On the relationship between the centroid and the footprint of uncertainty of Interval Type–2 fuzzy numbers

Juan Carlos Figueroa–García, Roman Neruda, Yurilev Chalco–Cano and Heriberto Román–Flores
Universidad Distrital Distrital Francisco José de Caldas, Bogotá - Colombia
Informatics Dept., Czech Academy of Sciences, Prague - Czech Republic
Universidad de Tarapacá, Arica - Chile

***Abstract*—This paper presents experimental results about the relationship between the Footprint Of Uncertainty (FOU) and the centroid of Interval Type–2 Fuzzy Numbers (IT2FN) which are a subclass of Type–2 fuzzy numbers. Four types of IT2FNs are analyzed: singleton–core triangular, interval–core triangular, singleton–core Gaussian and interval–core Gaussian IT2FNs. A quadratic relationship was detected, characterized, and their results are presented and discussed using numerical experiments.**

***Index Terms*—Type–2 fuzzy numbers, K–M algorithm, centroid**

I. Introduction and Motivation

THE centroid of an Interval Type–2 Fuzzy Number *IT2FN* (often called type–reduction) is an important measure in fuzzy logic systems such as fuzzy neural networks, fuzzy clustering, fuzzy rule–based systems, fuzzy control, fuzzy classification, etc. Among the most important methods for defuzzification of fuzzy sets/numbers are the Yager index (see Yager [1] and Figueroa-García, Chalco-Cano & Román-Flores [2]), the possibilistic mean (see Carlsson & R. Fullér [3]), and the centroid (see Wu & Mendel [4]), all of them with different advantages/disdvantages.

Fuzzy sets/numbers gained popularity into machine learning methods (see Neruda & Kudová [5], Kazík, Pilat & Neruda [6]) since they can represent uncertainty coming from data. Also some decision making techniques/methods use fuzzy measures as a base for practical applications (optimization, differential equations, classification, etc.). A popular way to comprise uncertainty of a fuzzy set/number into a single measure is by computing its centroid and other related measures like its variance, uncertainty/mathematical boundaries etc. in order to preserve part of the lost information when only using the centroid as a “fuzzy” measure.

The centroid of a Type–2 fuzzy set is a very popular measure in many real–world applications because it meets two highly desirable mathematical properties: it is monotonically increasing/decreasing and convex. One of the most important method to compute the centroid of an IT2FN is the K–M algorithm proposed by Karnik & Mendel [7] and Mendel & Liu [8], improved by Melgarejo [9], [10], [11] and defined for α–cuts by Figueroa-García [12]. Having said that, we consider the relationship between the centroid of an IT2FS and its footprint of uncertainty a natural relationship to analyze since common sense dictates it exists so the focus of this paper is to analyze the behavior of some interesting ratios between the centroid and the FOU of an IT2FN.

The paper is organized as follows: a first introductory section; Section 2 presents some basics on fuzzy sets/numbers; Section 3 presents the centroid of an IT2FN and its relationship to its FOU; some numerical examples are analyzed in Section 4, and some concluding remarks are presented in Section 5.

II. Basics of interval Type–2 fuzzy numbers

A Type-1 fuzzy set A is denoted by capital letters with a membership function $\mu_A(x)$; Type-2 fuzzy sets (T2FS) are denoted by emphasized capital letters $\tilde{A}$ with a membership function $\mu_{\tilde{A}}(x)$. $\mu_A(x)$ measures the degree of affinity of a particular value $x \in X$ to the concept/word/label A, and $\mu_{\tilde{A}}(x)$ measures the degree of uncertainty of the same value $x \in X$ regarding A, so A measures imprecision and $\tilde{A}$ measures uncertainty. $\mathcal{P}(X)$ is the class of all crisp sets, $\mathcal{F}_1(X)$ is the class of all Type-1 fuzzy sets, and $\mathcal{F}_2(X)$ is the class of all Type-2 fuzzy sets (see Mendel [13], [14]). Let $\mathcal{L}([0,1])$ be the set of all closed subintervals over $[0,1]$:

$$\mathcal{L}([0,1]) = \{\mathbf{x} = [x_l, x_r] \,|\, (x_l, x_r) \in [0,1]^2, x_l \leqslant x_r\}.$$

A Type-2 fuzzy set is then the following ordered pair:

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) : x \in X,\, u \in J_x \subseteq [0,1]\},$$

where $\tilde{A}$ represents uncertainty around a linguistic label/concept/word A. An *Interval Type–2 fuzzy set* (IT2FS) $\tilde{A}$

Juan Carlos Figueroa-García is Assistant Professor of the Engineering Dept. of Universidad Distrital Francisco José de Caldas, Bogotá - Colombia, email: jcfigueroag@udistrital.edu.co

Roman Neruda is Assistant Professor of the Institute of Computer Science of the Czech Academy of Sciences, Prague - Czech Republic, email: roman@cs.cas.cz

Yurilev Chalco–Cano is full time professor of the Math. Dept. of Universidad de Tarapacá, Casilla 7D, Arica, Chile, email: ychalco@uta.cl

Heriberto Román–Flores is full time professor of the Instituto de Alta Investigación of Universidad de Tarapacá, Casilla 7D, Arica, Chile, email: hroman@uta.cl

978-1-7281-6932-3/20/$31.00 ©2020 IEEE

is a simpler representation of a T2FS in which $\mu_{\tilde{A}}(x, u) = 1$ and it encloses infinite Type–1 fuzzy sets into its *Footprint of Uncertainty* (FOU) (see Mendel, John and Liu [15]):

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \,|\, x \in X\}$$

where $\mu_{\tilde{A}}(x)$ is fully characterized by $J_x \subseteq [0, 1]$ using a *Upper* membership function $UMF(\tilde{A}) = \overline{\mu}_{\tilde{A}} \equiv \bar{A}$ and a *Lower* membership function $LMF(\tilde{A}) = \underline{\mu}_{\tilde{A}} \equiv \underline{A}$ (see Fig. 1).

A T2FS $\tilde{A}$ is a *Type-2 fuzzy number* (T2FN) only if both its UMF and LMF are fuzzy numbers (e.g. normal and convex fuzzy subsets of $\mathbb{R}$, see Zadeh [16]), ${}^{\alpha}\tilde{A}$ must be a closed interval for all $\alpha \in [0, 1]$, and $supp(\tilde{A}) \in \mathbb{R}$ (see Figueroa–García et al [2]). Let us denote $\mathcal{P}(\mathbb{R})$ as the class of all crisp numbers, $\mathcal{F}_1(\mathbb{R})$ as the class of all Type-1 fuzzy numbers, and $\mathcal{F}_2(\mathbb{R})$ as the class of all Type-2 fuzzy numbers.

Definition 1: Let $\tilde{A} : \mathbb{R} \to \mathcal{L}([0, 1])$ be a fuzzy subset of the reals. Then $\tilde{A} \in \mathcal{F}_2(\mathbb{R})$ is a Type-2 Fuzzy Number (T2FN) iff there exists a closed interval $[x_l, x_r] \neq \emptyset$ with a membership function $\mu_{\tilde{A}}(x)$ such that:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{for } x \in [x_l, x_r], \\ \tilde{l}(x) & \text{for } x \in [-\infty, x_l], \\ \tilde{r}(x) & \text{for } x \in [x_r, \infty], \end{cases} \quad (1)$$

where $\tilde{l} : (-\infty, x_l) \to \mathcal{L}([0, 1]), u \in J_x \subseteq [0, 1]$ is monotonic non-decreasing, continuous from the right, i.e. $\tilde{l}(x) = 0$ for $x < \overline{\varpi}_1$; $\tilde{r} : (x_r, \infty) \to \mathcal{L}([0, 1]), u \in J_x \subseteq [0, 1]$ is monotonic non-increasing, continuous from the left, i.e. $\tilde{r}(x) = 0$ for $x > \overline{\varpi}_2$.

Now, this paper is focused to two main subclasses of IT2FNs: *singleton–core* and *interval–core* IT2FNs. Note that those subclasses of IT2FNs include some of the most important shapes like Gaussian, triangular, exponential, quadratic, etc. which are among the most used in practical applications.

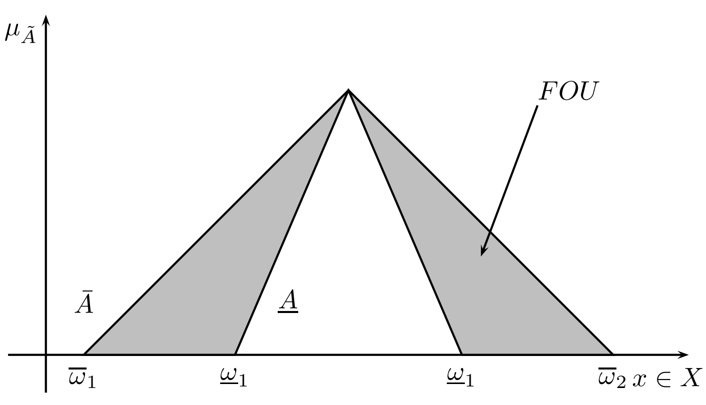

Fig. 1. Singleton–core Interval Type-2 Fuzzy number $\tilde{A}$

Figure 1 shows a triangular singleton–core IT2FN $\tilde{A}$ defined over the reals i.e. $x \in \mathbb{R}$; $\mu_{\tilde{A}}$ is a Type–2 triangular membership function divided into two bounds: $\underline{A}$ and $\overline{A}$. The *support* of $\tilde{A}$, $supp(\tilde{A})$ is the interval $x \in [\overline{\varpi}_1, \overline{\varpi}_2]$ and its core is $core(\tilde{A}) = x_c$. Also note that $\overline{\varpi}_1$ and $\overline{\varpi}_2$ are finite (see Definition 1).

Figueroa-García, Hernández-Pérez & Yurilev Chalco-Cano [17] and Figueroa-García, Chalco-Cano & Román-Flores [2] defined the size of the *footprint of uncertainty* (FOU) of an IT2FS $\tilde{A}$, FOU($\tilde{A}$) as follows:

$$\text{FOU}(\tilde{A}) = \int_{x \in X} \overline{\mu}_{\tilde{A}}(x)dx - \int_{x \in X} \underline{\mu}_{\tilde{A}}(x)dx \equiv |\overline{A}| - |\underline{A}| \quad (2)$$

where $|\bar{A}|$ is the total area of UMF($\tilde{A}$), and $|\underline{A}|$ is the total area of LMF($\tilde{A}$). FOU($\tilde{A}$) shows how ambiguous a set $\tilde{A}$ is, so it helps to measure how much uncertainty $\tilde{A}$ has. We point out that a Type–1 fuzzy set $A \in \mathcal{F}(X)$ has no FOU, so higher values of $FOU(\tilde{A})$ mean a more ambiguous set $\tilde{A}$.

An alternative way to represent a fuzzy number $A \in \mathcal{F}_1(\mathbb{R})$ is via α-cuts (see Hamrawi et al. [18], Figueroa-García [12], and Figueroa-García et al. [19]). The α-cut of a A, namely ${}^{\alpha}A$, is defined as:

$${}^{\alpha}A = \{x \,|\, \mu_A(x) \geqslant \alpha\}, \quad (3)$$

where ${}^{\alpha}A$ for a fuzzy number is:

$${}^{\alpha}A = \Big[\inf_{x \in X}\{\mu_A(x) \geq \alpha\}, \sup_{x \in X}\{\mu_A(x) \geq \alpha\}\Big] = [\check{a}_{\alpha}, \hat{a}_{\alpha}] \quad (4)$$

Thus, an IT2FN is the union of its α-cuts, $\bigcup_{\alpha \in [0,1]} \alpha \cdot {}^{\alpha}A$, where $\cup$ denotes union (Klir & Yuan [20]), so its extension to the α-cut of $\tilde{A}$ (see Figueroa–García [12] and Figueroa–García, Chalco–Cano & Román–Flores [19]) allows us to say that the primary α-cut of an IT2FN ${}^{\alpha}\tilde{A}$ is the set of all $x \in X$ whose J_x are greater than α, $J_x \geqslant \alpha$, this is:

$${}^{\alpha}\tilde{A} = \{x \,|\, \mu_{\tilde{A}}(x) \geqslant \alpha;\ u \in J_x \subseteq [0, 1]\}, \quad (5)$$

$${}^{\alpha}\tilde{A} = \{x \,|\, u \geqslant \alpha\}\ ;\ \alpha \in [0, 1], u \in J_x \subseteq [0, 1], \quad (6)$$

where the boundaries of ${}^{\alpha}\tilde{A}$ for IT2FNs are as follows:

$${}^{\alpha}\tilde{A} = \Big[\inf_{x \in X}\{\mu_{\tilde{A}}(x) \geqslant \alpha\}, \sup_{x \in X}\{\mu_{\tilde{A}}(x) \geqslant \alpha\}\Big], \quad (7)$$

$$\inf_{x \in X}\{\mu_{\tilde{A}}(x) \geqslant \alpha\} = [\check{a}^u_{\alpha}, \check{a}^l_{\alpha}], \quad (8)$$

$$\sup_{x \in X}\{\mu_{\tilde{A}}(x) \geqslant \alpha\} = [\hat{a}^l_{\alpha}, \hat{a}^u_{\alpha}], \quad (9)$$

which finally lead to:

$${}^{\alpha}\tilde{A} = \big[[\check{a}^u_{\alpha}, \check{a}^l_{\alpha}], [\hat{a}^l_{\alpha}, \hat{a}^u_{\alpha}]\big]\,, \quad (10)$$

III. The centroid of an IT2FN

The centroid of a fuzzy set $C(\tilde{A})$ (see Mendel [13]) is defined as follows.

Definition 2: Let $A_e \in \mathcal{F}_1(X)$ be an *embedded* Type–1 fuzzy set into FOU($\tilde{A}$), then the centroid $C(\tilde{A})$ of an IT2FS is the union of the centroids of all its embedded A_e, i.e.,

$$C(\tilde{A}) = \bigcup_{A_e} C(A_e) = [C_l(\tilde{A}), C_r(\tilde{A})]$$

where:

$$C_l(\tilde{A}) = \min_{A_e} C(A_e),$$

$$C_r(\tilde{A}) = \max_{A_e} C(A_e),$$

$$C_c(\tilde{A}) = \frac{C_l(\tilde{A}) + C_r(\tilde{A})}{2}.$$

The computation of the centroid of a Type–2 fuzzy set is basically an optimization problem without exact equations/solutions so far (in general, it is an NP–Hard problem). Widely known solutions to this problem have been proposed by Mendel [13], Karnik & Mendel [7], Mendel & Liu [8], Melgarejo [9], [10], [11] and Figueroa-García [12].

A. Relationship to FOU($\tilde{A}$)

There are different ways to analyze the behavior of the K–M algorithm, one of the most important results was provided by Mendel & Liu [8] who proven that the K–M algorithm does exponentially converge to a set of two min/max bounds. This result have boosted its use and practical applications.

Other non–linear behaviors are exhibited by $C(\tilde{A})$ when changing its parameters (Mendel & Liu [8] have proven its convergence for a fixed set of parameters), so what we have found after experimenting different relationships between $C(\tilde{A})$ and FOU($\tilde{A}$) is summarized into the following two ratios $R_1(\tilde{A})$ and $R_2(\tilde{A})$.

Definition 3: Let $\tilde{A} \in \mathcal{F}_2(\mathbb{R})$ be an IT2FN then $R_1(\tilde{A})$ and $R_2(\tilde{A})$ are defined as follows:

$$C_s(\tilde{A}) = C_l(\tilde{A}) - C_l(\tilde{A}),$$

$$R_1(\tilde{A}) = \frac{C_s(\tilde{A})}{\text{FOU}(\tilde{A})},$$

$$R_2(\tilde{A}) = \frac{C_s(\tilde{A}) \cdot |\underline{A}|}{|\overline{A}|}.$$

Those ratios are interesting since they do not linearly depend on the shape of $\overline{A}$ and $\underline{A}$. Due to the non–linear nature of the K–M algorithm, we do not expect any linear relationship, as it can be seen in next section.

IV. Numerical experiments

Now, the ratios $R_1(\tilde{A})$ and $R_2(\tilde{A})$ (see Definition 3) are applied to four basic examples: Gaussian, triangular, interval–Gaussian and interval–triangular IT2FNs. Its results are shown as follows.

A. Gaussian IT2FN

A singleton–core Gaussian IT2FN namely $G(m, \delta_{ul}, \delta_{ur}, \delta_{ll}, \delta_{lr})$ is shown in Figure 2:

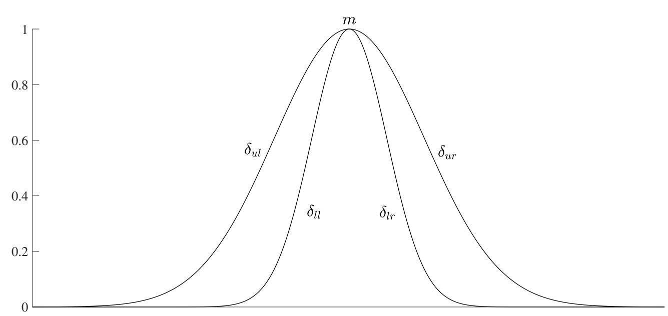

Fig. 2. Gaussian IT2FN

Two sets of experiments were performed: asymmetric and symmetric. The results of an asymmetric experiment for $m = 50, \delta_{ul} = 20, \delta_{ur} = 50, \delta_{ll} = 10$ with symmetric changes over δ_{lr} are summarized in Table I.

δ_{ll}	δ_{lr}	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU($\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
10	2	36.37	87.17	72.69	0.70	8.71
10	4	38.06	85.36	70.19	0.67	9.46
10	6	39.70	83.95	67.68	0.65	10.11
10	8	41.31	82.86	65.17	0.64	10.68
10	10	42.91	82.02	62.67	0.62	11.18
10	12	44.49	81.39	60.16	0.61	11.60
10	14	46.07	80.93	57.65	0.60	11.95
10	16	47.65	80.59	55.15	0.60	12.23
10	18	49.22	80.33	52.64	0.59	12.44
10	20	50.80	80.14	50.13	0.59	12.57
10	22	52.39	79.99	47.63	0.58	12.62
10	24	53.97	79.87	45.12	0.57	12.58
10	26	55.55	79.77	42.61	0.57	12.46
10	28	57.13	79.70	40.11	0.56	12.25
10	30	58.70	79.64	37.60	0.56	11.97
10	32	60.25	79.59	35.09	0.55	11.60
10	34	61.78	79.54	32.59	0.55	11.16
10	36	63.28	79.51	30.08	0.54	10.66
10	38	64.74	79.48	27.57	0.53	10.11
10	40	66.14	79.45	25.07	0.53	9.51
10	42	67.49	79.43	22.56	0.53	8.87
10	44	68.77	79.41	20.05	0.53	8.20
10	46	69.99	79.39	17.55	0.54	7.52
10	48	71.15	79.38	15.04	0.55	6.82

TABLE I
ASYMMETRIC EXAMPLE FOR SINGLETON–CORE GAUSSIAN IT2FN

The results of a symmetric experiment for $m = 50, \delta_{ul} = 50, \delta_{ur} = 50$ with symmetric changes over δ_{ll}, δ_{lr} are summarized in Table II.

δ_{ll}	δ_{lr}	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU($\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
2	2	-12.70	112.70	120.32	1.04	5.02
4	4	-0.70	100.70	115.30	0.88	8.11
6	6	6.23	93.77	110.29	0.79	10.50
8	8	11.04	88.96	105.28	0.74	12.47
10	10	14.67	85.33	100.27	0.70	14.13
12	12	17.57	82.43	95.25	0.68	15.57
14	14	19.98	80.02	90.24	0.67	16.81
16	16	22.08	77.92	85.23	0.66	17.87
18	18	23.98	76.02	80.21	0.65	18.73
20	20	25.77	74.23	75.20	0.64	19.38
22	22	27.48	72.52	70.19	0.64	19.81
24	24	29.15	70.85	65.17	0.64	20.01
26	26	30.79	69.21	60.16	0.64	19.98
28	28	32.41	67.59	55.15	0.64	19.70
30	30	34.02	65.98	50.13	0.64	19.17
32	32	35.63	64.37	45.12	0.64	18.40
34	34	37.23	62.77	40.11	0.64	17.37
36	36	38.83	61.17	35.09	0.64	16.09
38	38	40.42	59.58	30.08	0.64	14.55
40	40	42.02	57.98	25.07	0.64	12.77
42	42	43.62	56.38	20.05	0.64	10.72
44	44	45.21	54.79	15.04	0.64	8.43
46	46	46.81	53.19	10.03	0.64	5.87
48	48	48.40	51.60	5.01	0.64	3.06

TABLE II
SYMMETRIC EXAMPLE FOR SINGLETON–CORE GAUSSIAN IT2FN

The ratio $R_2(\tilde{A})$ for both examples is shown in Figure 4 where Example 1 and Example 2 stand for asymmetric and symmetric examples respectively.

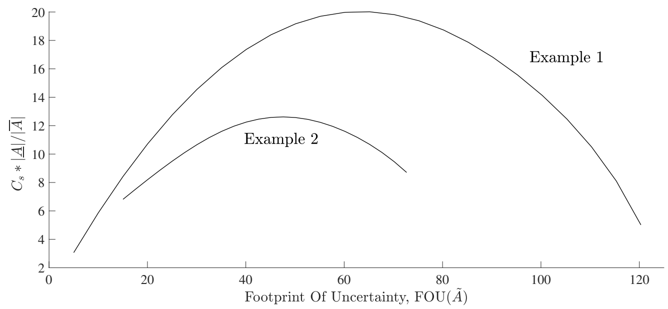

Fig. 3. Ratio $R_2(\tilde{A})$ for the two Gaussian examples

B. Triangular IT2FN

A triangular IT2FN namely $T(a_u, a_l, b, c_l, c_u)$ is shown in Figure 4:

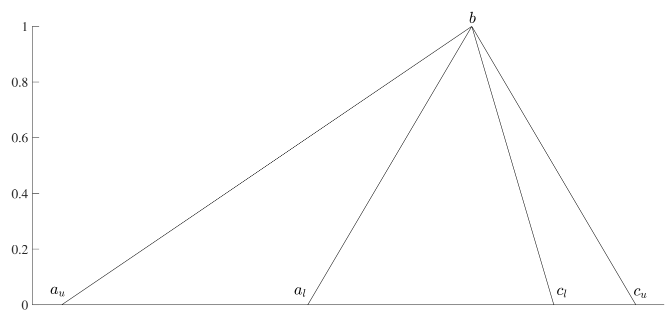

Fig. 4. Triangular IT2FN

The results of a symmetric experiment for $a_u = 0, b = 50, c_u = 100$ with symmetric changes over a_l, c_l are summarized in Table III.

a_l	c_l	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU$\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
48	52	24.75	75.25	48.00	1.05	2.02
46	54	29.22	70.78	46.00	0.90	3.33
44	56	31.92	68.08	44.00	0.82	4.34
42	58	33.85	66.15	42.00	0.77	5.17
40	60	35.32	64.68	40.00	0.73	5.87
38	62	36.50	63.50	38.00	0.71	6.48
36	64	37.47	62.53	36.00	0.70	7.01
34	66	38.33	61.67	34.00	0.69	7.47
32	68	39.11	60.89	32.00	0.68	7.84
30	70	39.85	60.15	30.00	0.68	8.12
28	72	40.57	59.43	28.00	0.67	8.30
26	74	41.27	58.73	26.00	0.67	8.38
24	76	41.96	58.04	24.00	0.67	8.36
22	78	42.64	57.36	22.00	0.67	8.24
20	80	43.32	56.68	20.00	0.67	8.02
18	82	43.99	56.01	18.00	0.67	7.69
16	84	44.66	55.34	16.00	0.67	7.26
14	86	45.33	54.67	14.00	0.67	6.72
12	88	46.00	54.00	12.00	0.67	6.08
10	90	46.67	53.33	10.00	0.67	5.33
8	92	47.33	52.67	8.00	0.67	4.48
6	94	48.00	52.00	6.00	0.67	3.52
4	96	48.67	51.33	4.00	0.67	2.45
2	98	49.33	50.67	2.00	0.67	1.28

TABLE III
SYMMETRIC EXAMPLE FOR SINGLETON–CORE TRIANGULAR IT2FN

The results of an asymmetric experiment for $a_u = 0, b = 50, c_u = 70, c_l = 60$ with asymmetric changes over a_l is summarized in Table IV.

a_l	c_l	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU$(\tilde{A})$	$R_1(\tilde{A})$	$R_2(\tilde{A})$
48	60	32.89	56.18	29.00	0.80	3.99
46	60	33.73	55.45	28.00	0.78	4.34
44	60	34.39	54.73	27.00	0.75	4.65
42	60	34.91	54.04	26.00	0.74	4.92
40	60	35.32	53.35	25.00	0.72	5.15
38	60	35.63	52.68	24.00	0.71	5.36
36	60	35.86	52.00	23.00	0.70	5.54
34	60	36.02	51.33	22.00	0.70	5.69
32	60	36.14	50.67	21.00	0.69	5.81
30	60	36.23	50.00	20.00	0.69	5.90
28	60	36.30	49.33	19.00	0.69	5.96
26	60	36.36	48.67	18.00	0.68	5.98
24	60	36.41	48.00	17.00	0.68	5.96
22	60	36.45	47.34	16.00	0.68	5.91
20	60	36.48	46.67	15.00	0.68	5.82
18	60	36.51	46.00	14.00	0.68	5.69
16	60	36.54	45.34	13.00	0.68	5.53
14	60	36.56	44.67	12.00	0.68	5.33
12	60	36.58	44.01	11.00	0.68	5.09
10	60	36.60	43.34	10.00	0.67	4.82
8	60	36.62	42.67	9.00	0.67	4.50
6	60	36.63	42.01	8.00	0.67	4.15
4	60	36.64	41.34	7.00	0.67	3.76
2	60	36.66	40.67	6.00	0.67	3.33

TABLE IV
ASYMMETRIC EXAMPLE FOR SINGLETON–CORE TRIANGULAR IT2FN

A third asymmetric experiment for $a_u = 40, b = 50, c_u = 100, a_l = 45$ with asymmetric changes over c_l are summarized in Table V.

a_l	c_l	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU$(\tilde{A})$	$R_1(\tilde{A})$	$R_2(\tilde{A})$
45	52	47.28	71.36	26.50	0.91	2.81
45	54	47.98	69.89	25.50	0.86	3.29
45	56	48.66	68.78	24.50	0.82	3.69
45	58	49.33	67.92	23.50	0.79	4.03
45	60	50.00	67.26	22.50	0.77	4.32
45	62	50.67	66.75	21.50	0.75	4.56
45	64	51.33	66.36	20.50	0.73	4.76
45	66	52.00	66.07	19.50	0.72	4.93
45	68	52.66	65.86	18.50	0.71	5.06
45	70	53.32	65.70	17.50	0.71	5.16
45	72	53.98	65.58	16.50	0.70	5.22
45	74	54.65	65.49	15.50	0.70	5.24
45	76	55.31	65.41	14.50	0.70	5.22
45	78	55.97	65.34	13.50	0.69	5.16
45	80	56.63	65.29	12.50	0.69	5.05
45	82	57.30	65.24	11.50	0.69	4.90
45	84	57.96	65.20	10.50	0.69	4.70
45	86	58.63	65.16	9.50	0.69	4.46
45	88	59.30	65.13	8.50	0.69	4.18
45	90	59.97	65.10	7.50	0.68	3.85
45	92	60.64	65.08	6.50	0.68	3.48
45	94	61.31	65.06	5.50	0.68	3.06
45	96	61.98	65.04	4.50	0.68	2.60
45	98	62.66	65.02	3.50	0.67	2.08

TABLE V
ASYMMETRIC EXAMPLE FOR SINGLETON–CORE TRIANGULAR IT2FN

The ratio $R_2(\tilde{A})$ for the three examples are shown in Figure 6 where Example 1, Example 2 and Example 3 stand for symmetric and the two asymmetric examples respectively.

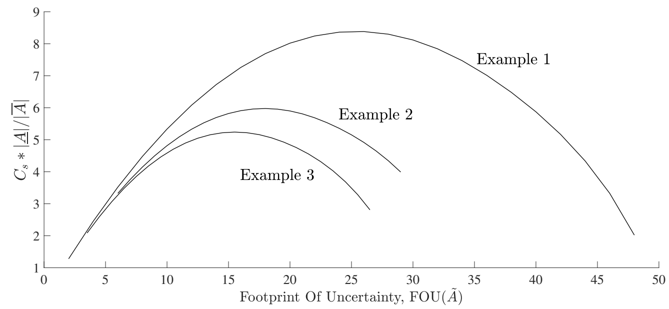

Fig. 5. Ratio $R_2(\tilde{A})$ for the three triangular examples

C. Interval–Gaussian IT2FN

An interval–Gaussian IT2FN namely $G(m_l, m_r, \delta_{ll}, \delta_{rl}, \delta_{lr}, \delta_{rr})$ is shown in Figure 6:

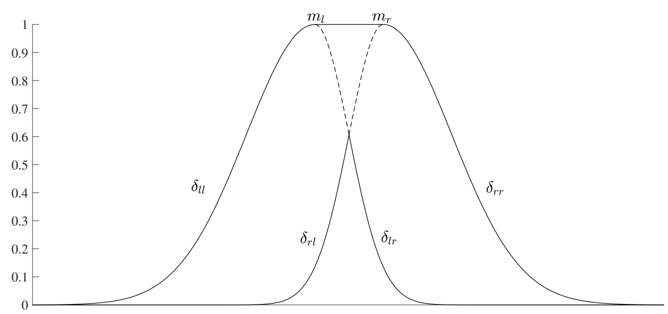

Fig. 6. Interval–Gaussian IT2FN

A symmetric experiment for $m_l = 50, m_r = 60, \delta_{ll} = 50$, $\delta_{rr} = 50$ with asymmetric changes over δ_{rl}, δ_{lr} is summarized in Table VI.

δ_{rl}	δ_{lr}	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU($\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
48	48	48.40	61.60	25.00	0.53	10.76
46	46	46.80	63.20	30.01	0.55	12.76
44	44	45.21	64.79	35.02	0.56	14.52
42	42	43.60	66.40	40.03	0.57	16.05
40	40	42.00	68.00	45.04	0.58	17.35
38	38	40.39	69.61	50.05	0.58	18.41
36	36	38.78	71.22	55.06	0.59	19.24
34	34	37.16	72.84	60.07	0.59	19.85
32	32	35.52	74.48	65.08	0.60	20.23
30	30	33.86	76.14	70.09	0.60	20.38
28	28	32.18	77.82	75.09	0.61	20.32
26	26	30.46	79.54	80.10	0.61	20.03
24	24	28.67	81.33	85.10	0.62	19.54
22	22	26.80	83.20	90.10	0.63	18.85
20	20	24.79	85.21	95.10	0.64	17.96
18	18	22.58	87.42	100.08	0.65	16.89
16	16	20.06	89.94	105.06	0.67	15.63
14	14	17.07	92.93	110.03	0.69	14.18
12	12	13.39	96.61	114.97	0.72	12.52
10	10	8.62	101.38	119.86	0.77	10.60
8	8	2.08	107.92	124.66	0.85	8.35
6	6	-7.89	117.89	129.24	0.97	5.66
4	4	-26.32	136.32	133.21	1.22	2.55
2	2	-80.65	190.65	135.27	2.01	0.13

TABLE VI
SYMMETRIC EXAMPLE FOR INTERVAL–CORE GAUSSIAN IT2FN

An asymmetric experiment for $m_l = 50, m_r = 60, \delta_{ll} = 10$, $\delta_{rr} = 50, \delta_{rl} = 10$ with asymmetric changes over δ_{lr} is summarized in Table VII.

δ_{rl}	δ_{lr}	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU($\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
50	10	81.75	92.32	19.95	0.53	8.10
48	10	80.16	92.38	22.46	0.54	9.00
46	10	78.57	92.44	24.96	0.56	9.80
44	10	76.99	92.50	27.46	0.57	10.52
42	10	75.40	92.58	29.97	0.57	11.14
40	10	73.82	92.67	32.47	0.58	11.67
38	10	72.23	92.77	34.97	0.59	12.11
36	10	70.65	92.89	37.47	0.59	12.46
34	10	69.07	93.03	39.97	0.60	12.72
32	10	67.48	93.20	42.47	0.61	12.90
30	10	65.90	93.40	44.96	0.61	12.99
28	10	64.32	93.64	47.46	0.62	12.99
26	10	62.73	93.93	49.95	0.62	12.91
24	10	61.14	94.29	52.44	0.63	12.74
22	10	59.56	94.74	54.93	0.64	12.50
20	10	57.97	95.30	57.42	0.65	12.17
18	10	56.38	96.01	59.90	0.66	11.77
16	10	54.78	96.92	62.37	0.68	11.29
14	10	53.18	98.07	64.84	0.69	10.73
12	10	51.57	99.53	67.29	0.71	10.08
10	10	49.93	101.38	69.73	0.74	9.34
8	10	48.24	103.72	72.15	0.77	8.50
6	10	46.47	106.71	74.53	0.81	7.54
4	10	44.57	110.59	76.86	0.86	6.46

TABLE VII
SYMMETRIC EXAMPLE FOR INTERVAL–CORE GAUSSIAN IT2FN

The ratio $R_2(\tilde{A})$ for both examples is shown in Figure 8 where Example 1 and Example 2 stand for symmetric and asymmetric examples respectively.

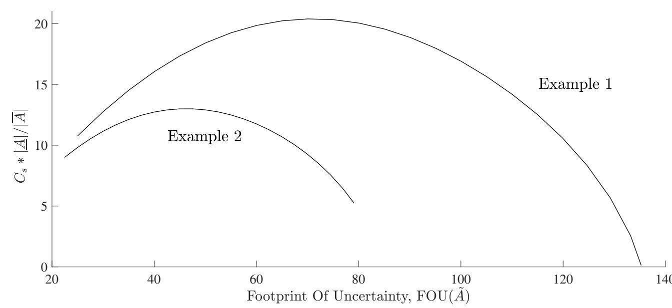

Fig. 7. Ratio $R_2(\tilde{A})$ for both Gaussian examples

D. Interval-triangular IT2FN

An interval–triangular IT2FN namely $T(a_l, a_r, b_l, b_r, c_l, c_r)$ is shown in Figure 8:

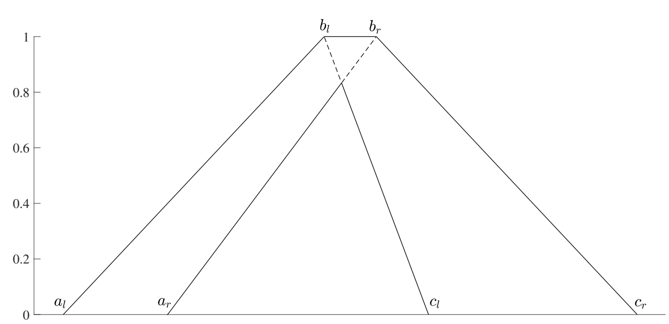

Fig. 8. Interval–triangular IT2FN

The results of a symmetric experiment for $a_l = 0, b_l = 50, b_r = 60, c_r = 110$ with symmetric changes over a_r, c_l are summarized in Table VIII.

a_r	c_l	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU$\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
54	56	12.83	97.17	59.83	1.41	0.23
52	58	22.27	87.73	58.87	1.11	1.23
50	60	27.44	82.56	57.50	0.96	2.30
48	62	30.91	79.09	55.92	0.86	3.28
46	64	33.45	76.55	54.21	0.80	4.16
44	66	35.42	74.58	52.44	0.75	4.94
42	68	37.01	72.99	50.61	0.71	5.63
40	70	38.32	71.68	48.75	0.68	6.25
38	72	39.44	70.56	46.86	0.66	6.81
36	74	40.41	69.59	44.96	0.65	7.31
34	76	41.30	68.70	43.04	0.64	7.74
32	78	42.13	67.87	41.11	0.63	8.10
30	80	42.92	67.08	39.17	0.62	8.39
28	82	43.68	66.32	37.22	0.61	8.59
26	84	44.42	65.58	35.26	0.60	8.72
24	86	45.15	64.85	33.31	0.59	8.77
22	88	45.86	64.14	31.34	0.58	8.73
20	90	46.56	63.44	29.37	0.57	8.61
18	92	47.26	62.74	27.40	0.57	8.41
16	94	47.94	62.06	25.43	0.55	8.13
14	96	48.63	61.37	23.46	0.54	7.76
12	98	49.31	60.69	21.48	0.53	7.31
10	100	49.98	60.02	19.50	0.51	6.77
8	102	50.66	59.34	17.52	0.50	6.15

TABLE VIII
SYMMETRIC EXAMPLE FOR INTERVAL–CORE TRIANGULAR IT2FN

An asymmetric experiment for $a_l = 0, b_l = 50, b_r = 60, a_r = 60, c_r = 110$ with symmetric changes over c_l is summarized in Table IX.

a_r	c_l	$C_l(\tilde{A})$	$C_r(\tilde{A})$	FOU$\tilde{A}$)	$R_1(\tilde{A})$	$R_2(\tilde{A})$
60	48	29.10	59.72	36.73	0.83	2.51
60	46	30.42	58.92	35.92	0.79	2.91
60	44	31.48	58.17	35.08	0.76	3.28
60	42	32.35	57.45	34.21	0.73	3.63
60	40	33.05	56.75	33.33	0.71	3.95
60	38	33.62	56.07	32.44	0.69	4.24
60	36	34.08	55.39	31.53	0.68	4.51
60	34	34.43	54.73	30.61	0.66	4.76
60	32	34.71	54.07	29.68	0.65	4.99
60	30	34.93	53.41	28.75	0.64	5.20
60	28	35.11	52.76	27.81	0.63	5.38
60	26	35.26	52.11	26.86	0.63	5.53
60	24	35.39	51.46	25.91	0.62	5.66
60	22	35.50	50.81	24.96	0.61	5.76
60	20	35.59	50.16	24.00	0.61	5.83
60	18	35.68	49.51	23.04	0.60	5.87
60	16	35.75	48.86	22.07	0.59	5.88
60	14	35.81	48.21	21.11	0.59	5.86
60	12	35.87	47.56	20.14	0.58	5.80
60	10	35.93	46.91	19.17	0.57	5.72
60	8	35.97	46.26	18.19	0.57	5.61
60	6	36.02	45.61	17.22	0.56	5.47
60	4	36.06	44.96	16.24	0.55	5.29
60	2	36.09	44.31	15.26	0.54	5.08

TABLE IX
ASYMMETRIC EXAMPLE FOR INTERVAL–CORE TRIANGULAR IT2FN

The ratio $R_2(\tilde{A})$ for both examples is shown in Figure 9 where Example 1 and Example 2 stand for symmetric and asymmetric examples respectively.

Now, it is clear that all experiments did lead to a quadratic–shaped relationship between FOU$(\tilde{A})$ and the ratio $R_2(\tilde{A})$. To verify this finding, we have performed a quadratic regression for FOU$(\tilde{A})$ as the independent variable (x_i) and $R_2(\tilde{A})$ as

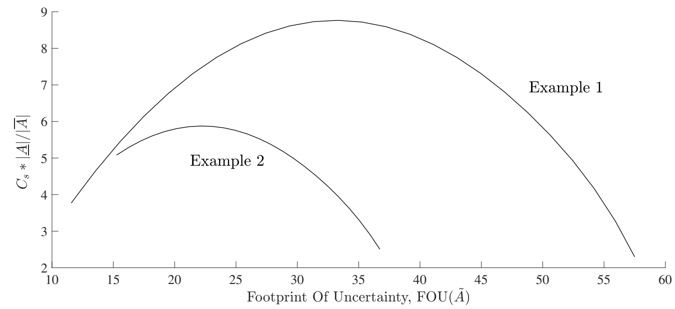

Fig. 9. Ratio $R_2(\tilde{A})$ for both triangular examples

dependent (y_i) defined as follows:

$$y_i = a_2 x_i^2 + a_1 x_i + a_0 + \epsilon_i$$

where ϵ_i is a random error between $\hat{y}_i = a_2 x_i^2 + a_1 x_i + a_0$ and y_i.

The goodness of fit is measured using the mean absolute deviation measure namely MAD:

$$\text{MAD} = \frac{\sum_i |y_i - \hat{y}_i|}{n \cdot y_i} \cdot 100\%$$

where n is the sample size.

The obtained results for the nine experiments (as shown in last section) are shown in Table X

Exp.	a_2	a_1	a_0	MAD (%)
1	-0.0209	0.6576	0.0773	0.5300
2	-0.0126	0.6458	0.1013	1.2457
3	-0.0174	0.6353	0.1884	0.4853
4	-0.0058	0.5454	-0.3823	0.8563
5	-0.0047	0.6125	0.3009	1.3777
6	-0.0045	0.6433	-2.5628	2.6470
7	-0.0071	0.6576	-2.2267	0.4427
8	-0.0107	0.7112	-3.0339	0.7272
9	-0.0160	0.7117	-2.0585	0.2298

TABLE X
RESULTS OF THE REGRESSION FIT

E. Analysis of the results

It is very interesting to find out that there exists an almost perfect quadratic relationship between $C_s(\tilde{A})$ and FOU$(\tilde{A})$ computed via $R_2(\tilde{A})$. This helps to find exact methods to obtain $C(\tilde{A})$ since most of popular membership functions have closed–form equations for $\overline{A}$ and $\underline{A}$.

It is also remarkable to obtain such very low MAD values in all experiments (regarding a quadratic regression model). This is an interesting evidence of the relationship between FOU$(\tilde{A})$ and some of the uncertainty measures that can be computed for Type–2 fuzzy sets/numbers.

On the other hand, the ratio $R_1(\tilde{A})$ shows a monotonic non–increasing behavior (no particular fit was found) which implies that FOU$(\tilde{A})$ has no a linear relation to $C_s(\tilde{A})$. This also implies that the size of $C(\tilde{A})$ does not linearly depend on the size of FOU$\tilde{A}$), it shows a non–linear behavior instead.

V. Concluding Remarks.

A quadratic regression between $C_s(\tilde{A})$ and FOU$(\tilde{A})$ of interval Type–2 fuzzy numbers show a clear non–linear relationship between the centroid $C(\tilde{A})$ of an IT2FN and its uncertainty computed via its FOU$(\tilde{A})$. The obtained MAD measures were low enough to say there is a good quadratic fit between FOU$(\tilde{A})$ and $R_2(\tilde{A})$.

The presented results can be used as a reference point for finding closed equations (or at least approximations) for $C(\tilde{A})$. Since $\overline{A}$ and $\underline{A}$ are easy to compute in many cases, a possible way to obtain $C(\tilde{A})$ and other uncertainty measures could be by using non–linear relationships.

Further analysis on how FOU$(\tilde{A}), a_2, a_1$ and a_0 are related one another is needed. A similar analysis applied to General/Interval Type-2 fuzzy sets (see Figueroa-García et al. [2]) is a step forward from the presented results.

Acknowledgments

Roman Neruda was supported by the Czech Science Foundation project number 18-23827S.

Juan Carlos Figueroa-García would like to thank to Prof. Miguel Melgarejo, Prof. José Jairo Soriano Mendez and *LAMIC* research group of Universidad Distrital for their invaluable support.

References

[1] R. Yager, "A procedure for ordering fuzzy subsets of the unit interval," *Information Sciences*, vol. 24, no. 1, pp. 143–161, 1981.

[2] J. C. Figueroa-García, Y. Chalco-Cano, and H. Román-Flores, "Yager index and ranking for Interval Type-2 fuzzy numbers," *IEEE Transactions on Fuzzy Systems*, vol. 81, no. 1, pp. 93–102, 2018.

[3] C. Carlsson and R. Fullér, "On possibilistic mean value and variance of fuzzy numbers," *Fuzzy Sets and Systems*, vol. 122, no. 1, pp. 315–326, 2001.

[4] D. Wu and J. M. Mendel, "A comparative study of ranking methods, similarity measures and uncertainty measures for interval type-2 fuzzy sets," *Information Sciences*, vol. 179, no. 1, pp. 1169–1192, 2009.

[5] R. Neruda and P. Kudová, "Learning methods for radial basis function networks," *Future Generation Computer Systems*, vol. 21, no. 7, pp. 1131–1142, 2005.

[6] O. Kazík, M. Pilat, and R. Neruda, "Meta learning in multi–agent systems for data mining," in *2011 IEEE/WIC/ACM International Conferences on Web Intelligence and Intelligent Agent Technology*, vol. 2. IEEE, 2011, pp. 433–434.

[7] N. N. Karnik and J. M. Mendel, "Centroid of a type-2 fuzzy set," *Information Sciences*, vol. 132, no. 1, pp. 195–220, 2001.

[8] J. M. Mendel and F. Liu, "Super-exponential convergence of the Karnik-Mendel algorithms for computing the centroid of an interval type-2 fuzzy set," *IEEE Transactions on Fuzzy Systems*, vol. 15, no. 2, pp. 309–320, 2007.

[9] C. Celemin and M. Melgarejo, "A proposal to speed up the computation of the centroid of an interval Type-2 fuzzy set," *Advances in Fuzzy Systems*, vol. 2013, 1993.

[10] M. A. Melgarejo, "A Fast Recursive Method to compute the Generalized Centroid of an Interval Type-2 Fuzzy Set," in *Annual Meeting of the North American Fuzzy Information Processing Society (NAFIPS)*. IEEE, 2007, pp. 190–194.

[11] ——, "Implementing Interval Type-2 Fuzzy processors," *IEEE Computational Intelligence Magazine*, vol. 2, no. 1, pp. 63–71, 2007.

[12] J. C. Figueroa-García, "An approximation method for type reduction of an interval Type-2 fuzzy set based on α-cuts," in *Proceedings of FEDCSIS 2012*, IEEE, Ed. IEEE, 2012, pp. 1–6.

[13] J. Mendel, *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, N. Upper Saddle River, Ed. Prentice Hall, 1994.

[14] J. M. Mendel and D. Wu, *Perceptual Computing: Aiding People in Making Subjective Judgments*, I. Press, Ed. John Wiley & Sons, 2010.

[15] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808–821, 2006.

[16] L. Zadeh, "The concept of a linguistic variable and its application to approximate reasoning-i," *Information Sciences*, vol. 8, pp. 199–249, 1975.

[17] J. C. Figueroa-García, G. J. Hernández-Pérez, and Y. Chalco-Cano, "On computing the footprint of uncertainty of an Interval Type-2 fuzzy set as uncertainty measure," *Communications in Computer and Information Science*, vol. 657, no. 1, pp. 247–257, 2016.

[18] H. Hamrawi, S. Coupland, and R. john, "Type-2 fuzzy alpha-cuts," *IEEE Transactions on Fuzzy Systems*, vol. 25, no. 3, pp. 682–692, 2017.

[19] J. C. Figueroa-García, Y. Chalco-Cano, and H. Román-Flores, "Distance measures for Interval Type-2 fuzzy numbers," *Discrete Applied Mathematics*, vol. 197, no. 1, pp. 93–102, 2015.

[20] G. J. Klir and T. A. Folger, *Fuzzy Sets, Uncertainty and Information*, P. Hall, Ed. Prentice Hall, 1992.