# The Natural Transformations with the Multi-Fuzzy Commutativity Condition

Krystian Adam Jobczyk*
AGH University of Science and Technology
Department of Applied Computer Science
al. Mickiewicza 30, 30-059 Krakow

Antoni Ligęza
AGH University of Science and Technology
Department of Applied Computer Science
al. Mickiewicza 30, 30-059 Krakow

***Abstract*—The natural transformation – as a pilar of a functional dynamism in category theory – forms a unique transformation between the so-called functors, which operate between categories and their morphisms. The natural transformations are determined by the appropriate commutativity conditions in diagrams, which co-define them and their general form may be predicted by the so-called Yoneda's lemma. The situation seems to change radically if we exchange single diagrams for multi-diagrams. This paper is aimed at proposing a new concept of multi-fuzzy natural transformation as based on the concept of fuzzy natural transformation, which may be just defined by the scenario with multi-diagrams. It seems to be noteworthy that such a multi-fuzzy natural transformation may be referred to coding theory. In addition, a multi-fuzzy version of Yoneda's lemma is formulated and proved.**

## I. Introduction

A methodologically exhaustive specification of category theory seems to be hardly feasible. It forms a natural consequence of its permanent and vigorous development outside its parent and purely theoretic environment of algebraic topology, but also – a consequence of its still increasing application area. Finally, a gravity of the realistic postulate to adopt category theory in a role of a dominating paradigm in foundation of mathematics instead of set theory cannot be omitted in such an specification attempt. The category theory – primary initiated as an independent research branch by S. Eilenberg an S. McLane in [1] – has recently found a broader reception what founds its reflection in a number of different book positions devoted to it, such as: [2], [3], [4], [5]. The functional and dynamic attitude of reasoning in this area manifests itself in many ways and seems to oscillate around the main conceptual line – leading from the concept category itself via a functorial analysis towards a concept of the so-called natural transformation and its properties. In fact, a nature of the categorial approach to the foundations of mathematics and computer science is aimed at different mappings that may be regarded not only between categories (as the most 'static' entities of category theory), but also between the so-called (covariant and contravariant) functors as unique mappings between categories. Finally – from the most abstract perspective – one can introduce the *natural transformations* as unique and conceptually sophisticated mappings between functors themselves and venture to predict their general form.

The main formal tool – exploited to it – is recognized as the so-called Yoneda's lemma. It allows us to predict a general form of the natural transformations between considered functors relatively quickly – taking only a piece of knowledge about 'behaviour' of functors – and it usually simplifies the whole reasoning with respect to them. For example – for algebraic structures of a cyclic nature [1] (such as monoids or grupoids), the form of the natural transformation between functors – defined with respect to elements of the structure – may be simply determined by a value of a unique functor (the so-called *representable functor* defined in Section II) for an initial object of the structure.

### A. The paper motivation

The condition of diagram commutativity constitutes a distinctive sign of many categorial types of reasoning. In fact, a majority of categorial entities and their duals, such as: products, co-products or functors[2], is involved in a kind of commutativity and many of their properties are just warranted by a commutativity of the appropriate diagrams (see, for example: [1], [5]). Meanwhile, the diagram commutativity forms a two-valued logic-based property. It exactly means that – if the expected commutativity holds – a given categorial entity constitutes the structure of the appropriate type (a product, a co-product, a category, etc.). Otherwise – the same entity does not constitute any structure of this type. Seemingly – this dichotomy seems to form a universaly convenient foundation for categorial reasoning.

Nevertheless, a broader look at category theory as a promising and a pretty general paradigm of thinking – not only in foundation of mathematics and theoretic computer science, but also in reasoning in different situational today's life contexts and its modeling (see:[6], [7], [8]) – disinclines to this two-valued comprehension of commutativity. In fact, many logical concepts and mental mechanisms often show their fuzzy nature. In addition, plenty of commonly exploited mathematical entities have their fuzzy or fuzzified counterparts, such as:

* The first author is also an external member of the Saul Kripke Center at CUNY, New York.

[1] They may be generated by a single element.

[2] All of these fundamental concepts may be found in every handbooks of category theory, for example see: [1], [5]

978-1-7281-6932-3/20/$31.00 ©2020 IEEE

fuzzy groups (see: [9]), fuzzy graphs (see: [10]) or – even C*-algebras – recently described in [11].

Thus, category theory – as potentially suitable to be addressed to such fuzzy structures from a general point of view – should be equipped by formal tools and a conceptual tissue to grasp different aspects of fuzziness.

This methodological expectation persuades us to decide for one of the following solutions: either to fuzzify the categorial concepts themselves or to fuzzify the principles which define them. It seems that the first solution (to introduce a fuzzified versions of the fundamental concepts of category theory[3]) is too radical decision – as so far fuzzification may almost immediately and radically change the whole categorial scenario.

The second proposal (to fuzzify the categorial principles only) seems to be more reasonable – as it allows us to preserve most of the categorial structures and results. In addition, the categorial entities with fuzzified properties seem to naturally correspond to some combinatorial entities such as Hamming's distances – what will be shown in the paper. This approach (to fuzzify the categorial principles) allows us to make several steps further and consider a sequence of fuzzy-commutative diagrams – as described in [12] – and prove a multi-Yoneda's lemma.

These circumstances and shortcoming constitute the main motivating factors to propose a new concept of the natural transformations as based on a multi-fuzzy commutativity – due to earlier ideas from [12] – and to infer some consequences from it.

### B. *Paper objectives and organization*

According to these facts and the motivation factors, this paper is aimed at:

1) proposing a new concept of natural transformation with the multi-fuzzy commutativity condition,
2) formulating and proving the corresponding multi-fuzzy Yoneda's lemma for such a multi-fuzzy transformation and
3) giving an outline of possible application area of this construction.

Rest of the paper is organized as follows. Section II introduces a terminological background of current analysis and give some guidelines for further constructions. Section III forms the proper paper body and it contains the proper definitions of the multi-fuzzy natural transformation in different variants. Section IV contains a new approach to multi-fuzzy natural transformations in terms of the so-called Hamming's distances and fundamental concepts of coding theory. Section V includes the formulation of a multi-fuzzy Yoneda's lemma and an outline of the proof of its proof. In Section VI the leading problem of the paper analysis is solved by means of the proposed conceptual framework. Section VII contains closing remarks.

[3]For example: for a fuzzy category, a fuzzy product, a fuzzy functor etc.

## II. Terminological framework and the leading problem formulation

In this section both the terminological framework of the paper analysis and the leading problem are put forward[4].

### A. *Terminological background*

The fundamental concept of category itself forms a generalization of a concept of the group of transformation, thus we begin with its definition. The term 'transformation' is used here in a general sense as a synonym of a map.

**Definition 1 (Group of transformations).** If $X$ is an arbitrary set and $A = \{\alpha : X \to X\}$ be a set of transformations from $X$ to $X$. Then the structure $G = \langle |G|, \bullet \rangle$, with a domain $|G|$ of $G$ and a group operation $\bullet : |G|^2 \to |G|$, is said to be a `group of transformations`, if it satisfies the following conditions:

1) $G$ is closed on $\bullet$ (if $\alpha, \beta \in G$, then $\beta \bullet \alpha \in G$),
2) There exists a **neutral element** in $G$ (the identity transformation $i_X \in G$),
3) Each transformation $\alpha$ from $G$ is **reversible** in $G$ (In other words, for each transformation $\alpha$ there exists an in inverse transformation $\alpha^{-1} \in G$).

If we reject the condition 3) and admit the transformations from $X \to Y$ (for $Y \neq X$), we achieve the following definition of a category of transformation.

**Definition 2 (Category of transformations).** If $X, Y$ are some arbitrary sets and $K = \{\alpha : X \to Y\}$ be a class of transformations from $X$ to $Y$. Then the structure $\mathcal{K} = \langle K, \bullet \rangle$ is said to be a `category of transformations`, if it satisfies the following conditions:

1) $\mathcal{K}$ is closed on $\bullet$ (if $\alpha, \beta \in K$, then $\beta \bullet \alpha \in K$),
2) there exists a **neutral element** in $K$ (the identity transformation $i_X \in K$),
3) the associativity for $\bullet$ holds.

If $\mathcal{K}$ is a category (of transformations), then the transformations are said to be **morphisms** or – simply –**arrows** and sets $X, Y, \ldots$ are **objects** (of the category). Thus:

$$\mathcal{K} = (M, O), \tag{1}$$

where $M$ is a class of morphisms (from $K$) and $O$ is a class of objects.

**Example.** *This table contains a couple of the most common examples of categories.*

| *Category* | *Objects* | *Arrows* |
|---|---|---|
| ***Set*** | *sets* | *total functions* |
| ***RelA*** | *sets* | *binary relations* |
| ***Pos*** | *posets* | *monotone functions* |
| ***Grp/Gr*** | *groups* | *morphisms* |
| ***Top*** | *topological spaces* | *continuous maps* |
| ***Metr*** | *metric spaces* | *contractions* |

[4]All the definitions may be easily found in each handbook of category theory, for example in [6], [4]

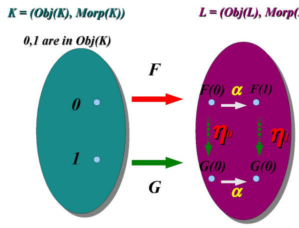


Fig. 1. A visual presentation of an idea of the natural transformation $\eta$ for two functors $F$ and $G$ operating between categories $K$ and $L$.

**Definition 3** (**Small and big category**). A category is said to be **small** if and only if a set of its morphisms forms a set. Otherwise, if the set of the categorial morphisms forms a proper class, then the category is said to be a **big** one.

**Example.** *The categories: **Top, Metr, Set, Gr** are the big ones.*

**Definition 4.** Assume that $K = \langle O_K, Morp_K \rangle$ and $L = \langle O_L, Morp_L \rangle$ are categories. A morphism $F : K \to L$ is said to be called a *covariant functor/homomorphism* if and only if it forms the pair:

$$F = (Hom^{Obj}, Hom^{Mor}), \tag{2}$$

such that: $Hom^{0bj}$ forms a homomorphism between objects of $K$ and $L$ and $Hom^{Morf}$ forms a homomorphism between morphisms of $K$ and $L$, i.e:

1) for each object $X$ of $K$ it holds: $F(id_X) = id_{F(X)}$.
2) for each morphism: $f : X \to Y$ and $g : Y \to Z$, it holds: $F(g \bullet f) = F(g) \bullet F(f)$.

**Example.** *The power set functor $P$ : **Set** $\to$ **Set** maps each set to its power set and each function $f : X \to Y$ to the map which sends $U \subseteq X$ to its image $f(U) \subseteq Y$.*

**Definition 5.** If a functor $F : K \to L$ reverses the direction of arrows (morphisms) in categories, then is said to be a *contravariant functor*.

**Example.** *The same, but a functor $P$ sends each $f : X \to Y$ to the map which sends $V \subseteq Y$ to its preimage $f^{-1}(V) \subseteq X$.*

Independently of a nature of functors to be considered (the covariant or the contravariant ones), one can venture to map one of them into the second one – as depicted in Fig. 1.

**Definition 6.** Assume that $F$ and $G$ are two (in general: different) functors between categories $\mathcal{C}$ and $\mathcal{D}$. Then a *natural transformation* $\eta$ from $F$ to $G$ is a family of morphisms that satisfy the following conditions:

1) to each object $X \in C$ a morphism $\eta_X : F(X) \to G(X)$ between objects of $D$ is associated (it is said to be a **component** on $\eta$ at $X$).
2) The commutativity: $\eta_Y \bullet F(f) = G(f) \bullet \eta_X$ holds.

**Example.** *Assume that two functors $F$ and $G$ (between some categories $\mathcal{C}$ and $\mathcal{D}$) are defined as depicted in Fig. 2 a) (the upward diagram), such that $F(0) = 0, F(1) = 0, G(0) = 1, G(1) = 1$ – as depicted in 2a) (the right diagram). The components of the only possible natural transformation are: $\alpha_0 = f$, $\alpha_1 = f$ because this defining ensures that the diagram commutes. If $F(0,0), F(1,0), G(0,0)$ and $G(1,0)$, the only possible solution is $\alpha_0 = id_0$, $\alpha_0 = id_0$.*

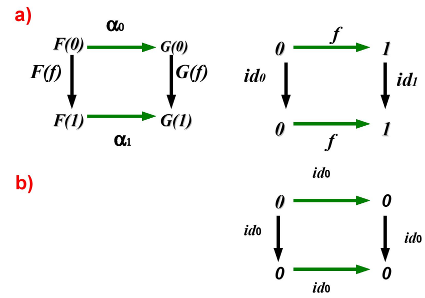


Fig. 2. Two examples of the natural transformations with components marked in green.

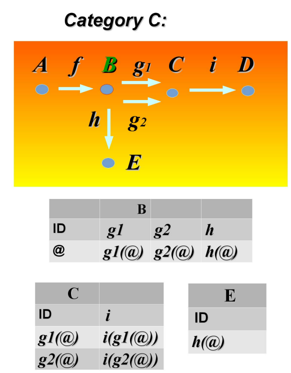


Fig. 3. A visual and a table-based presentation a Hom-functor in the category object $B$.

**Definition 7.** Let $\mathcal{C}$ be a `locally small category` and let **Set** be the category of sets. For each object $A \in \mathcal{C}$ let $Hom(A, -)$ be the so-called hom-functor that maps object $X$ to the set $Hom(A, X)$ (of homomorphisms between $A$ and $X$). A functor $F : \mathcal{C} \to$ **Set** is said to be **representable** if it is `naturally isomorphic` to $Hom(A, -)$ for some object $A$ of $\mathcal{C}$. A representation of $F$ is a pair $(A, \phi)$, where:

$$\phi : Hom(A, -) \to F. \tag{3}$$

**Example.** *Consider a category $\mathcal{C}$ with objects denoted by $A, B, C, D, E$ and morphisms $f, g_1, g_2, h$ and $i$ between them as depicted in Fig. 3. The hom-functors: $Hom(B, -), Hom(C, -)$ and $Hom(E, -)$ are represented by the tables.*

**Theorem.** *(**Yoneda's Lemma**) Let $\mathcal{C}$ be a locally small category, $F$ be a functor from $\mathcal{C} \to$ **Sets** and $Hom(c, -)$ – the representable functor for $c \in \mathcal{C}$. Then the natural*

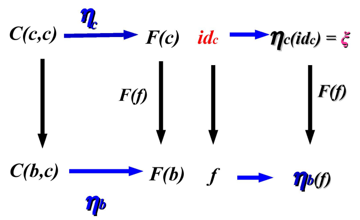


Fig. 4. The illustration of Yoneda's lemma for the natural transformation with the components $\eta_b, \eta_c$. $\mathcal{C}(c,c)$ denotes a class of $\mathcal{C}$-morphisms from $c$ object to $c$ itself, $\mathcal{C}(b,c)$ – the same for functors from $c$ to $b$. The upward diagram describes the general situations, the right one – the particular one, as: $id_c \in \mathcal{C}(c,c)$, $f \in \mathcal{C}(b,c)$, etc. The lacking functors between $\mathcal{C}(c,c)$ and $\mathcal{C}(b,c)$ are omitted for a transparency of the picture.

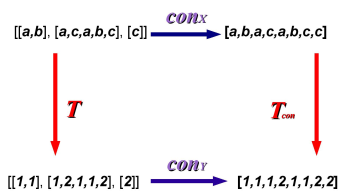


Fig. 5.

*transformations* $\eta$ *from* $Hom(c,-)$ *to* $F$ *corresponds by a bijection to the set* $F(c)$*, that is:*

$$\eta(Hom(c,-),F) \simeq F(c). \tag{4}$$

The sens of the Yoneda's Lemma is the folowing one: if you want to predict how to transform a given Hom-functor to another functor $F$ (that maps a given category $\mathcal{C}$ to **Set** category), then it is enough to take $F$-values for the initially chosen object $c \in \mathcal{C}$.

**An idea of the proof:** It consists in a demonstration that the entire transformation $\eta : H(-,c) \to F$ is already completely determined by a single value $\xi ;= \eta_c(id_c) \in F(c)$, for any object $c \in \mathcal{C}$. In order to prove this, one needs to exploit the naturality of $\eta$. This diagrams shows that not only $\eta_c$ is determined by $\xi$, but also $\eta_b$ as $\eta_b(f) = F(f)(\xi)$. In the same way the natural transformation $\eta$ may be defined for other objects from the category $\mathcal{C}$, what finishes the proof.

### B. The leading problem formulation

Consider the situation of the list (consisting of internal lists): $[[a,b],[a,c,a,b,c],[c]]$ that is firstly transformed via a functor concatenation $con_X$ and the translation $T_{con}$ and – secondly – via a translation $T$ and the functor concatenation $con_Y$ at the end. In both cases the final list after the first stage $[1,1,1,2,1,1,2,2]$ is achieved because of the diagram commutativity – as depicted in Fig.4. Decide

**A** how to fuzzify the natural transformation $\eta = (T, T_{con})$ between functors $con_X$ and $con_Y$ and

**B** what are the defuzzification of the transformation.

**C** how to control the potential errors in the translation process if the translation will be continued up to $k$-stage (a multi-diagram containing $k$ single diagrams) will be created?

## III. The natural transformation with fuzzified commutativity

Before we introduce the proper definition of a (multi-fuzzy) natural transformation with multi-fuzzy commutativity, let us repeat the fact that the definition of natural transformation relies on commutativity of the appropriate diagram (For example – see: Fig. 5). Let us note that the diagram commutativity – which defines the natural transformation – is uniquely representable by the equation condition (see: an outline of the proof of Yoneda's lemma):

$$\eta_b(f) = F(f)(\xi), \tag{5}$$

which formally represents the fact that there are two alternative ways from the initial object $c$ (of a category $\mathcal{C}$) to the final point of the diagram transformation. The first one leads via the composition $\eta_b$ with $f$-mapping. The second one – via the composition functor $F(f)$ for the argument $\xi = \eta_c(id_c)$ (see: the right diagram in Fig. 4.)

It was also underlined in [12] that the most natural way to fuzzify the commutativity leads via either the embedding:

$$\eta_b(f) \subseteq F(f)(\xi) \quad \text{or} \quad F(f)(\xi) \subseteq \eta_b(f). \tag{6}$$

In fact, conditions (5) and (6) allow us to introduced a relaxation of the diagram commutativity towards its fuzzification. In fact, the following cases of relative sets (/ denotes a difference between two sets) are possible:

1) $F(f)(\xi)/\eta_b(f) = \emptyset$,
2) $F(f)(\xi)/\eta_b(f) = A$, where $A$ is a finite set (of values),
3) $F(f)(\xi)/\eta_b(f) = A$, where $A$ is a denumerable set,
4) $F(f)(\xi)/\eta_b(f) = A$, where $A$ is uncountable.

The case 1) determines the normal commutativity of the diagram for natural transformation. In case 2) – the cardinality $card(A)$ may be one of the values $1, 2 \ldots, k$, for an arbitrary large, but finite $k$.

The establishments enabled of introducing the concept of fuzzy natural transformation in two depictions (the upward and a downward one), which were further specified due to the cardinality of the relative set [5]. The upward fuzzy natural transformation was introduced as follows.

**Definition 8.** (**The upward fuzzy natural transformation**[6].) Assume that $F$ and $G$ are two (in general: different) functors between categories $C$ and $D$. Then *the upward fuzzy natural transformation* $\eta$ from $F$ to $G$ is a family of morphisms that satisfy the following requirements:

[5] We introduced, for example, the finite-valued (upward/downward) fuzzy natural transformations and their denumerable and uncountable counterparts – see:[12].

[6] The name is motivated by the fact that the composition $G(f) \bullet \eta_X$ – as achieved via the upward part of the diagram as in Fig. 6 – determines the set of values with the greater cardinality than the set $\eta_Y \bullet F(f)$ – achieved via the downward part of the same diagram.

1) to each object $X \in C$ a morphism $\eta_X : F(X) \to G(X)$ between objects of $D$ is associated (it is said to be a **component** on $\eta$ at $X$).
2) The following upward fuzzy-commutativity:

$$\eta_Y \bullet F(f) \subset G(f) \bullet \eta_X \tag{7}$$

holds.

### A. Towards a multi-fuzzy natural transformation

In this subsection, we introduce a multi-variant of the fuzzy natural transformations, earlier proposed in [12]. In order to make it, let us assume that the right (single) fuzzy commutativity diagram in Fig. 4 has just been enlarged to a multi-diagram determining the following new situation as depicted in Fig. 6. The novelty relies of the existence of the following two sequences of functors: $\{F_i\}$ and $\{G_i\}$, for $i = 1, 2, ...k$. Finally, a corresponding sequence of values $(\xi_0, \dots, \xi_k)$ is given, where – as earlier in the original Yoneda's lemma $\xi_0 = \eta_1(id)$, but also $\xi_1 = \eta_2(f)$, etc.

Due to the idea of an ordinary natural transformation – we intend to view a multi-natural transformation (still a non-fuzzy one) as a sequence of the components $\eta = (\eta_1, \eta_2, \dots, \eta_k)$, for a fixed finite $k$, which warranties commutativity of the largest external diagram in Fig. 6, i.e. $\eta_k \bullet (F_k \bullet ... \bullet F_1(f)) = (G_k \bullet ... \bullet G_1(f)) \bullet \eta_1$.

In order to fuzzify the multi-natural transformation (to introduce a multi-fuzzy natural transformation) it remains to consider a sequence of inequalities (a sequence of non-empty relative sets) instead of equalities in each of single the commutative diagrams in the multi-diagram.

The only problem is to decide whether the results obtained via 'upward compositions' of functors $(G_k \bullet ... \bullet G_1(f)) \bullet \eta_1$ form the set which is included in the set obtained via 'downward' path $\eta_k \bullet (F_k \bullet ... \bullet F_1(f))$ or conversely[7]. This piece of knowledge is enough to introduce a series of new definitions for different variants of multi-fuzzy natural transformations.

In next sections – in order to preserve information about relative sets at each stage of the multi-diagram, we should rather to define the final relative sets as the simple sums of the appropriate relative sets. (It also allows us to prove the multi-fuzzy variant of Yoneda's lemma.) More precisely, if denote the first relative set (for the first diagram in the multi-diagram) as $A_1$, the second one as $A_2, \dots$, the $k$-th by $A_k$, then the final relative sets $A = \bigoplus_{i=1}^{k} A_i$[8].

[7]One can generalize this criterion towards a comparison of cardinalities of the sets. However, the solution will be to general and much more elusive. In addition, considering their cardinalities only might be informatively misleading from the point of view of measurement of fuzziness, if the obtained sets have an empty intersection.

[8]Obviously, we should find the relative sets at each stage staring from the points, for which the corresponding diagram normally commutes, for which the composition via the upward path gives the same result as the composition via downward path.

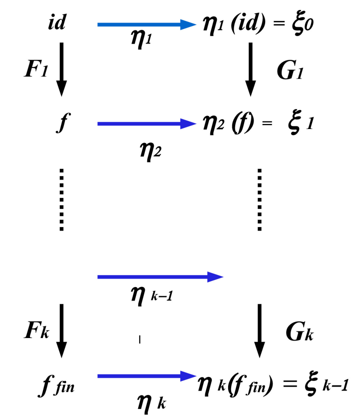


Fig. 6. A multi-diagram with two sequences of functors $\{F_i\}_{i=1}^{k}$ and $\{G_i\}_{i=1}^{k}$ as a construction basis for the multi-fuzzy natural transformation.

**Definition 9.** (**The upward multi-fuzzy natural transformation**[9].) Assume that, for a fixed finite $k$, $\{F_i\}_{i=1}^{k-1}$ and $\{G_i\}_{i}^{k-1}$are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$ (*resp.*), for $i = 1, 2, \dots, k$. Then *the upward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_{i}^{k-1}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \to G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \dots, k$.
2) The following upward fuzzy commutativity:

$$\eta_k^X \bullet (F_k \bullet \dots \bullet F_1(f)) \subset (G_k \bullet \dots \bullet G_1(f)) \bullet \eta_1^X \tag{8}$$

holds (the set obtained via the 'upward' functor composition contains the set obtained via the 'downward' functor composition.)

**Definition 10.** (**The downward multi-fuzzy natural transformation**.) Assume that for a fixed finite $k$, $\{F_i\}_{i}^{k-1}$ and $\{G_i\}_{i}^{k-1}$ are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$, for $i = 1, 2 \dots, k$. Then *the downward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_{i}^{k-1}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \to G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \dots, k$,
2) the following downward multi-fuzzy commutativity:

$$(G_k \bullet \dots \bullet G_1(f)) \bullet \eta_1^X \subset \eta_k^X \bullet (F_k \bullet \dots \bullet F_1(f)) \tag{9}$$

[9]As in [12], the name is motivated by the fact that the composition $G(f) \bullet \eta_X$ via the upward part of the diagram gives the set of values with the greater cardinality than the set $\eta_Y \bullet F(f)$ – achieved via the downward part of the same diagram.

holds (the set obtained via the 'downward' functor composition contains the set obtained via the 'upward' functor composition.).

Because of the cases 1)-4) – the following 3 more-specified definitions may be put forward for each of the earlier definitions. We introduce the definitions for the upward case only. The downward counterparts may be easily introduced by analogy.

**Definition 11.** (**The finite-valued upward multi-fuzzy natural transformation**.) Assume that, for a fixed finite $k$, $\{F_i\}_{i=1}^{k-1}$ and $\{G_i\}_{i=1}^{k-1}$, are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$ (resp.) for $i = 1, 2 \ldots, k$. Then *the finite-valued upward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_i^{k-1}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \rightarrow G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \ldots, k$,
2) the following upward multi-fuzzy commutativity:
$$\eta_k^X \bullet (F_k \bullet \ldots \bullet F_1(f)) \subset (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X \quad (10)$$
holds,

where the relative set determined by (10) satisfies the condition $card(A) < \infty$ ($A$ is simply finite).

In a similar way, a *denumerable-valued upward multi-fuzzy natural transformation* and the *uncountable-valued upward multi-fuzzy natural transformation* are defined.

**Definition 12.** (**The denumerable-valued upward multi-fuzzy natural transformation**.)

Assume that, for a fixed finite $k$, $\{F_i\}_{i=1}^{k-1}$ and $\{G_i\}_{i=1}^{k-1}$, are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$, for $i = 1, 2 \ldots, k$. Then *the denumerable-valued upward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_i^{k-1}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \rightarrow G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \ldots, k$,
2) the following upward multi-fuzzy commutativity:
$$\eta_k^X \bullet (F_k \bullet \ldots \bullet F_1(f)) \subset (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X \quad (11)$$
holds,

where the relative set determined by condition (11) is denumerable, i.e $card(A) = \aleph_0$.

**Definition 13.** (**The uncountable-valued upward fuzzy natural transformation**.) Assume that, for a fixed finite $k$, $\{F_i\}_{i=1}^{k-1}$ and $\{G_i\}_{i=1}^{k-1}$, are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$, for $i = 1, 2 \ldots, k$. Then *the upward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ to $\{G_i\}_i^{k-1}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \rightarrow G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \ldots, k$,
2) the following upward multi-fuzzy commutativity:
$$\eta_k^X \bullet (F_k \bullet \ldots \bullet F_1(f)) \subset (G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X \quad (12)$$
holds,

where the relative set determined by condition (12) is equinumerous with set of reals, i.e $card(A) = 2^{\aleph_0}$.

## IV. The natural transformation with multi-fuzzy commutativity in terms of Hamming's distances

It is not difficult to observe that – even in the context of multi-fuzzy commutativity – some situations (such as situations 2) and 3) in Section III) preserve a convenient computationally nature. In addition, they may be considered somehow together as 'countable' cases. In [12], all the countable cases were specified together in terms of Hamming's distances and they allowed the authors to distinguish a unique subclass of the Hamming's countable-valued fuzzy natural transformations. The main motivation to associate the Hamming's distances to fuzzy commutativity diagrams was inspired by a observation that the final relative sets – fuzzifying the normal diagram commutativity – may be interpreted as a transmission of errors in coding of words. More precisely, the difference/relative sets were identified with sets $\{i : a_i \neq b_i\}$ – the sets of indices $i$, for which the words **a** =$\langle a_1 \ldots, a_n\rangle$ and **b** = $\langle b_1 \ldots, b_n\rangle$ in $A^m$ are different.

The exact definition of Hamming's countable-valued upward fuzzy natural transformation is as follows.

**Definition 14.** (**Hamming's countable upward fuzzy natural transformation**, see: [12].) Assume that $F, G$ are functors between categories $C$ and $D$ as in Definition 8 and let $\eta$ be an upward fuzzy natural transformation between $F$ and $G$. $\eta$ is said to be *Hamming's countable-valued upward fuzzy natural transformation* if the following holds:
$$G(f) \bullet \eta_X / \eta_Y \bullet F(f) = \{i : a_i \neq b_i\}, \quad (13)$$
where $card(\{i : a_i \neq b_i\}) < \infty$ or $card(\{i : a_i \neq b_i\}) = \aleph_0$ (i.e. the set is either finite or denumerable).

In the similar way *Hamming's countable-values downward fuzzy natural transformation* was defined provided that the the difference set $\eta_Y \bullet F(f)/G(f) \bullet \eta_X$ is taken instead of
$$G(f) \bullet \eta_X / \eta_Y \bullet F(f). \quad (14)$$

Due to this idea – a new definition of Hamming's countable upward/downward multi-fuzzy natural transformation will be proposed. It will also adopt the following technical definitions of Hamming's distance and Hamming's ball. (see: [13], pp. 375-378).

**Definition 15.** Let A be a finite alphabet and $n \in \mathbf{N}$. `The Hamming's distance` between words **a** =$\langle a_1 \ldots, a_n\rangle$ and **b** = $\langle b_1 \ldots, b_n\rangle$ in $A^m$ is a cardinality of the set $\{i : a_i \neq b_i\}$. This number is denoted by $d_H(\mathbf{a}, \mathbf{b})$.

**Definition 16.** The Hamming's ball of radius d and center **a** – denoted by $S_d(\mathbf{a})$ – is defined as

$$S_d(\mathbf{a}) = \{\mathbf{b} : d_H(\mathbf{a}, \mathbf{b}) \leq d\}. \tag{15}$$

**Theorem.** *The necessary and sufficient condition to* detect *at most d errors is the condition:*

$$\boldsymbol{a} \neq \boldsymbol{b} \rightarrow \boldsymbol{a} \notin S_d(\boldsymbol{b}). \tag{16}$$

Nevertheless, a multi-diagram scenario and an expectation of a computational graspability of the relative sets force a need to introduce an operationally effective method to compute the final relative set from the earlier relative sets for each stage of the multi-diagram. As earlier mentioned – we adopt the idea to define the final relative set as a simple sum of the relative sets for each single diagram. Namely, let us assume that a multi-diagram with $k$-single diagrams – as depicted in Fig.6 (or Fig. 7) – is given. If $A_1$ is a relative set for the 1-st diagram from the beginning (for example: in $id_0$ in a multi-diagram in Fig.6), $A_2$ – for the 2nd diagram, etc., then the final relative set $A$ after $k$-steps will be defined as a simple sum of them, i.e. $A = \bigoplus_{i=1}^{k} A_i$[10]. It leads to the following definition.

**Definition 17.** (**Hamming's countable downward multi-fuzzy natural transformation**) Assume that, for a fixed finite $k$, $\{F_i\}_{i=1}^{k-1}$ and $\{G_i\}_{i=1}^{k}$, are two finite sequences of (in general: different) functors between categories $C_i$ and $D_i$, for $i = 1, 2 \ldots, k$. Then *Hamming's countable upward multi-fuzzy natural transformation* $\eta$ from $\{F_i\}_{i=1}^{k-1}$ from $\{G_i\}_{i=1}^{k}$ is a family of morphisms that satisfy the following requirements:

1) to each object $X \in C_i$ a morphism $\eta_i^X : F_i(X) \rightarrow G_i(X)$ between objects of $D_i$ is associated (it is said to be an $i$-**component** of $\eta$ at $X$), for $i = 1, 2 \ldots, k$,
2) the following condition holds:

$$max_{i=1,2\ldots,k-1}\{card(A_i)\} = \aleph_0 \text{ or} \tag{17}$$

$$max_{i=1,2\ldots,k-1}\{card(A_i)\} < \infty \tag{18}$$

for sets $A_i$ taken from $\bigotimes_{i=1}^{k-1} A_i$[11], where

$$\bigoplus_{i=1}^{k-1} A_i = \eta_k^X \bullet (F_k \bullet \ldots \bullet F_1(f))/(G_k \bullet \ldots \bullet G_1(f)) \bullet \eta_1^X \tag{19}$$

In other words, we consider the maximal cardinality of the sets from the simple sum (as the final relative/difference set). If it is no greater than $\aleph_0$ (i.e. the set with the greater cardinality is at most denumerable), then the multi-fuzzy natural transformation is of Hamming's sort.

In the similar way, the upward counterpart is defined. The only difference is a need to exchange the order of composition of functors in (19) for the final relative set.

[10]Since the simple sum has a finite number of elements, we can identify $\bigoplus_{i=1}^{k} A_i$ with $(A_1, A_2, \ldots, A_k)$

[11]Let us note that the number of the relative sets is equal to $k-1$ as we have $k-1$ single diagrams by $k$ number of $\eta$-components.

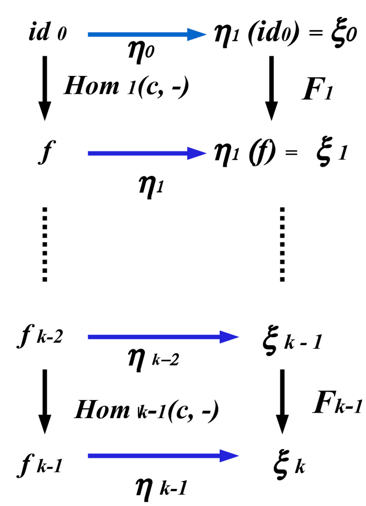


Fig. 7.

## V. Fuzzy Yoneda's Lemma

The Hamming's representation (of multi-fuzzy natural transformations) enables of formulating the multi-fuzzy version of Yoneda's lemma – due to earlier ideas for fuzzy Yoneda's lemma from [12] in a smarter and a more concise way.

Let us repeat that the main difference between the classical Yoneda's lemma and the fuzzy Yoneda's lemma consists in a *degree of similarity* between the natural transformation (between the initial functors: $Hom(c, -)$ [12] and its representable functor, say $F$) and $F(c)$. In other words, we deal with an isomorphism in the classical Yoneda's lemma and – with a similarity $\sim$ (measured by cardinality of difference sets) in a case of the fuzzy Yoneda's lemma. We say here about *similarity up to the difference set* $A$ and denote it by '$\sim uptod.s.A$'.

In a case of multi-fuzzy Yoneda's lemma we will deal with a multi-similarity up to the difference set. Meanwhile, the difference set (the final relative set) has already been defined as a simple sum of the 'local' relative sets for the single fuzzy commutative diagrams. This fact should be also reflected in our new definition.

**Definition 18.** (**Multi-similarity up to the difference set A**.) Assume that $K, M$ are two (not necessary non-empty) sets. We say that $K$ *is multi-similar to*$M$ *up to the difference set* $A$ if and only if $K/M = A$, where $A$ is a finite simple sum of some components, i.e. $A = \oplus_i^k A_i$ for some $A_i$, for a fixed $k$[13]. We denote this fact by: $K \sim M$ (up to d.s. $A$). Conversely,

[12]Hom-functor for a fixed object $c$ in a given category $\mathcal{C}$.

[13]In practice, we will consider finite simple sums. Pedantically – because we consider $k$-components of $\eta$ multi-fuzzy natural transformations – we take $k-1$ relative sets and components in their simple sum. However, this fact is not important from the purely merit-related point of view.

$M$ is *multi-similar to*[14] $K$ *up to the difference set* $A$ if and only if $M/K = A$, where $A$ is a finite simple sum of some component.

It allows us to formulate the multi-fuzzy Yoneda's Lemma[15]. The main difference between fuzzy Yoneda's lemmma and its multi-fuzzy counterpart manifests itself in a fact that the considered natural transformation does not map a given functor to its Hom-representable functor, but the whole sequence of functors to their Hom-representable functors.

**Theorem.** ***(The multi-fuzzy Yoneda's Lemma.)*** *Let* $\mathcal{C}_{\rangle}{}_{i=1}^{k}$ *be a sequence of locally small categories,* $\{F_i\}_{i=1}^{k-1}$ *be a sequence of functors from* $\mathcal{C}_{\rangle} \to$ ***Sets*** *and* $Hom_i(c, -)$ *– the representable functor for* $c \in \mathcal{C}_{\rangle}$*. Then the downward fuzzy natural transformations* $\eta$ *from the sequence* $\{Hom_i(c, -)\}_{i=1}^{k-1}$ *to the sequence* $\{F_i\}_{i=1}^{k-1}$ *is similar to the set* $F_{k-1} \bullet \ldots F_1(c)$ *up to a difference set A, what we denote by:*

$$F_{k-1}\bullet \ldots F_1(c) \sim \eta(\{Hom_i(c,-)\}_{i=1}^{k-1}, \{F_i\}_{i=1}^{k}) \;\; (up\ to\ d.s.\ A) \quad (20)$$

*for i = 1,2. . . , k.*

**The proof outline:** The proof is inductive over the complication of the multi-diagram. For $i = 1$, i.e. when the multi-fuzzy commutative diagram is a single fuzzy diagram, the proof runs as in [12].

In this case – as in a proof of the classical variant of Yoneda's lemma – we must find the general form of both the components, say $\eta_c$ and $\eta_b$ of $\eta$-transformation, where $c, b$ are fixed objects of the initial category $\mathcal{C}$ (as in Fig.4). As earlier, one path leads from the object $c$ via $\eta_c$. Establishing the values $\eta_c(id_c)$ as $\xi$, we can finally get – via $F(f)$ – the value $F(f)\xi$. The second path from $c$ leads to $f$ composed with $\eta_b$, i.e. $\eta_b(f)$. If $\eta$ constitutes only the upward fuzzy natural transformation, then $F(f)\xi/\eta_b(f) = A$, where $A$ is a not-empty set, so – due to the Definition 18 – $F(f)\xi \sim \eta_b(f)$. It means that $\eta_b$ is also established by $\xi$, with exception the points from the difference set $A$, thus: $\eta(Hom(c, -), F) \sim F(c)\ (up\ to\ d.s.\ A)$[16].

In further inductive steps (see: Fig.7) we exploit the fact that each such $\xi_i$ up to $\xi_k$ may be established in such a way that the relative differences $F_i\xi_i \sim \eta_i(f)$ (up to d.s $A_i$) – because all functors $Hom_i(c, -)$ and $F_i$ satisfy the same assumptions as $F$ and its Hom-representable functorat at each single diagram stage like in the original Yoneda's lemma. One only needs to restrict the analysis to the elements, for which the diagram commutes (and omit the points from the relative sets). Having already defined $\xi_{k-1}$, it is easy to define the final $\xi_k$ and repeat the same reasoning in order to establish the last needed component $\eta_k$ of our fuzzy natural transformation as earlier. It means that at each stage $i$ we establish similarities of the form: $\eta_i(Hom_i(c, -), F_i) \sim F_i(c)$. The thesis of theorem follows from it and from the fact that $\xi_k$ is obtained from $\xi_0$ via the composition $F_{k-1} \bullet \ldots \bullet F_1$.

## VI. The problem solving and closing remarks

Being equipped with the definition of fuzzy natural transformations in terms of Hamming's representability let us return to our leading problem to answer the questions **A**, **B** and **C**.

**Ad. A**. Due to the arrangements from Section IV – a possible fuzzification of the natural transformation $\eta = (T, T_{con})$ may be identified with a similarity up to a difference set $A$ for the relative complement (relative set) $con_Y(T)(l)/T_{con}(con_X(l)$, where $l = [[a, b], [a, c, a, b, c], [c]]$ (see: Fig. 5). Since the final lists [1,1,1,2,1,1,2,2], [1,1,2,3,2,2,1,1] – as Hamming's representable – may be seen as two words **a** = $(a_1, \ldots, a_8)$ and **b** = $(b_1, \ldots, b_8)$ (over an alphabet $\Sigma = \{1, 2, 3\}$) with the Hamming's distance $d_H(\mathbf{a}, \mathbf{b}) = \{i : a_i \neq b_i\}$. Since the lists are mutually different in 6 places, their $d_H = 6$.

**Ad. B**. Deffuzification (as a return to the natural transformation from the initial fuzzy one) requires – at first – a detection of all the 6 errors in our case. Meanwhile – due to the necessary and sufficient condition to detect of at most $d$ errors, we only need to check whether the condition: $\mathbf{a} \neq \mathbf{b} \to \mathbf{a} \notin S_6(\mathbf{b}) = \{\mathbf{c} : d_H(\mathbf{b}, \mathbf{c}) \leq 6\}$ is satisfied. Since the lists **a, b** are different, it is enough to verify whether $a \notin S_6(\mathbf{b})$. Nevertheless $d_H(\mathbf{a}, \mathbf{b}) = 6$, so $a \in S_6(\mathbf{b})$, so a fuzzification is impossible with this pair of lists. We need a pair of list with at least 7 different values.

**Ad. C** Due to our earlier arrangements and a method of creating the relative sets in multi-fuzzy commutative diagrams – we are in a position to (at least partially) control the error propagation – even having a restricted knowledge about further translation process. In fact, it is enough that we preserve a knowledge about errors (relative sets) at each stage, say $A_i, i = 1, 2, , k$ – due to solution in point B. In fact, the final the error propagation $A$ is a simple sum of them $A = \bigoplus_i^k A_i$[17].

## VII. Closing remarks

It has been already shown how to define a multi-fuzzy natural transformations in different variants. As the main distinction criterion for the construction of this taxonomy – the cardinality of the difference sets was adopted. In a case of finite or denumerable difference sets we can venture to propose the Hamming's representation of the multi-fuzzy natural transformations. In a case of difference set with greater cardinality – such a representation is not considered. Finally, a multi-fuzzy Yoneda's lemma was formulated with an outline of the proof. Obviously – as in the original Yoneda's lemma and a fuzzy Yoneda's lemma– it allows us to predict a general form of multi-fuzzy natural transformations, but only in some approximative way – as we have a tendency to say: 'up to

[14]The sens of the name may be elucidated by the fact that each finite simple sum may be identified with a sequence of its components. In fact, the difference set $A$ forms a sequence of the appropriate 'local' relative sets and it does not constitute any single set.

[15] The multi-fuzzy Yoneda's lemma may be also specified in two variants – dependently of a nature of the multi-diagram. However, for a brevity of the presentation we decide to formulate a multi-fuzzy Yoneda's lemma in a more general (generic) form.

[16]The same reasoning may be repeated for the downward version.

[17]Let us note that we have $k + 1$ components of the corresponding multi-fuzzy natural transformation

the difference sets'. Simultaneously, we have had a chance to observe some novelty of this theorem with respect to it predecessor. In fact, a multi-fuzzy Yoneda's lemma enables of predicting the form of multi-fuzzy natural transformation between the whole sequence of functors and its corresponding sequence of their representable functors. By the way, the multi-fuzzy Yoneda's lemma introduces a piece of optimism: we are in a position to preserve a knowledge about relatives difference sets at each stage of our runs on multi-diagrams.

It seems that the purely theoretic considerations may find their application area in a programming framework of Haskell.

## REFERENCES


[1] S. Eilenberg and S. MacLane, "General theory of natural equivalences," *Transactions of the American Mathematical Society*, vol. 28, p. 247, 1945.

[2] F. Lawvere and R. Rosebrugh, *Sets for Mathematics*. Oxford University Press, 2003.

[3] F. Lawvere and S. Schnauel, *Conceptual Mathematics: A First Introduction to Categories*. Cambridge University Press, 2009.

[4] S. Awoday, *Category Theory*. Oxford University Press, 2010.

[5] S. Mac Lane, *Categories for the Working Mathematician*. Springer-Verlag, 1998.

[6] D. Spivak, *Category theory for the sciences*. Cambridge Press, 2014.

[7] R. Fagin, J. Halpern, Y. Moses, and M. Vardi, *Reasoning about Knowledge*. Cambridge, 1995.

[8] A. Lomuscio and J. Michaliszyn, "An epistemic halpern-shoham logic," in *Proceedings of IJCAI-2013*, 2013, pp. 1010–1016.

[9] A. Rosenfeld, J. Moderson, and K. Bhutani, *Fuzzy Group Theory*. ISBN 0-691-12993-2: Springer, 2006.

[10] J. Moderson, S. Mathew, and D. Malik, *Fuzzy Graph Theory with Applications to Human Trafficking*. Springer, 2018.

[11] F. Fattahi, "Fuzzy quasi c*-algebra," *Fuzzy Information and Engineering*, vol. 5(3), pp. 327–33, 2013.

[12] K. Jobczyk and P. Gał czyński, "The natural transformation with fuzzified commutativity," *Proc. of FUZZ-IEEE*, pp. 1–6, 2019.

[13] W. Lipski and W. Marek, *Combinatorial analysis*. PWN, 1986.