on a WNM-chain we have $a^2 = 0$ if $a \leq \neg a$ and $a^2 = 0$ if $a < a$, for all $a \in C$. In the former case, we have $a^2 \lor (a \land \neg a) = 0 \lor a = a$. In the latter, $a^2 \lor (a \land \neg a) = a \lor \neg a = a$.

Corollary 5.2: Every NM-algebra is a pre-linear Nelson algebra. Hence Lemma 4.3 entails $\text{NM} = \text{PN}$.

Proof: Let $A$ be a NM-algebra. Then $A$ is involutive and, by Lemma 5.1, $A$ satisfies the equation $x \approx x^2 \lor (x \land \neg x)$. It is shown in [26, Theorem 6.1] that, for an involutive CIBRL, this is equivalent to being a Nelson algebra.
Corollary 5.2, together with the structural results recalled in Section III, entails that NM-algebras are (as rotations) in a one-to-one correspondence with pairs \( \langle G, f \rangle \) where \( G \) is a G-algebra and \( f \) a filter of the Boolean skeleton of \( G \), and (as twist-structures) are also in a one-to-one correspondence with pairs \( \langle G, \nabla \rangle \) where \( G \) is a G-algebra and \( \nabla \) a dense filter of \( G \). To see that the two perspectives indeed match, it is sufficient to observe that, on every G-algebra \( G \), the filters of the Boolean skeleton are in one-to-one correspondence with the dense filters\(^4\). On account of space limitations, a detailed analysis of this result will be deferred to a future publication.

It may be worth mentioning that Example 4.6, together with Lemma 5.1, provides an answer to a problem that was left open in [24]: namely, whether the equation \( x \approx x^2 \lor (x \land \neg x) \) may be proven to be equivalent, in a non-involutive setting, to the Nelson equation. (The answer is, of course, negative.)

Another open problem mentioned in [24] can be recast (and resolved) in the present context. As observed earlier, every G-algebra is a (pre-linear)QN-algebra, and (by Corollary 5.2) every NM-algebra is also a (pre-linear)QN-algebra. Thus \( G \cup NM \subseteq PQN \). Indeed, in a fuzzy setting, one could motivate the introduction pre-linear QN-algebras as ‘a common generalisation of Gödel and NM-algebras’. One might further enquire whether this is a ‘minimal’ generalisation, in the sense, for instance, that the variety \( V(G \cup NM) \) generated by \( G \cup NM \) is precisely PQN. Also in this case the answer is negative.

Let us begin by observing that, since \( PQN = V(PQN) \), we have \( V(G \cup NM) \subseteq PQN \). Thus \( V(G \cup NM) \) is also a subvariety of \( WNM \), i.e. a variety of algebras of fuzzy logic. Let us further note that the equation \( x \approx x^2 \lor (\neg \neg y \Rightarrow y) \approx 1 \) is clearly satisfied by every algebra in \( G \cup NM \), and therefore in \( V(G \cup NM) \). However, there are algebras in PQN that do not satisfy it, as the following Example shows (see also [3, Def. 11]).

**Example 5.3:** Consider the (uniquely determined) three- and the two-element G-chains:

\[
G_+ = \langle G_+ = \{0_+, a_+, 1_+\}; \land_+, \lor_+, \rightarrow_+, 0_+, 1_+\rangle
\]

\[
G_- = \langle G_- = \{0_-, 1_-\}; \land_-, \lor_-, \rightarrow_-, 0_-, 1_-\rangle.
\]

Let \( \nabla = G_- \). Define \( n : G_+ \rightarrow G_- \) by \( n(a) = n(1_+) = 1_- \) and \( n(0) = 0_- \). We have \( n(a) = 0 \) and \( p(n(a)) = 1_+ \). These determine a QN twist-structure \( A = Tw(G_+, G_-, n, p, \nabla) \). Observe that \( \langle 0_+, 0_+ \rangle, \langle a_+, 0_- \rangle \in A \). We have \( \langle a_+, 0_- \rangle = \langle 0_+, 1_- \rangle \) and \( \neg\neg\langle a_+, 0_- \rangle = \langle 1_+, 0_+ \rangle \), which give us the following:

\[
\left(\langle 0_+, 0_+ \rangle \Rightarrow \langle 0_+, 0_- \rangle \right) \lor \neg\neg\langle a_+, 0_- \rangle = \langle a_+, 0_- \rangle = \langle 0_+, 1_- \rangle \lor \neg\neg\langle a_+, 0_- \rangle = \langle a_+, 0_- \rangle = \langle 0_+, 0_- \rangle \lor \langle 0_+, 0_- \rangle = \langle 0_+, 1_- \rangle.
\]

The following lemma entails that \( V(G \cup NM) \) is axiomatised precisely by adding \( x \Rightarrow x^2 \lor (\neg \neg y \Rightarrow y) \approx 1 \) to the equational presentation of PQN.

**Lemma 5.4:** Let \( A \) be a linearly ordered pre-linear QN-algebra. The following are equivalent:

(i) \( A \) satisfies \((x \Rightarrow x^2) \lor (\neg \neg y \Rightarrow y) \approx 1\).

(ii) \( A \) is either a Gödel algebra or a Nelson algebra.

**Proof:** The non-trivial implication is from (i) to (ii). Let then \( A \) be a subdirectly irreducible algebra in PQN. Suppose \( A \) is neither Gödel nor Nelson. Then there are elements \( a, b \in A \) such that \( a \neq a^2 \) (thus \( a^2 < a \)) and \( b \neq \neg\neg b \) (thus \( b < \neg\neg b \)). This means that \( a \Rightarrow a^2 \approx 1 \) and \( \neg\neg b \Rightarrow b < 1 \). Since \( A \) is linearly ordered, the preceding considerations imply \((a \Rightarrow a^2) \lor (\neg\neg b \Rightarrow b) \neq 1\).

Lemma 5.4 applies, in particular, to subdirectly irreducible \( QN \)-algebras (which are linearly ordered, by Proposition 4.10.i). The following result is thus an immediate consequence (as well as an instance of [14, Lemma 5.25]).

**Corollary 5.5:** \( V(G \cup NM) \) is the subvariety of PQN axiomatised by:

\[(x \Rightarrow x^2) \lor (\neg \neg y \Rightarrow y) \approx 1.\]

**Proof:** Observe that the subvariety of PQN axiomatised by \((x \Rightarrow x^2) \lor (\neg \neg y \Rightarrow y) \approx 1 \) and \( V(G \cup NM) \) have the same subdirectly irreducible members; therefore, they must coincide [7, II, Cor. 9.7].

The next (and last) corollary entails standard completeness (see item 3. in Section VI) of the logic associated to the class \( WNM \). Denote by \( [0, 1]_G \) and by \( [0, 1]_{NM} \), respectively, the G-algebra and NM-algebra having as universe the real interval \([0, 1]\); both algebras are unique up to isomorphism. It is well known that \( V([0, 1]_G) \) is the variety of G-algebras and \( V([0, 1]_{NM}) \) is the variety of NM-algebras.

**Corollary 5.6:** \( V(G \cup NM) = V(\{[0, 1]_G, [0, 1]_{NM}\}) \).

**Proof:** Our previous considerations entail that \( V(\{[0, 1]_G, [0, 1]_{NM}\}) \subseteq V(G \cup NM) \). For the converse inclusion we proceed as in Corollary 5.5. Let \( A \) be a subdirectly irreducible member of \( V(G \cup NM) \). Then, by Lemma 5.4, \( A \) is either a G-algebra or an NM-algebra. Thus either \( A \in V([0, 1]_G) \) or \( A \in V([0, 1]_{NM}) \). In both cases we have \( A \in V(\{[0, 1]_G, [0, 1]_{NM}\}) \), as claimed.

**VI. Future Work**

As mentioned in the Introduction, this paper has been a first attempt at establishing a connection between the theory of quasi-Nelson algebras/logics and fuzzy systems extending MTL. Future research may take several directions, among which we mention a few below.

1. Is it possible to extend (some form of) twist-structure representation to the whole class of WNM-algebras? Recent results [22] show that twist-structures can be used to represent algebras in the implication-free language \( \land, \lor, \neg \), including De Morgan and Kleene lattices [21], as well as more general sub(quasi)varieties of semi-De Morgan algebras [25]. The algebras representable in this way have been dubbed semi-Kleene lattices in [22]. It is easy to check that the \( \langle *, \Rightarrow \rangle \)-free reduct of every WNM-algebra is a semi-Kleene lattice; as we have seen, it is this reduct that determines (on WNM-chains) the behaviour of the remaining operations. These considerations suggest that representing WNM-algebras via twist-structure may indeed be a feasible project.
2. The above-mentioned question suggests another structural relation that may be worthwhile exploring: namely the one between WNM-algebras (and thus PQN) and Sugihara monoids. These structures, that are algebraic models of relevance logic, have also been studied from a twist representation point of view; interestingly, the twist representation proposed (e.g.) in [15] also decomposes a Sugihara monoid as a special binary power of a Gödel algebra. To make this even more intriguing, we may further observe that every Sugihara monoid (indeed, even every generalised Sugihara monoid in the sense of [15]) also has a semi-Kleene lattice reduct.

3. From a fuzzy logic point of view on the logic of pre-linear QN-algebras (i.e. the extension of WNM-logic by the axiom \((\neg\neg\varphi \land \neg\varphi) \Rightarrow \varphi\)), an obvious (and open) question concerns so-called standard completeness. This property means for a logic to be complete not only with respect to its linearly ordered algebraic models (which we know to be true, by pre-linearity), but also with respect to a class of (possibly non-isomorphic) algebras defined over the real interval \([0,1]\).

Standard completeness is known to hold for WNM-logic [12, Thm. 3], but it is also well known that the property need not be preserved by axiomatic extensions. An even stronger form of standard completeness is single-chain real completeness, that is, completeness with respect to a unique algebraic model over the real unit interval. This property, which holds for Gödel and NM-logic, is entailed by the observation that the G-algebra (resp. NM-algebra) over \([0,1]\) is unique up to isomorphism. Since every G-algebra and every NM-algebra is a pre-linear QN-algebra, we have over \([0,1]\) at least two non-isomorphic QN-algebras (in fact, Example 5.3 suggests that there may be more). This however, does not destroy all hope of proving single-chain real completeness for the logic of pre-linear QN-algebras. Therefore, both standard and single-chain real completeness are currently open problems, which we intend to address in future work.

4. Lastly, let us mention a possible extension of our approach outside the setting of integral residuated lattices. Alongside constructive logic with strong negation, a later paper by David Nelson [1] introduced a paraconsistent weakening of N-logic that is nowadays known as N4. The algebraic models of N4 (called N4-lattices) are residuated structures related to algebras of relevance logics, which can also be represented as twist-structures by a straightforward generalisation of the construction presented in this paper. Indeed, the twist construction for N-algebras (though not the one for QN) may be seen as a special case of that for N4-lattices; abstractly, N-lattices correspond precisely to the subvariety of N4-lattices defined by the equation \(x \Rightarrow x = y \Rightarrow y\). N4-lattices that are twist-structures over Gödel algebras have already been studied in the paper [4], but the algebras considered there are not pre-linear in the usual sense. We speculate that a more thorough investigation of ‘pre-linear N4-lattices’ (along the lines of the present paper, as well as extending the results of [4]) may turn out to be an intriguing project for future research.