Spatial Data Types for Heterogeneously Structured Fuzzy Spatial Collections and Compositions

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***Abstract*—Fuzzy set theory has found increasing interest in the geosciences, geographic information systems, and spatial database systems to represent geometric objects in the two-dimensional space that reveal an intrinsically vague or fuzzy nature. A spatial object is fuzzy if it contains locations that cannot be assigned completely to the object or to its complement. From a conceptual perspective, *fuzzy spatial data types* for 0-dimensional *fuzzy points*, 1-dimensional *fuzzy lines*, and 2-dimensional *fuzzy regions* in the plane have been introduced, e.g., by the authors formal *Fuzzy Spatial Algebra* (*FUSA*). But the limitation of fuzzy spatial objects to a fixed geometric dimension turns out to be sometimes too restrictive since such objects could benefit from a characterization in terms of several fuzzy spatial sub-objects of different dimensions. An example is a river that consists of 1-dimensional linear parts and 2-dimensional areal parts. For this purpose, this paper introduces a new *fuzzy spatial composition* type with corresponding operations. It allows one to accommodate fuzzy spatial sub-objects that are either adjacent or disjoint. As a generalization of this type, this paper provides a *fuzzy spatial collection* type with corresponding operations. Fuzzy collection objects allow one to keep an arbitrary, finite number of fuzzy spatial objects of possibly different dimensions without any topological constraints in a single object. Application examples show how these new data types can be deployed.**

***Index Terms*—Fuzzy spatial data types, heterogeneous spatial data, fuzzy spatial collection, fuzzy spatial composition**

I. Introduction

Geographical Information Systems (GIS) and spatial database systems are sophisticated tools to represent, manage, and query *crisp spatial objects* that are characterized by an exact location and a precisely defined extent, shape, and boundary in space. Examples are the positions of lighthouses and countries with their political boundaries. *Spatial data types* for crisp *points*, *lines*, and *regions* have been introduced for their representation, including geometric operations such as topological relationships (e.g., *overlap*), geometric set operations (e.g., *union*), and numerical operations (e.g., *distance*).

But increasingly, geoscientists have shown interest in modeling spatial phenomena characterized by *spatial fuzziness*. It captures the inherent property of many spatial objects in reality that have inexact locations, vague boundaries, and/or blurred interiors, and hence cannot be adequately represented by crisp spatial objects. Examples are air polluted areas, temperature zones, soil strata, oceans, agricultural cultivation areas, and habitats of species. In the geosciences, GIS, and spatial database systems, fuzzy set theory has become a popular formal tool for modeling such *fuzzy spatial objects*. For representing them, the authors' *Fuzzy Spatial Algebra* (*FUSA*) provides a formal definition of the *fuzzy spatial data types fpoint* for 0-dimensional *fuzzy points*, *fline* for 1-dimensional *fuzzy lines*, and *fregion* for 2-dimensional *fuzzy regions*. It relaxes the strict decision of belonging (membership degree 1) or non-belonging (membership degree 0) of a point to an object. Instead, partial membership is allowed and expressed by a membership degree in the interval $[0, 1]$. Further, multiple belonging to several spatial objects is possible with different membership degrees. A number of *fuzzy spatial operations* has been formally defined like *fuzzy geometric set operations* (e.g., *fuzzy geometric union*), *fuzzy numerical operations* (e.g., *fuzzy length*, *fuzzy area*), and *fuzzy topological relationships* (e.g., *fuzzy overlap*, *fuzzy inside*).

Despite this progress, it turns out that the limitation of fuzzy spatial objects to a single and fixed geometric dimension is sometimes too restrictive conceptually since many fuzzy real-world phenomena could be better characterized by several fuzzy sub-objects of different geometric dimension. An example is a fuzzy spatial object that represents the pollution of a river whose geometry consists of 1-dimensional linear parts and 2-dimensional areal parts. For this purpose, the first goal of this paper is to introduce a new *fuzzy spatial composition* type as a *heterogeneous* fuzzy spatial data type. A *fuzzy spatial composition object* contains one fuzzy point sub-object, one fuzzy line sub-object, and one fuzzy region sub-object to represent spatial components of each of the three geometric dimensions. Each pair of fuzzy sub-objects has to fulfill the topological constraint that its two sub-objects are either disjoint or adjacent.

The second goal is a generalization of this data type to a *fuzzy spatial collection* type. This type abandons any kind of constraints on its sub-objects. For instance, it allows several fuzzy spatial sub-objects of the same and/or different fuzzy spatial data types (*fpoint*, *fline*, *fregion*) as well as overlapping sub-objects. Further, it permits other fuzzy spatial collection objects and fuzzy spatial composition objects as sub-objects.

The third goal of this paper consists in the definition of a selected set of operations on the two new data types. Examples are aggregation operations (collect, fuzzy union) on fuzzy spatial collection objects as well as fuzzy geometric set operations (union, intersection, and difference) on fuzzy spatial collection objects and fuzzy spatial composition objects.

The fourth goal is to show how the new data types can be used in applications. For this purpose, we specify a simple application scenario and pose some SQL queries that apply some of the operations introduced in this paper.

Section II discusses related work. Section III provides a brief overview of the fuzzy spatial data types *fpoint*, *fline*, *fregion* and their operations in FUSA. Section IV formally defines the data type $fcomposition$ for fuzzy spatial composition objects. Section V extends the fuzzy geometric set operations that are currently defined on two objects of the same fuzzy spatial data type to two objects of different fuzzy spatial data types. This enables the definition of fuzzy geometric set operations on two fuzzy spatial composition objects in Section VI. Section VII formally defines the data type $fcollection$ for fuzzy spatial collection objects. Operations on two fuzzy spatial collection objects are provided in Section VIII. Spatial SQL queries that illustrate the new concepts are shown in Section IX. Finally, Section X draws some conclusions and discusses future work.

II. Related Work

In this section we discuss related work that refers to the handling of collections of heterogeneous spatial objects. We characterize it according to the following criteria: (i) underlying approach, (ii) flexibility, and (iii) applicability.

As for the underlying approach, we distinguish three approaches. The first approach exclusively deals with crisp spatial objects. It provides a spatial collection type called GEOMETRYCOLLECTION specified by OGC [1] and implemented in many spatial database systems and GIS such as PostGIS and QGIS. But this data type is unable to represent vague spatial phenomena. The second approach models vague spatial objects on the basis of a three-valued logic with the truth values *true*, *false*, and *maybe*. The Qualitative Min-Max Model (QMM) [2] represents a fuzzy spatial object as a pair of crisp spatial objects that represent the minimum and maximum extents of the fuzzy spatial object. The QMM allows the specification of a fuzzy line as a pair of mixed-dimensional spatial entities (i.e., all binary combinations of points, lines, and regions). However, this model does not leverage a many-valued logic and hence limits the representation of spatial fuzziness. The third approach employs fuzzy set theory to assign membership degrees to each point of a fuzzy spatial object. The fuzzy approach in [3], [4] defines two fuzzy spatial data types for handling heterogeneous spatial objects, called VExt and VLDim (for details see below).

As for the flexibility, we discuss how the aforementioned references represent heterogeneous data types. A GEOMETRYCOLLECTION object may contain any list of spatial objects of all crisp spatial data types (i.e., crisp points, lines, and regions) without any restrictions. The QMM characterizes the interior and the boundary of a fuzzy line object by using the four adverbs "weakly", "fairly", "strongly", and "completely". However, if both the interior and the boundary are completely vague, then a fuzzy line object is represented by a simple region. We consider this as a "break of dimensional abstraction". Finally, the fuzzy approach specifies a VExt object as a collection of fuzzy lines and fuzzy regions, and a VLDim object as a collection of fuzzy points and fuzzy lines. A data type that includes collections of objects of all three fuzzy spatial data types together is not available.

As for the applicability, we point out how heterogeneous data types can be used in spatial operations. The main operations on two GEOMETRYCOLLECTION objects are *collect*, which gathers the components of both objects, and *union*, which geometrically merges the components of both objects. The QMM specifies topological relationships (e.g., *disjoint*, *meet*) between fuzzy spatial objects by extending the 9-Intersection model [5] on crisp spatial objects. Essentially, a topological relationship is determined by using the minimal and maximal extents of the fuzzy spatial objects as independent crisp spatial objects in the intersection matrix. Similar configurations of the matrix are clustered and then associated to the four adverbs. Finally, the fuzzy approach does not define spatial operations for handling VExt and VLDim objects. A VExt object is obtained when extracting the fuzzy boundary of a fuzzy region object, while a VLDim object is the result of the fuzzy boundary of a fuzzy line object.

Our approach distinguishes itself because of the following characteristics. First, it is based on FUSA [6], which leverages a many-valued logic for representing spatial fuzziness. Second, our two heterogeneous data types, *fcollection* and *fcomposition*, are flexible and expressive to represent several spatial phenomena that cannot be well represented by available approaches. Third, these data types can be handled by fuzzy spatial operations and integrated into database systems by extending the SPA [7].

III. Fuzzy Spatial Data Types and Their Operations

This section gives a brief overview of FUSA and outlines its fuzzy spatial data types (Section III-A) and its fuzzy spatial operations (Section III-B).

A. Fuzzy Points, Fuzzy Lines, and Fuzzy Regions

The fuzzy spatial data types *fpoint* for fuzzy points, *fline* for fuzzy lines, and *fregion* for fuzzy regions in FUSA build the foundation of the data types $fcomposition$ and $fcollection$ in this paper. Their formal definitions, special properties, and constraints leverage concepts from fuzzy set theory and fuzzy topology [8], [9] and are presented in [6], [7].

Intuitively, fuzzy spatial objects have the same geometric structure as crisp spatial objects. But, in addition, each of their points is associated with a membership degree in $]0,1]$ indicating to which extent a point belongs to the spatial object. A fuzzy point object of type *fpoint* is a set of disjoint *simple fuzzy points*. A simple fuzzy point $\tilde{p}$ at (a,b) in $\mathbb{R}^2$, written $\tilde{p}(a,b)$, is a fuzzy singleton in $\mathbb{R}^2$ defined by the membership function $\mu_{\tilde{p}(a,b)}(x,y) = m \in]0,1]$ if $(x,y) = (a,b)$, and

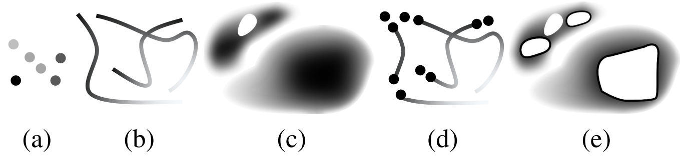

Fig. 1. A *fpoint* object (a), a *fline* object (b), a *fregion* object (c), the fuzzy boundary of the *fline* object (d), and the fuzzy boundary of the *fregion* object (e). Darker areas indicate higher membership degrees than lighter areas.

Fig. 2. Example of modeling the pollution of a river as a fuzzy spatial composition object consisting of a fuzzy line sub-object (all linear parts) and a fuzzy region object (all areal parts).

$\mu_{\tilde{p}(a,b)}(x, y) = 0$ otherwise. Figure 1a shows a *fpoint* object composed of six simple fuzzy points.

A fuzzy line object of type *fline* is a set of adjacent or disjoint *simple fuzzy lines*, with special properties. A simple fuzzy line object $\tilde{l}$ is given as the fuzzy point set $\tilde{l} = \{(q, \mu_{\tilde{l}}(q)) \,|\, q \in f_{\tilde{l}}([0, 1])\}$, where $f_{\tilde{l}}$ is a continuous function that models a simple *crisp* line and $\mu_{\tilde{l}}$ is a (semi-)continuous function that enforces a smooth transition of membership values with possibly finitely many discontinuities. Figure 1b shows a *fline* object composed of five simple fuzzy lines.

A fuzzy region object of type *fregion* is defined as a set of disjoint *simple fuzzy regions*. A *simple fuzzy region* $\tilde{r}$ is a bounded, connected, and regular closed fuzzy set with a membership function $\mu_{\tilde{r}} : \mathbb{R}^2 \to]0, 1]$ that assigns a membership degree to each point in $\tilde{r}$, that is, $\tilde{r} = \{(q, \mu_{\tilde{r}}(q)) \,|\, q \in \mathbb{R}^2\}$. Regularity avoids that $\tilde{r}$ has geometric anomalies like isolated or dangling line and point features as well as missing lines and points in the form of cuts and punctures [5]. Figure 1c shows a *fregion* object composed of two simple fuzzy regions.

The two operations *core* and *support* map fuzzy sets to crisp sets. Let $\tilde{A} \in \{fpoint, fline, fregion\}$. The *core* comprises all points with membership degree 1, i.e., $core(\tilde{A}) = \{q \in \mathbb{R}^2 \,|\, \mu_{\tilde{A}}(q) = 1\}$. The *support* contains all points with membership degree greater than 0, i.e., $supp(\tilde{A}) = \{q \in \mathbb{R}^2 | \mu_{\tilde{A}}(q) > 0\}$.

The *fuzzy boundary* of *fline* and *fregion* objects result in heterogeneous data types. Let $\tilde{S} \in \{fline, fregion\}$. The *fuzzy boundary* of $\tilde{S}$, denoted by $\partial\tilde{S}$, is the union of the crisp boundary of the core of $\tilde{S}$, denoted by $\partial_c\tilde{S} = \{(q, 1) \in \tilde{S} \,|\, q \in \partial(core(\tilde{S}))\}$, and the fuzzy non-core part of $\tilde{S}$, denoted by $\partial_f\tilde{S} = \{(q, \mu_{\tilde{S}}(q)) \in \tilde{S} \,|\, 0 < \mu_{\tilde{S}}(q) < 1\}$. That is, $\partial\tilde{S} = \partial_c\tilde{S} \cup \partial_f\tilde{S}$. Further, $\partial_c\tilde{S} \in fline$ if $\tilde{S} \in fregion$ and $\partial_c\tilde{S} \in fpoint$ otherwise; and $\partial_f\tilde{S} \in fregion$ if $\tilde{S} \in fregion$ and $\partial_c\tilde{S} \in fline$ otherwise. Figure 1d shows the fuzzy boundary of the *fline* object depicted in Figure 1b. It is formed by a *fpoint* object and a *fline* object. Figure 1e shows the fuzzy boundary of the *fregion* object depicted in Figure 1c. It is formed by a *fline* object and a *fregion* object.

B. Fuzzy Spatial Operations

For handling heterogeneous data types, we make use of the fuzzy geometric set operations of two fuzzy spatial objects [6], such as fuzzy geometric intersection, union, and difference. They have the signature $\alpha \times \alpha \to \alpha$ with $\alpha \in \{fpoint, fline, fregion\}$, that is, their arguments are of the *same* fuzzy spatial data type. Let $\tilde{A}, \tilde{B} \in \alpha$, and let reg be a regularization function that corrects any kind of arising geometric anomaly (see Section V, [6]). The *fuzzy geometric intersection* of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \otimes \tilde{B} = \{(q, \mu_{\tilde{A}\otimes\tilde{B}}(q)) \,|\, q \in \mathbb{R}^2 \wedge \mu_{\tilde{A}\otimes\tilde{B}}(q) = \min(\mu_{\tilde{A}}(q), \mu_{\tilde{B}}(q))\}$. The *fuzzy geometric union* of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \oplus \tilde{B} = \{(q, \mu_{\tilde{A}\oplus\tilde{B}}(q)) \,|\, q \in \mathbb{R}^2 \wedge \mu_{\tilde{A}\oplus\tilde{B}}(q) = \max(\mu_{\tilde{A}}(q), \mu_{\tilde{B}}(q))\}$. The *fuzzy bounded difference* of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \dot{-} \tilde{B} = \{(q, \mu_{\tilde{A}\dot{-}\tilde{B}}(q)) \,|\, q \in \mathbb{R}^2 \wedge \mu_{\tilde{A}\dot{-}\tilde{B}}(q) = \max(0, \mu_{\tilde{A}}(q) - \mu_{\tilde{B}}(q))\}$. The *fuzzy symmetric difference* of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A}\Delta\tilde{B} = \{(q, \mu_{\tilde{A}\Delta\tilde{B}}(q)) \,|\, q \in \mathbb{R}^2 \wedge \mu_{\tilde{A}\Delta\tilde{B}}(q) = |\mu_{\tilde{A}}(q) - \mu_{\tilde{B}}(q)|\}$.

IV. Definition of a Fuzzy Spatial Data Type for Fuzzy Spatial Compositions

The fuzzy spatial data types *fpoint*, *fline*, and *fregion* in FUSA provide fuzzy spatial objects whose components are all of the same dimension 0, 1, or 2, respectively. From a modeling perspective, this often limits the expressiveness of spatial reality since many real-world phenomena could be better characterized by a *fuzzy spatial composition object* consisting of fuzzy spatial sub-objects of *different* geometric dimension that fulfill the topological constraints of pairwise disjointedness or adjacency. For example, if we plan to model the pollution of a river as a fuzzy spatial composition object, we can distinguish narrow 1-dimensional linear parts of the river and broader 2-dimensional areal parts possibly with holes (islands) of the river (Figure 2). An assembly of a fuzzy line sub-object and a fuzzy region sub-object, which meet each other in common points of equal membership degree, to a fuzzy spatial composition object would sufficiently capture spatial reality. Formally, we define the fuzzy spatial data type *fcomposition* for fuzzy spatial composition objects as follows:

$$
\begin{aligned}
& fcomposition = \\
& \{(\tilde{p}, \tilde{l}, \tilde{r}) \,| \\
& \text{(i)}\ \ \tilde{p} \in fpoint, \tilde{l} \in fline, \tilde{r} \in fregion \\
& \text{(ii)}\ \ \forall\, o_1, o_2 \in \{supp(\tilde{p}), supp(\tilde{l}), supp(\tilde{r})\}, o_1 \neq o_2 : \\
& \qquad meet_c(o_1, o_2) \vee disjoint_c(o_1, o_2) \\
& \text{(iii)}\ \forall\, \tilde{o_1} \in \{\tilde{p}, \tilde{l}\}\ \forall\, \tilde{o_2} \in \{\tilde{l}, \tilde{r}\}, \tilde{o_1} \neq \tilde{o_2}, \\
& \qquad \forall\, q \in supp(\tilde{o_1}) \cap \partial supp(\tilde{o_2}) : \mu_{\tilde{o_1}}(q) = \mu_{\tilde{o_2}}(q)\}
\end{aligned}
$$

Each fuzzy spatial composition object is represented as a triple that contains a fuzzy point object, a fuzzy line object, and a fuzzy region object (Condition (i)). Condition (ii) requires that any pair of supports of these three fuzzy spatial objects may only be either adjacent or disjoint. The reason is that a point of a fuzzy spatial composition object should not belong to the interiors of more than one fuzzy sub-object since this would contradict the composition idea. However, common boundary points are allowed since they serve as "connection points" between the fuzzy sub-objects. The topological relationships are expressed by the well-defined *topological cluster predicates* $meet_c$ and $disjoint_c$ [5] on complex crisp spatial objects. In the case of common boundary points between the three fuzzy sub-objects, in order to ensure semantic consistency, Condition (iii) requires that the membership degree of each intersection point must be the same in each intersecting sub-object. One or more intersection points are given if the support of the fuzzy point sub-object intersects the boundary of the support of the fuzzy line sub-object or fuzzy region sub-object, or if the support of the fuzzy line sub-object intersects the boundary of the support of the fuzzy region sub-object.

The fact that each data type α is a subtype of the data type $\mathit{fcomposition}$ is given by the following equivalences: (i) $\tilde{p} \in \mathit{fpoint} \Leftrightarrow (\tilde{p}, \varnothing, \varnothing) \in \mathit{fcomposition}$, (ii) $\tilde{l} \in \mathit{fline} \Leftrightarrow (\varnothing, \tilde{l}, \varnothing) \in \mathit{fcomposition}$, and (iii) $\tilde{r} \in \mathit{fregion} \Leftrightarrow (\varnothing, \varnothing, \tilde{r}) \in \mathit{fcomposition}$.

V. Extending the Fuzzy Geometric Set Operations to Operands of Different Fuzzy Spatial Data Types

The fuzzy geometric set operations on fuzzy spatial compositions in Section VI will require an extension of the fuzzy geometric set operations (Section III-B) on two objects of the *same* fuzzy spatial data type (Section III-A) to two objects of *different* fuzzy spatial data types. Since this can produce fuzzy spatial objects with sub-objects of different geometric dimensions, that is, fuzzy spatial composition objects, the result type of the extended operations is the type $\mathit{fcomposition}$. Formally, the signature $\alpha \times \alpha \rightarrow \alpha$ is modified to the signature $\alpha \times \beta \rightarrow \mathit{fcomposition}$ with $\alpha, \beta \in \{\mathit{fpoint}, \mathit{fline}, \mathit{fregion}\}$. This implies that also for $\alpha = \beta$ we obtain an object of type $\mathit{fcomposition}$ now.

First we define some needed auxiliary operations. The first operation is $\mathit{extract} : X \rightarrow \mathit{point} \times \mathit{line} \times \mathit{region}$, where X is any point set in $\mathbb{R}^2$. This function extracts three crisp spatial objects from a given point set as follows. Let $A \subseteq X$ and $\mathit{extract}(A) = (p, l, r)$. First, we identify the set $A_{region} \subseteq A$ such that holds $A_{region} \in \mathit{region}$ and $\forall\, A_{region} \subset B \subseteq A : B \notin \mathit{region}$. This means that A_{region} can be either an empty region object or the largest region object that exists in A. We identify the set $A_{point} = \{q \in A \,|\, \forall\, \epsilon > 0 : N(q, \epsilon) - \{q\} = \varnothing\}$ such that holds $A_{point} \in \mathit{point}$. That is, each point q of A_{point} is an isolated point of A since its neighborhood does not contain any other points of A. We can conclude that $B = A - A^{\circ}_{region} - A_{point} \in \mathit{line}$. The components of B are non-self-intersecting lines [5] that possibly share boundary points. Let $P = \{q \in B \,|\, \mathit{degree}(q) \geq 3\}$ where the function degree returns the number of line components that contain a given point. Thus, P contains boundary points that are shared by at least three line components. Points with a degree equal to 2 denote the end of closed lines representing the boundaries of the components of A_{region}. We are not interested in them since we aim to form a crisp line object that possibly shares boundary points with the crisp region object. This crisp line object is given as $A_{line} = B - A_{region} \cup P \in \mathit{line}$. This means that A_{line} can be either an empty crisp line object or a crisp line object that shares single boundary points with the boundary of A_{region}. Finally we make the following assignments: $p = A_{point}$, $l = A_{line}$, and $r = A_{region}$.

Other auxiliary functions are typeid, type, $\mathit{project}$, and reg. Let $\tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\}$. The auxiliary function typeid, with the signature $\omega \rightarrow \mathit{text}$ with $\omega \in \{\mathit{point}, \mathit{line}, \mathit{region}\}$, receives a crisp spatial object as input and yields the name of its crisp spatial data type as a string value. Similarly, the auxiliary function type yields the crisp spatial data type of a given input. For instance, if $\tilde{o} \in \mathit{fpoint}$, then $\mathit{typeid}(\mathit{supp}(\tilde{o})) = \text{“}\mathit{point}\text{”}$ and $\mathit{type}(\mathit{supp}(\tilde{o})) = \mathit{point}$. The auxiliary function $\mathit{project}$ has the signature $\mathit{point} \times \mathit{line} \times \mathit{region} \times \mathit{text} \rightarrow \omega$. It retrieves one of the three components of a triple (p, l, r) according to the specified data type given as the last parameter. For instance, $\mathit{project}((p, r, l), \text{“}\mathit{point}\text{”}) = p$.

The last auxiliary function *reg* aims at avoiding isolated discontinuities in the transitions of the membership degrees of fuzzy spatial objects. Hence, it assumes that the underlying geometric structure of the input point set is a valid crisp spatial object without geometric anomalies (e.g., cuts and punctures) and yields a valid fuzzy spatial object. The function *reg* does not modify membership degrees if the geometric structure consists of single points since they do not have a smooth transition of their membership degrees. Thus, if $\mathit{supp}(\tilde{o}) \in \mathit{point}$ then $\mathit{reg}(\tilde{o}) = \tilde{o} \in \mathit{fpoint}$. For linear and areal objects, the function *reg* fixes *removable discontinuities* in their membership functions.

A removable discontinuity exists at a point $q_0 \in \mathit{supp}(\tilde{o})$ if both $\mu_{\tilde{o}}(q_0)$ and $\lim_{q \to q_0} \mu_{\tilde{o}}(q) = L < \infty$ exist while $\mu_{\tilde{o}}(q_0) \neq L$. This means that the function is not continuous at that point. This discontinuity can be removed to make $\mu_{\tilde{o}}$ continuous at q_0. For this, we define an almost everywhere identical function named fix such that $\mathit{fix}(\mu_{\tilde{o}}(q))$ is $\mu_{\tilde{o}}(q)$ for $q \neq q_0$ and L for $q = q_0$. If $\mathit{supp}(\tilde{o}) \in \mathit{line}$ or $\mathit{supp}(\tilde{o}) \in \mathit{region}$, then $\mathit{reg}(\tilde{o}) = \{(q, \mathit{fix}(\mu_{\tilde{o}}(q))) \,|\, q \in \mathit{supp}(\tilde{o})\}$. As a result, we obtain fuzzy line objects and fuzzy region objects with smooth transitions of their membership degrees possibly with a finite number of jump discontinuities. Further, such objects are free of geometric anomalies because of their valid underlying geometric structures.

Let $\tilde{A} \in \alpha$ and $\tilde{B} \in \beta$. Let I be the point set intersection of the supports of $\tilde{A}$ and $\tilde{B}$, i.e., $I = \mathit{supp}(\tilde{A}) \cap \mathit{supp}(\tilde{B})$. The *fuzzy geometric intersection* of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \otimes \tilde{B} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that

$$\begin{aligned} \forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\} : \tilde{o} = \mathit{reg}(\{(q, \mu_{\tilde{o}}(q)) \,|\, & \\ & q \in \mathit{project}(\mathit{extract}(I), \mathit{typeid}(\mathit{supp}(\tilde{o}))) \wedge \\ & \mu_{\tilde{o}}(q) = \min(\mu_{\tilde{A}}(q), \mu_{\tilde{B}}(q))\}) \end{aligned}$$

The key strategy is to separately compute each fuzzy spatial sub-object of the yielded *fcomposition* object. In a first step, we retrieve the crisp spatial object that has the same spatial data type as the support of the fuzzy spatial object $\tilde{o}$ to be determined from the triple returned by $\mathit{extract}(I)$. For instance, in the first iteration, $\tilde{o}$ is equal to $\tilde{p}$ and we retrieve the crisp point object from I, i.e., $\mathit{project}(\mathit{extract}(I), \text{“}\mathit{point}\text{”})$.

TABLE I

POSSIBLE CONFIGURATIONS OF FUZZY GEOMETRIC SET OPERATIONS.

Fuzzy Geometric Set Operation	Operands	Possible Results
Fuzzy geometric intersection	*fpoint* $\times$ α	$(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \varnothing, \varnothing)$
	fline $\times$ *fline*	$(\tilde{p}, \tilde{l}, \varnothing)$, $(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \tilde{l}, \varnothing)$, $(\varnothing, \varnothing, \varnothing)$
	fline $\times$ *fregion*	$(\tilde{p}, \tilde{l}, \varnothing)$, $(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \tilde{l}, \varnothing)$, $(\varnothing, \varnothing, \varnothing)$
	fregion $\times$ *fregion*	$(\tilde{p}, \tilde{l}, \tilde{r})$, $(\tilde{p}, \tilde{l}, \varnothing)$, $(\tilde{p}, \varnothing, \tilde{r})$, $(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \tilde{l}, \varnothing)$, $(\varnothing, \varnothing, \tilde{r})$, $(\varnothing, \varnothing, \varnothing)$
Fuzzy geometric union	*fpoint* $\times$ *fpoint*	$(\tilde{p}, \varnothing, \varnothing)$
	fpoint $\times$ *fline*	$(\tilde{p}, \tilde{l}, \varnothing)$, $(\varnothing, \tilde{l}, \varnothing)$
	fpoint $\times$ *fregion*	$(\tilde{p}, \varnothing, \tilde{r})$, $(\varnothing, \varnothing, \tilde{r})$
	fline $\times$ *fline*	$(\varnothing, \tilde{l}, \varnothing)$
	fline $\times$ *fregion*	$(\varnothing, \tilde{l}, \tilde{r})$, $(\varnothing, \varnothing, \tilde{r})$
	fregion $\times$ *fregion*	$(\varnothing, \varnothing, \tilde{r})$
Fuzzy bounded difference and fuzzy symmetric difference	*fpoint* $\times$ α	$(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \varnothing, \varnothing)$
	fline $\times$ α	$(\varnothing, \tilde{l}, \varnothing)$, $(\varnothing, \varnothing, \varnothing)$
	fregion $\times$ α	$(\varnothing, \varnothing, \tilde{r})$, $(\varnothing, \varnothing, \varnothing)$

Then, in a second step, each point of the retrieved crisp spatial object is associated with a membership degree calculated by the minimum function applied to the corresponding membership degrees of $\tilde{A}$ and $\tilde{B}$. Unfortunately, this can introduce isolated removable discontinuities along the distribution of the membership degrees. The execution of the function *reg* corrects this problem.

The *fuzzy geometric union* is defined in a similar way. Let U be the point set union of the supports of $\tilde{A}$ and $\tilde{B}$, i.e., $U = supp(\tilde{A}) \cup supp(\tilde{B})$. Then this operation is defined as $\tilde{A} \oplus \tilde{B} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that

$$\begin{aligned}\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\} : \tilde{o} = reg(\{(q, \mu_{\tilde{o}}(q)) \mid \\ q \in project(extract(U), typeid(supp(\tilde{o}))) \wedge \\ \mu_{\tilde{o}}(q) = \max(\mu_{\tilde{A}}(q), \mu_{\tilde{B}}(q))\})\end{aligned}$$

The operations for the *fuzzy bounded difference* and *fuzzy symmetric difference* of $\tilde{A}$ and $\tilde{B}$ employ the same principle. The fuzzy bounded difference of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \dot{-} \tilde{B} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that

$$\begin{aligned}\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\} : \tilde{o} = reg(\{(q, \mu_{\tilde{o}}(q)) \mid \\ q \in supp(\tilde{A}) \in type(supp(\tilde{o})) \wedge \\ \mu_{\tilde{o}}(p) = \max(0, \mu_{\tilde{A}}(q) - \mu_{\tilde{B}}(q))\})\end{aligned}$$

The fuzzy symmetric difference of $\tilde{A}$ and $\tilde{B}$ is defined as $\tilde{A} \Delta \tilde{B} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that

$$\begin{aligned}\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\} : \tilde{o} = reg(\{(q, \mu_{\tilde{o}}(q)) \mid \\ q \in supp(\tilde{A}) \in type(supp(\tilde{o})) \wedge \\ \mu_{\tilde{o}}(q) = |\mu_{\tilde{A}}(q) - \mu_{\tilde{B}}(q)|\})\end{aligned}$$

Table I depicts the possible operand combinations of each fuzzy geometric set operation together with the possible triple arrangements of the resulting *fcomposition* object for each operand combination. Only one possible triple arrangement can be returned by a fuzzy geometric set operation. For instance, the fuzzy geometric intersection of a *fline* object and a *fregion* object can only yield one of the following four *fcomposition* objects: $(\tilde{p}, \tilde{l}, \varnothing)$, $(\tilde{p}, \varnothing, \varnothing)$, $(\varnothing, \tilde{l}, \varnothing)$, and $(\varnothing, \varnothing, \varnothing)$. Figure 3 shows some examples of fuzzy geometric set operations where the rows represent the following combinations of operands: *fpoint* $\times$ *fline* ($\tilde{C}$ and $\tilde{D}$), *fpoint* $\times$ *fregion* ($\tilde{E}$ and $\tilde{F}$), and *fline* $\times$ *fregion* ($\tilde{G}$ and $\tilde{H}$).

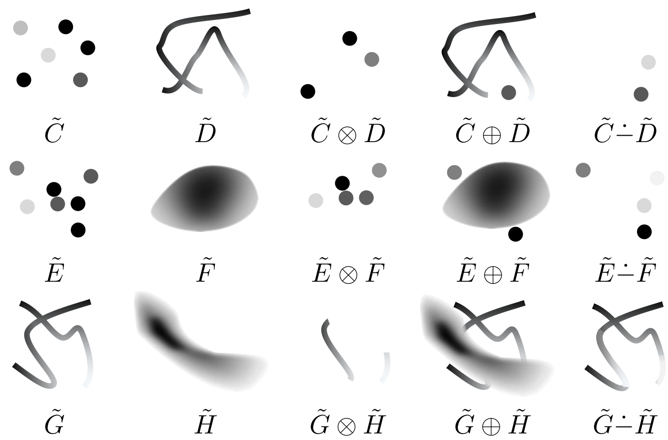

Fig. 3. Fuzzy geometric set operations on different fuzzy spatial data types.

Table I also reveals some important properties. First, if a fuzzy geometric set operation is processed on operands of the same fuzzy spatial data type, there exists a possible resulting *fcomposition* object that only contains one non-empty fuzzy spatial object that corresponds to the result of the fuzzy geometric set operation defined on operands of equal data types (Section III-B). Second, Table I only lists a specific set of triple arrangements for each fuzzy geometric set operation and a given combination of operands. This is based on the following dimension argument. Let m be the dimension of $\tilde{A}$ and n be the dimension of $\tilde{B}$, where $m, n \in \{0, 1, 2\}$. The fuzzy spatial sub-objects obtained from the fuzzy geometric set intersection of $\tilde{A}$ and $\tilde{B}$ can have all dimensions $\leq \min(m, n)$. The sub-objects obtained from the fuzzy geometric set union of $\tilde{A}$ and $\tilde{B}$ can have all dimensions in $\{m, n\}$. Finally, the fuzzy bounded difference and fuzzy symmetric difference yield a *fcomposition* object that can only contain one non-empty fuzzy spatial sub-object which is of the same data type as the first operand.

VI. OPERATIONS ON FUZZY SPATIAL COMPOSITIONS

In this section we define a few type-specific operations (Section VI-A) and several fuzzy geometric set operations

(Section VIII-B) on fuzzy spatial composition objects. Other possible operations are beyond the scope of this paper.

A. Type-Specific Operations

The operation *fboundary* yields the fuzzy boundary of a fuzzy spatial object as a fuzzy spatial composition object. As defined in Section III-A, the fuzzy boundaries of fuzzy line objects and fuzzy region objects consist of sub-objects of heterogeneous data types (Figures 1d and e). The fuzzy boundary of a fuzzy point object is an empty fuzzy spatial composition object. We now are able to define the operation $\mathit{fboundary} : \alpha \rightarrow \mathit{fcomposition}$ with $\tilde{A} \in \alpha$ as

$$\mathit{fboundary}(\tilde{A}) = \begin{cases} (\varnothing, \varnothing, \varnothing) & \text{if } \tilde{A} \in \mathit{fpoint} \\ (\partial_c \tilde{A}, \partial_f \tilde{A}, \varnothing) & \text{if } \tilde{A} \in \mathit{fline} \\ (\varnothing, \partial_c \tilde{A}, \partial_f \tilde{A}) & \text{if } \tilde{A} \in \mathit{fregion} \end{cases}$$

We overload the type-specific operation *project* from Section V and allow it to retrieve a particular fuzzy spatial sub-object from a *fcomposition* object. Its signature is $\mathit{fpoint} \times \mathit{fline} \times \mathit{fregion} \times \mathit{text} \rightarrow \alpha$. For instance, $\mathit{project}((\tilde{p}, \tilde{l}, \tilde{r}), \text{“}\mathit{fpoint}\text{”}) = \tilde{p}$.

B. Fuzzy Geometric Set Operations

We overload the fuzzy geometric set operations *union*, *intersection*, *bounded difference*, and *absolute difference* (Section V) to handle two *fcomposition* objects as operands. Thus, the signature $\alpha \times \beta \rightarrow \mathit{fcomposition}$ is overloaded by $\mathit{fcomposition} \times \mathit{fcomposition} \rightarrow \mathit{fcomposition}$. Let $\tilde{X} = (\tilde{p}_1, \tilde{l}_1, \tilde{r}_1), \tilde{Y} = (\tilde{p}_2, \tilde{l}_2, \tilde{r}_2) \in \mathit{fcomposition}$. To compute a fuzzy geometric set operation on $\tilde{X}$ and $\tilde{Y}$ we take into account all nine possible combinations among the fuzzy spatial objects of both operands. For each combination we process the corresponding fuzzy geometric set operation. For this purpose, we introduce the *9-combination matrix* (*9CM*) as

$$\mathit{9CM}(\tilde{X}, \tilde{Y}, \sigma) = \begin{bmatrix} \tilde{p}_1 \ \sigma \ \tilde{p}_2 & \tilde{p}_1 \ \sigma \ \tilde{l}_2 & \tilde{p}_1 \ \sigma \ \tilde{r}_2 \\ \tilde{l}_1 \ \sigma \ \tilde{p}_2 & \tilde{l}_1 \ \sigma \ \tilde{l}_2 & \tilde{l}_1 \ \sigma \ \tilde{r}_2 \\ \tilde{r}_1 \ \sigma \ \tilde{p}_2 & \tilde{r}_1 \ \sigma \ \tilde{l}_2 & \tilde{r}_1 \ \sigma \ \tilde{r}_2 \end{bmatrix}$$

where $\sigma \in \{\otimes, \oplus, \dot{-}, \Delta\}$ and each cell of this matrix is an *fcomposition* object.

Using a similar strategy as in Section V, we build the fuzzy spatial sub-objects of the *fcomposition* object returned by a fuzzy geometric set operation one by one. Each fuzzy spatial sub-object is created from the aggregation of all fuzzy spatial sub-objects of the same data type disposed in the 9-combination matrix. For instance, to compute the fuzzy point sub-object $\tilde{p}$ of the resulting *fcomposition* object, we aggregate all fuzzy point sub-objects of the cells of the 9-combination matrix. This aggregation iteratively calls the fuzzy geometric union on 9 fuzzy spatial objects. However, after computing the first fuzzy geometric union, an *fcomposition* object is obtained (Section V), which in turn is the operand of the next iterative fuzzy geometric union. Because of this, we need to define the fuzzy geometric union between a fuzzy spatial object and a fuzzy spatial composition object. Let $\tilde{A} \in \alpha$. We overload the signature of the fuzzy geometric union to $\alpha \times \mathit{fcomposition} \rightarrow \mathit{fcomposition}$ by defining it as

$$\tilde{A} \oplus \tilde{X} = \begin{cases} (\mathit{project}(\tilde{p}_1 \oplus \tilde{A}, \text{“}\mathit{fpoint}\text{”}), \tilde{l}_1, \tilde{r}_1) & \text{if } \tilde{A} \in \mathit{fpoint} \\ (\tilde{p}_1, \mathit{project}(\tilde{l}_1 \oplus \tilde{A}, \text{“}\mathit{fline}\text{”}), \tilde{r}_1) & \text{if } \tilde{A} \in \mathit{fline} \\ (\tilde{p}_1, \tilde{l}_1, \mathit{project}(\tilde{r}_1 \oplus \tilde{A}, \text{“}\mathit{fregion}\text{”})) & \text{if } \tilde{A} \in \mathit{fregion} \end{cases}$$

Essentially, this definition uses the fact that the fuzzy geometric union between two operands of the same fuzzy spatial data type results in a fuzzy spatial composition object accommodating only one non-empty fuzzy spatial sub-object that has the same data type as the operands (Table I).

Let $M = \mathit{9CM}(\tilde{X}, \tilde{Y}, \sigma)$ where $M_{i,j}$ denotes the i-th row and the j-th column of the 9-combination matrix M. Hence, we are able to aggregate all values of a 9-combination matrix as $\bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(M_{i,j}, \gamma) = \mathit{project}(M_{1,1}, \gamma) \oplus \mathit{project}(M_{1,2}, \gamma) \oplus \mathit{project}(M_{1,3}, \gamma) \oplus \mathit{project}(M_{2,1}, \gamma) \oplus \mathit{project}(M_{2,2}, \gamma) \oplus \mathit{project}(M_{2,3}, \gamma) \oplus \mathit{project}(M_{3,1}, \gamma) \oplus \mathit{project}(M_{3,2}, \gamma) \oplus \mathit{project}(M_{3,3}, \gamma)$ with $\gamma \in \{\text{“}\mathit{fpoint}\text{”}, \text{“}\mathit{fline}\text{”}, \text{“}\mathit{fregion}\text{”}\}$.

We overload the auxiliary function typeid (Section V) so that it yields the name of the fuzzy spatial data type for a given fuzzy spatial object as input. Let $M^I = \mathit{9CM}(\tilde{X}, \tilde{Y}, \otimes)$. We define the fuzzy geometric set operation *intersection* as $\tilde{X} \otimes \tilde{Y} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that

$$\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\} : \tilde{o} = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(M^I_{i,j}, \mathit{typeid}(\tilde{o}))$$

Let $M^U = \mathit{9CM}(\tilde{X}, \tilde{Y}, \oplus)$, $M^B = \mathit{9CM}(\tilde{X}, \tilde{Y}, \dot{-})$, and $M^A = \mathit{9CM}(\tilde{X}, \tilde{Y}, \Delta)$. The operations for ***fuzzy geometric union***, ***fuzzy bounded difference***, and ***fuzzy symmetric difference*** are defined in a similar form as follows.

$\tilde{X} \oplus \tilde{Y} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that $\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\}$:

$$\tilde{o} = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(M^U_{i,j}, \mathit{typeid}(\tilde{o}))$$

$\tilde{X} \dot{-} \tilde{Y} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that $\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\}$:

$$\tilde{o} = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(M^B_{i,j}, \mathit{typeid}(\tilde{o}))$$

$\tilde{X} \Delta \tilde{Y} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that $\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\}$:

$$\tilde{o} = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(M^A_{i,j}, \mathit{typeid}(\tilde{o}))$$

The definitions show that our approach is very generic in the sense that we are able to define all fuzzy geometric set operations on fuzzy spatial composition objects as

$\tilde{X} \ \sigma \ \tilde{Y} = (\tilde{p}, \tilde{l}, \tilde{r})$ such that $\forall\, \tilde{o} \in \{\tilde{p}, \tilde{l}, \tilde{r}\}$:

$$\tilde{o} = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{3} \mathit{project}(\mathit{9CM}(\tilde{X}, \tilde{Y}, \sigma)_{i,j}, \mathit{typeid}(\tilde{o}))$$

VII. Definition of a Fuzzy Spatial Data Type for Fuzzy Spatial Collections

Fuzzy spatial collection objects are a generalization of fuzzy spatial composition objects and abandon any kind of topological and non-topological constraints on their sub-objects. They are the values of the new fuzzy spatial data type *fcollection*. This data type allows that several fuzzy spatial sub-objects may be of the same and/or different fuzzy spatial data types comprising the types *fpoint*, *fline*, *fregion*, *fcomposition*, and *fcollection*. That is, in particular, it permits other fuzzy spatial collection objects and fuzzy spatial composition objects as sub-objects. Therefore, this data type enables a user to store fuzzy geometries in a single object with largest flexibility and expressiveness.

For a set X let $\mathcal{P}^{\text{fin}}(X)$ be the powerset of all finite subsets of X, that is, $\mathcal{P}^{\text{fin}}(X) = \{Y \subseteq X \mid |Y| \text{ is finite}\}$. We then define the fuzzy spatial data type *fcollection* as follows:

$$\begin{aligned} \mathit{fcollection} = \{(P, L, R, C, D) \mid\ & P \in \mathcal{P}^{\text{fin}}(\mathit{fpoint}), \\ & L \in \mathcal{P}^{\text{fin}}(\mathit{fline}), R \in \mathcal{P}^{\text{fin}}(\mathit{fregion}), \\ & C \in \mathcal{P}^{\text{fin}}(\mathit{fcomposition}), \\ & D \in \mathcal{P}^{\text{fin}}(\mathit{fcollection})\} \end{aligned}$$

A fuzzy spatial collection object is defined as a quintuple whose components comprise arbitrary (and possibly empty) finite sets of fuzzy point objects, fuzzy line objects, fuzzy region objects, fuzzy spatial composition objects, and, recursively, fuzzy spatial collection objects. In particular, overlapping of arbitrary fuzzy spatial objects is allowed. The fifth component allows the creation of a recursive and hierarchical structure of a fuzzy spatial collection object. The recursive definition terminates if the leaf nodes of all branches of the recursion contain the empty set as a fifth component of a quintuple, that is, at some branch with recursion depth k we specify the fuzzy spatial collection object $(P_k, L_k, R_k, C_k, \varnothing)$.

The fact that the data type *fcomposition* is a subtype of the data type *fcollection* is given by the implication that $(\tilde{p}, \tilde{l}, \tilde{r}) \in \mathit{fcomposition} \Rightarrow (\{\tilde{p}\}, \{\tilde{l}\}, \{\tilde{r}\}, \varnothing, \varnothing) \in \mathit{fcollection}$.

VIII. Operations on Fuzzy Spatial Collections

In this section we define a few type-specific operations (Section VIII-A) and some fuzzy geometric set operations (Section VIII-B) on fuzzy spatial collection objects. Other possible operations are beyond the scope of this paper.

A. Type-Specific Operations

The operation $\mathit{collect} : \mathit{fcollection} \times \mathit{fcollection} \to \mathit{fcollection}$ gathers the fuzzy spatial objects of the components of two fuzzy spatial collection objects in a set-theoretic sense into a single fuzzy spatial collection object. Let $A = (P_1, L_1, R_1, C_1, D_1), B = (P_2, L_2, R_2, C_2, D_2) \in \mathit{fcollection}$. We define

$$\begin{aligned} \mathit{collect}(A, B) = (& P_1 \cup P_2, L_1 \cup L_2, R_1 \cup R_2, \\ & C_1 \cup C_2, D_1 \cup D_2) \end{aligned}$$

The operation $\mathit{flatten} : \mathit{fcollection} \to \mathit{fcollection}$ resolves the hierarchical structure of a fuzzy spatial collection object by gathering all fuzzy spatial objects of the whole hierarchy and rearranging them into a single "flat" fuzzy spatial collection object of the kind $(P, L, R, C, \varnothing)$ that preserves the identity of sub-objects. In this sense, it is a generalization of the operation *collect*. The operation *flatten* is recursively defined as follows:

(i) $\mathit{flatten}((P, L, R, C, \varnothing)) = (P, L, R, C, \varnothing)$

(ii) $\mathit{flatten}((P, L, R, C, \{(P_1, L_1, R_1, C_1, \varnothing), \ldots, (P_k, L_k, R_k, C_k, \varnothing)\})) =$
$$(P \cup \bigcup_{i=1}^{k} P_i, L \cup \bigcup_{i=1}^{k} L_i, R \cup \bigcup_{i=1}^{k} R_i, C \cup \bigcup_{i=1}^{k} C_i, \varnothing)$$

(iii) $\mathit{flatten}((P, L, R, C, \{D_1, \ldots, D_k\})) = \mathit{flatten}((P, L, R, C, \{\mathit{flatten}(D_1), \ldots, \mathit{flatten}(D_k)\}))$

Definition part (i) returns a fuzzy spatial collection object as it is if there is no set of fuzzy spatial collection sub-objects as a fifth component that would define a sub-hierarchy. Part (ii) deals with the case of a two-level hierarchy that at the lower level only contains fuzzy spatial collection sub-objects with no sub-hierarchies. In this case, the lower level is resolved by adding all fuzzy spatial sub-objects according to their fuzzy spatial data types to the corresponding sets of fuzzy spatial objects of the higher level. Part (iii) specifies the actual recursion by applying the operation *flatten* recursively to all fuzzy spatial collection sub-objects in the fifth component of a fuzzy spatial collection object.

B. Fuzzy Geometric Set Operations

Each fuzzy geometric set operation $\sigma \in \{\otimes, \oplus, \dot{-}, \Delta\}$ with $\sigma : \mathit{fcollection} \times \mathit{fcollection} \to \mathit{fcollection}$ is performed on all fuzzy sub-objects kept in the hierarchies of two fuzzy spatial collection objects. Let $\tilde{X} = (P_1, L_1, R_1, C_1, D_1), \tilde{Y} = (P_2, L_2, R_2, C_2, D_2) \in \mathit{fcollection}$. Then the operation σ on $\tilde{X}$ and $\tilde{Y}$ is defined as $\tilde{X} \sigma \tilde{Y} = (\{\tilde{p}\}, \{\tilde{l}\}, \{\tilde{r}\}, \varnothing, \varnothing)$ such that

(i) $\forall\, i \in \{1, 2\} : (P_i^f, L_i^f, R_i^f, C_i^f, \varnothing) = \mathit{flatten}((P_i, L_i, R_i, C_i, D_i)$

(ii) $\forall\, i \in \{1, 2\} : (\{\tilde{p}_i\}, \{\tilde{l}_i\}, \{\tilde{r}_i\}, \varnothing, \varnothing) =$
$$\begin{aligned} (& \bigoplus_{\tilde{q} \in P_i^f} \tilde{q} \oplus \bigoplus_{\tilde{c} \in C_i^f} \mathit{project}(\tilde{c}, \text{“}\mathit{fpoint}\text{”}), \\ & \bigoplus_{\tilde{l} \in L_i^f} \tilde{l} \oplus \bigoplus_{\tilde{c} \in C_i^f} \mathit{project}(\tilde{c}, \text{“}\mathit{fline}\text{”}), \\ & \bigoplus_{\tilde{r} \in R_i^f} \tilde{r} \oplus \bigoplus_{\tilde{c} \in C_i^f} \mathit{project}(\tilde{c}, \text{“}\mathit{fregion}\text{”}), \\ & \varnothing, \varnothing) \end{aligned}$$

(iii) $\forall\, i \in \{1, 2\} : (\{\tilde{p}_i^c\}, \{\tilde{l}_i^c\}, \{\tilde{r}_i^c\}, \varnothing, \varnothing) =$
$$\begin{aligned} (& \{\tilde{q} \mid \tilde{q} = \mathit{project}(\mathit{project}(\tilde{p}_i \oplus \tilde{l}_i, \text{“}\mathit{fpoint}\text{”}) \oplus \tilde{r}_i, \\ & \qquad \text{“}\mathit{fpoint}\text{”})\}, \\ & \{\tilde{q} \mid \tilde{q} = \mathit{project}(\tilde{l}_i \oplus \tilde{r}_i, \text{“}\mathit{fline}\text{”})\}, \{\tilde{r}_i\}, \varnothing, \varnothing) \end{aligned}$$

(iv) $(\tilde{p}, \tilde{l}, \tilde{r}) = (\tilde{p}_1, \tilde{l}_1, \tilde{r}_1)\ \sigma\ (\tilde{p}_2, \tilde{l}_2, \tilde{r}_2)$

Since we intend to provide the result of the fuzzy geometric set operations as fuzzy spatial collection objects on the basis of the fuzzy spatial data types *fpoint*, *fline*, and *fregion* only, definition part (i) removes the hierarchies of the two operand objects and flattens them. But this still leads to fuzzy spatial composition objects as sub-objects. Hence, to resolve them, in part (ii) we form the fuzzy geometric union of all fuzzy point objects, fuzzy line objects, and fuzzy region objects respectively that can be found in each of the two flattened fuzzy spatial collection objects and store them in the first three sets of the two resulting fuzzy spatial collection objects. Part (iii) takes these two objects and transforms each of them into another fuzzy spatial collection object. Their single fuzzy point, single fuzzy line, and single fuzzy region sub-objects fulfill the topological constraints of disjointedness or adjacency required in the definition of the data type $fcomposition$. Note that the fuzzy union operation has the effect of only preserving those lower-dimensional objects that are not located in higher-dimensional objects. For instance, this refers to points that are located outside of lines and regions (see Figure 3). Part (iv) computes the final result by applying the corresponding fuzzy geometric set operation on two fuzzy spatial composition objects. This means that ultimately the result is always a fuzzy spatial composition object represented as a fuzzy spatial collection object.

IX. Spatial Queries on Fuzzy Spatial Collections and Compositions

In this section, we show how fuzzy spatial collections and compositions can be integrated into a relational database system and its query language. For demonstration purposes, we make use of an ecological application about the presence of animals in a natural environment, air and water polluted areas, and forests. We assume that they are represented by the relational table schemas *animal*, *pollution*, and *forest* respectively as follows:

```
animal(species:varchar[50], loc:fcomposition)
pollution(id:int, extent:fcollection)
forest(id:int, type:varchar[30], area:fregion)
```

Each table schema has an attribute representing the respective geographic information as a fuzzy spatial object. The fuzzy spatial data types are used in the same manner as standard data types such as *int* or *date*. In particular, the table schemas *animal* and *pollution* make use of the data types *fcomposition* and *fcollection*, respectively. By using these data types, we can represent the geographic information as a single object, instead of scattering the information over a collection of relational tables. This concept is possible due to the use of extensible relational database systems that allow us to define *abstract data types* (ADTs). An ADT hides the complexity of the object structure and executes the operations as abstract methods whose internal specifications are invisible to users.

By employing such concepts, we are able to pose SQL queries such as the ones presented below. The first query asks for the locations (e.g., points, lines, and regions) of forests with animal population. For this, we compute the union of all forests by using the *aggregate* function *aggr_union* that yields a fuzzy spatial composition object. Then we perform the operation *intersection* ($\otimes$) of this result and the locations with presence of animals as follows:

```
SELECT A.species, intersection(F.a, A.loc)
FROM (SELECT aggr_union(area) a FROM forest) F,
     animal A
```

The next query aims to find out the forest areas that are not affected by any pollution. To this end, we process the operation *fbounded_difference* ($\dot{-}$) of each forest area and the aggregation of all polluted areas given by the aggregate function *aggr_collect* as follows:

```
SELECT F.type, fbounded_difference(P.c, F.area)
FROM (SELECT aggr_collect(extent) c
      FROM pollution) P, forest F
```

The final query asks for all non-empty and empty areal intersection parts of the presence of animals and the pollution areas. Hence, we use the operation *project* to retrieve the intersected fuzzy region objects as follows:

```
SELECT P.id, A.species, project(
      intersection(P.extent, A.loc), "fregion")
FROM animal A, pollution P
```

X. Conclusions and Future Work

In this paper we have provided the formal definitions of two new fuzzy spatial data types named $fcomposition$ for fuzzy spatial composition objects and $fcollection$ for fuzzy spatial collection objects. For processing them, we have defined a number of fuzzy geometric set operations and type-specific operations. In contrast to the well-known fuzzy spatial data types for fuzzy point, fuzzy line, and fuzzy region objects, the new data types allow heterogeneous fuzzy spatial objects as sub-objects and may have a highly complex structure in terms of sets and hierarchies of sub-objects.

Future work will deal with (i) the definition of other useful operations and predicates on both types, (ii) the implementation of these types and their operations in the context of our Spatial Plateau Algebra [7], (iii) the construction of such objects from real world data by deploying algorithms proposed in [10], and (iv) the study of optimization methods for processing fuzzy geometric set operations on fuzzy spatial composition objects since the evaluation of the 9-combination matrix is costly.

References

[1] "OpenGIS implementation standard for geographic information - simple feature access - part 1: Common architecture," Open Geospatial Consortium, 2011, http://www.opengeospatial.org/standards/ca.

[2] L. Bejaoui, F. Pinet, Y. Bedard, and M. Schneider, "Qualified topological relations between spatial objects with possibly vague shape," *Int. Journal of Geographical Information Science*, vol. 23, no. 7, pp. 877–921, 2009.

[3] A. Dilo, "Representation of and reasoning with vagueness in spatial information: A system for handling vague objects," Ph.D. dissertation, International Institute for Geo-information Science & Earth Observation, University of Twente, The Netherlands, 2006.

[4] A. Dilo, R. A. de By, and A. Stein, "A system of types and operators for handling vague spatial objects," *Int. Journal of Geographical Information Science*, vol. 21, no. 4, pp. 397–426, 2007.

[5] M. Schneider and T. Behr, "Topological relationships between complex spatial objects," *ACM Trans. on Database Systems*, vol. 31, no. 1, pp. 39–81, 2006.

[6] A. C. Carniel and M. Schneider, "A conceptual model of fuzzy topological relationships for fuzzy regions," in *IEEE Int. Conf. on Fuzzy Systems*, 2016, pp. 2271–2278.

[7] ——, "Spatial plateau algebra: An executable type system for fuzzy spatial data types," in *IEEE Int. Conf. on Fuzzy Systems*, 2018, pp. 1–8.

[8] Y. M. Liu and M. K. Luo, *Advances in Fuzzy Systems - Applications and Theory. Fuzzy Topology*. World Scientific, 1998, vol. 9.

[9] R. H. Warren, "Boundary of a fuzzy set," *Indiana University Mathematics Journal*, vol. 26, no. 2, pp. 191–197, 1977.

[10] A. C. Carniel and M. Schneider, "A systematic approach to creating fuzzy region objects from real spatial data sets," in *IEEE Int. Conf. on Fuzzy Systems*, 2019, pp. 1–9.