Developing Idea of Ordinal Sum of Fuzzy Implications

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***Abstract*—Fuzzy implication functions are one of the most widely studied class of operations investigated in fuzzy logic due to their importance in theory and also many different applications. One can find many different methods of obtaining new fuzzy implications. In this contribution we deal with the ordinal sums of fuzzy implications – one such method. We present some chosen aspects of the development of this idea through recent years. Two of the previously published methods are improved. New generalisations of the existing construction are also proposed.**

I. Introduction

Fuzzy implications and methods of their constructions and generalisations are constantly of great interest in recent studies (see e.g. [1], [2], [3]). One way of obtaining fuzzy implications on the basis of the given ones is considering ordinal sums of fuzzy implications (cf. e.g. [4], [5], [6]).

The contribution deals with such methods based on ordinal sums of fuzzy implications. It shows some aspects of development of this idea through recent years started with the article [7] and generalised later in [8], [9], [10]. In our paper two of the previously published constructions are improved and new generalisation of the existing construction are proposed. In the proposed constructions, the complement of the summands is not necessary Gödel or Rescher implication. Moreover, the problem of monotonicity of intervals in the ordinal sums of fuzzy implications are taken into account. Such methods allow us to better adapt the value of fuzzy implication for specific use. Additionally, preservation of the chosen basic properties by the ordinal sums are examined, in particular neutral, identity, and consequent boundary property.

First, in Section II, we put basic definitions and properties concerning fuzzy implications, and we recall some of the previously introduced methods of constructing ordinal sums of fuzzy implications. Next, in Section III, we present new constructions of ordinal sums of fuzzy implications and we analyse some of their properties.

This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge in Rzeszów, through Project Number RPPK.01.03.00-18-001/10.

II. Preliminaries

A. Fuzzy implications

Here, we recall the definition of a fuzzy implication and we list some basic properties connected with this notion.

Definition 1 (see [11], [12]). A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:
(I1) it is decreasing in its first variable,
(I2) it is increasing in its second variable,
(I3) $I(0,0) = 1$,
(I4) $I(1,1) = 1$,
(I5) $I(1,0) = 0$.

Definition 2 (cf. [12], [13], [14], [11], [15]). We say that a fuzzy implication I fulfils:

- the neutrality property (NP), if

$$I(1,y) = y, \quad y \in [0,1], \tag{NP}$$

- the identity principle (IP), if

$$I(x,x) = 1, \quad x \in [0,1], \tag{IP}$$

- the property (CB), if

$$I(x,y) \geq y, \quad x,y \in [0,1], \tag{CB}$$

- the strong boundary condition (SBC) if

$$I(x,0) = 0, \quad x \in (0,1], \tag{SBC}$$

- the strong corner condition for 0 (SCC0) if

$$I(x,y) = 0 \Rightarrow x = 1 \wedge y = 0, \quad x,y \in [0,1], \tag{SCC0}$$

- the strong corner condition for 1 (SCC1) if

$$I(x,y) = 1 \Rightarrow x = 0 \vee y = 1, \quad x,y \in [0,1]. \tag{SCC1}$$

Example 1 (cf. [12, pp. 4,5], [16]). Let us present the following family of fuzzy implications for $\alpha \in [0,1]$

$$I_\alpha(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0 \\ 1, & \text{if } x = 0 \text{ or } y = 1 \\ \alpha & \text{otherwise.} \end{cases}$$

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The operations I_0 and I_1 are the least and the greatest fuzzy implication, respectively, where

$$I_0(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ or } \ y = 1 \\ 0, & \text{otherwise,} \end{cases}$$

$$I_1(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0 \\ 1, & \text{otherwise.} \end{cases}$$

The following are other examples of fuzzy implications.

$$I_{\text{LK}}(x,y) = \min(1-x+y,1),$$

$$I_{\text{GD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ y, & \text{if } x > y, \end{cases}$$

$$I_{\text{RC}}(x,y) = 1-x+xy,$$

$$I_{\text{DN}}(x,y) = \max(1-x,y),$$

$$I_{\text{GG}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \frac{y}{x}, & \text{if } x > y, \end{cases}$$

$$I_{\text{RS}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ 0, & \text{if } x > y, \end{cases}$$

$$I_{\text{YG}}(x,y) = \begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0 \\ y^x, & \text{otherwise,} \end{cases}$$

$$I_{\text{FD}}(x,y) = \begin{cases} 1, & \text{if } x \le y \\ \max(1-x,y), & \text{if } x > y, \end{cases}$$

$$I_{\text{WB}}(x,y) = \begin{cases} 1, & \text{if } x \le 1 \\ y, & \text{if } x = 1, \end{cases}$$

$$I_{\text{DP}}(x,y) = \begin{cases} y, & \text{if } x = 1 \\ 1-x, & \text{if } y = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Except for I_α for $\alpha \in [0,1)$ and I_{RS}, the fuzzy implications from this example fulfil property (CB). Other properties of the above fuzzy implications are deeply analysed and can be found in the literature.

B. Ordinal sum of fuzzy implications

Let us start with the ordinal sum of fuzzy implications introduced by Su, Xie, and Liu in 2015 in [7]. In this construction linearly transformed values of summands (fuzzy implications I_k) are considered on closed squares $[a_k,b_k]^2$ and are complemented by Gödel implication.

Definition 3 ([7]). Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k,b_k]\}_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider the operation $I\colon [0,1]^2 \to [0,1]$ given by

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x,y \in [a_k,b_k] \\ I_{GD}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{1}$$

Theorem 1 ([7]). *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications. The operation I given by* (1) *is a fuzzy implication if and only if I_k satisfies* (CB)*, whenever $k \in A$ and $b_k < 1$.*

The general structure of an ordinal sum of fuzzy implications given by (1) are represented on Fig. 1.

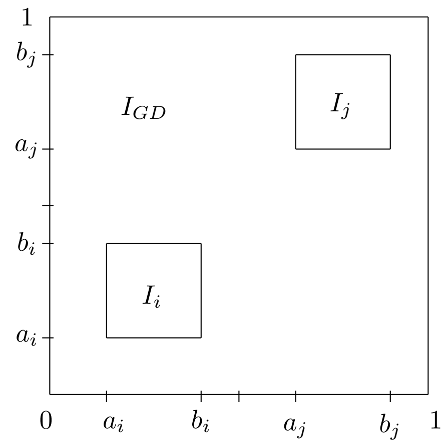

Fig. 1. The structure of an ordinal sum of fuzzy implications given by (1)

Next, let us recall other construction, where the Gödel implication is replaced by the Rescher implication. In this case the operation is always a fuzzy implication without any additional assumption on summands.

Definition 4 ([17]). Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k,b_k]\}_{k\in A}$ be a family of pairwise disjoint close subintervals of $[0,1]$ with $0 < a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider the operation $I\colon [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x,y \in [a_k,b_k] \\ I_{RS}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{2}$$

Theorem 2 ([17]). *The operation I given by* (2) *is a fuzzy implication.*

The general structure of an ordinal sum of fuzzy implications given by (2) are represented on Fig. 2.

Now, let us recall shortly two kinds of generalisations of these two constructions. One on the generalisations allow to consider instead of closed intervals $\{[a_k,b_k]\}$, intervals of different types (close, open or half-open) $|a_k,b_k|$ what enables to avoid the value of 1 on the interior of the main diagonal.

Definition 5 ([10]). Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k,b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k,b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider the operation $I\colon [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x,y \in |a_k,b_k| \\ I_{RS}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{3}$$

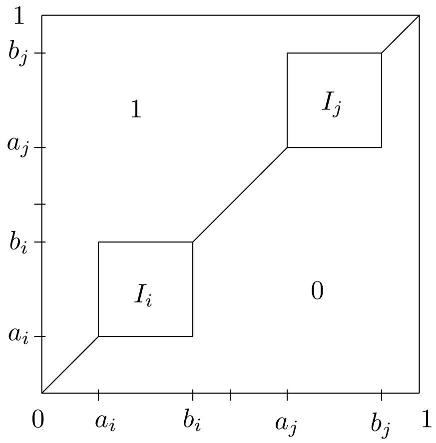

Fig. 2. The structure of an ordinal sum of fuzzy implications given by (2)

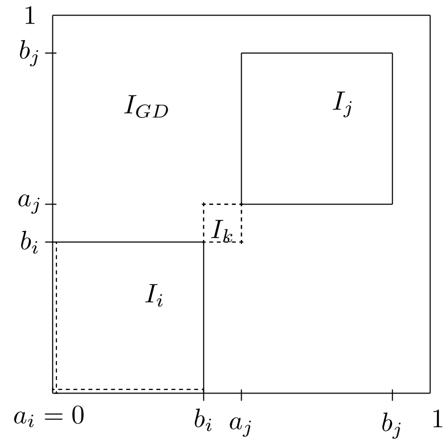

Fig. 4. The structure of an ordinal sum of fuzzy implications given by (4)

Theorem 3. *The operation I given by* (3) *is a fuzzy implication.*

The general structure of an ordinal sum of fuzzy implications given by (3) are represented on Fig. 3.

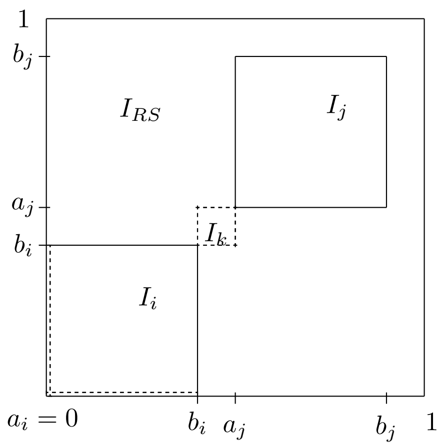

Fig. 3. The structure of an ordinal sum of fuzzy implications given by (3)

Definition 6 ([10])**.** Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Let us consider the operation $I\colon [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} a_k + (b_k - a_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x, y \in |a_k, b_k| \\ I_{GD}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{4}$$

Theorem 4. *The operation I given by* (4) *is a fuzzy implication if and only if I_k satisfies* (CB)*, whenever $k \in A$ and* $b_k < 1$.

The general structure of an ordinal sum of fuzzy implications given by (4) are represented on Fig. 4.

The other kind of generalisation of the operations (1) and (2) are the ones proposed in [8]. In these constructions it is not assumed that if arguments belong to an interval $[a_k, b_k]$, then values also need to belong to this interval. This means, that the values are not necessary increasing with respect to index set. However, the proposed construction is not always a fuzzy implication as it was stated in [8].

Indeed, let us consider the case when for some $k_0 \in A$ it is $b_{k_0} = 1$ and $d_{k_0} \neq 1$. Then

$$\begin{aligned} I(1,1) &= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{1-a_{k_0}}{1-a_{k_0}}, \frac{1-a_{k_0}}{1-a_{k_0}}\right) \\ &= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}(1,1) \\ &= c_{k_0} + (d_{k_0} - c_{k_0}) = d_{k_0} \neq 1. \end{aligned}$$

This is the reason for which we improved such definitions adding an assumption that if $1 \in [a_k, b_k]$, then $d_k = 1$. Below we present theses corrected versions.

Definition 7 ([8])**.** Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k, b_k]\}_{k\in A}$ be a family of pairwise disjoint subintervals of $(0,1)$, with $a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Moreover, let $\{[c_k, d_k]\}_{k\in A}$ be a family of subintervals of $[0,1]$, with $c_k \leq d_k$ for all $k \in A$ such that if $1 \in [a_k, b_k]$, then $d_k = 1$. Let us define an operation $I\colon [0,1]^2 \to [0,1]$ by the following formula:

$$I(x,y) = \begin{cases} c_k + (d_k - c_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x, y \in [a_k, b_k], \\ I_{RS}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{5}$$

The general structure of an ordinal sum of fuzzy implications given by (5) are represented on Fig. 5, wile 3d-visualisation is showed on Fig. 6.

Theorem 5 ([8])**.** *Let $\{I_k\}_{k\in A}$ be a family of implications. Then the operation I given by* (5) *is a fuzzy implication.*

Definition 8 ([8])**.** Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{[a_k, b_k]\}_{k\in A}$ be a family of pairwise disjoint subintervals of $(0,1)$, with $a_k < b_k$ for all $k \in A$, where A is a finite or countably infinite index set. Moreover, let $\{[c_k, d_k]\}_{k\in A}$ be a family of subintervals of $[0,1]$, with

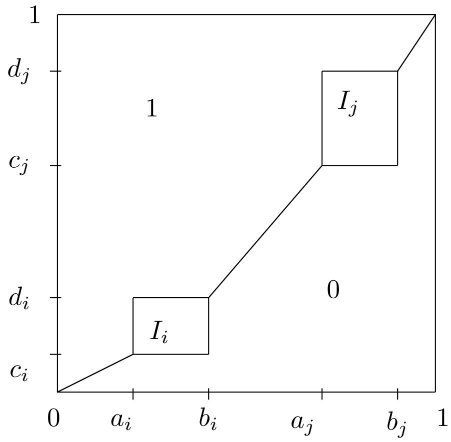

Fig. 5. The structure of an ordinal sum of fuzzy implications given by (5).

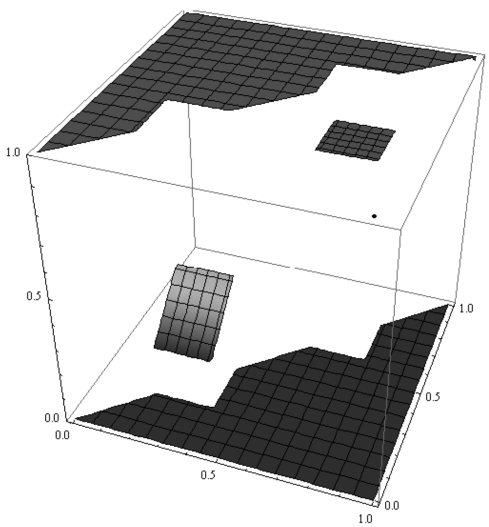

Fig. 6. 3D-visualization of the structure of a fuzzy implication given by (5).

$c_k \leq d_k$ for all $k \in A$ such that if $1 \in [a_k, b_k]$, then $d_k = 1$. Let us define an operation $I\colon [0,1]^2 \to [0,1]$ by the following formula:

$$I(x,y) = \begin{cases} c_k + (d_k - c_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right), \\ \qquad\qquad \text{if } x, y \in [a_k, b_k], \\ I_{GD}(x,y), \qquad\qquad \text{otherwise.} \end{cases} \tag{6}$$

The general structure of an ordinal sum of fuzzy implications given by (6) are represented on Fig. 7

Theorem 6 ([8]). *Let $\{I_k\}_{k\in A}$ be a family of implications. The operation I given by* (6) *fulfils (I3), (I4) and (I5).*

Theorem 7 ([8]). *Let $\{I_k\}_{k\in A}$ be a family of implications. If for all $k \in A$ we have $c_k \geq b_k$, then the operation I given by* (6) *is a fuzzy implication.*

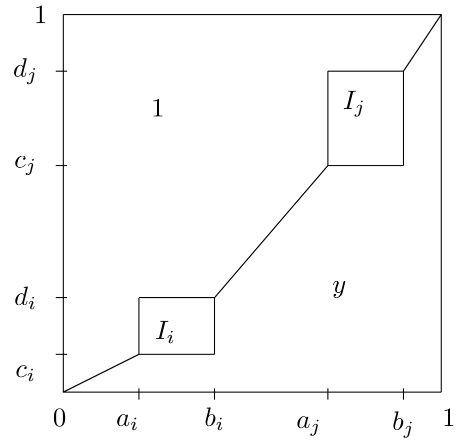

Fig. 7. The structure of an ordinal sum of fuzzy implications given by (6).

III. Main results

Here, we propose generalisations of all the constructions recalled in the previous section. On the one hand, the new constructions allow us to consider summands on intervals of different type (open, closed, or half-open), so they generalise ordinal sums proposed in [8]. On the other hand, they do not demand the values of the summands to be increasing with respect to index set, as for example in [17]. Additionally, in the both new constructions, the complement of the summands is not restricted to the one fuzzy implication, but the family of fuzzy implications described by the use of a unary function g.

A. Construction based on the Rescher implication

Let us start with a construction which is motivated by the ones with the use of the Rescher implication, however, in the definition the fuzzy implication I_{RS} is not involved.

Definition 9. Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ (open, closed, or half-open) with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Moreover, let $\{[c_k, d_k]\}_{k\in A}$ be a family of subintervals of $[0,1]$, with $c_k \leq d_k$ for all $k \in A$ such that if $(1,1) \in |a_k, b_k| \times |g(a_k), g(b_k)|$ then $d_k = 1$, and $g : [0,1] \to [0,1]$ such that $g(0) = 0$, be a strictly increasing, continuous function. Let us define an operation $I : [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} c_k + (d_k - c_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-g(a_k)}{g(b_k)-g(a_k)}\right), \\ \qquad\qquad \text{if } x \in |a_k, b_k|, y \in |g(a_k), g(b_k)| \\ 1, \qquad\qquad \text{if } y \geq g(x) \\ 0, \qquad\qquad \text{otherwise.} \end{cases} \tag{7}$$

The general structure of an ordinal sum of fuzzy implications given by (7) are represented on Fig. 8.

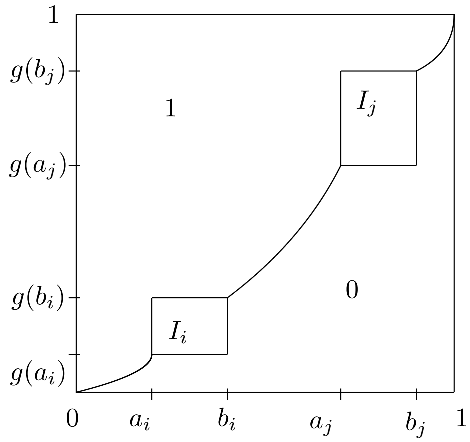

Fig. 8. The structure of an ordinal sum of fuzzy implications given by (7).

Theorem 8. *Let $\{I_k\}_{k\in A}$ be a family of implications. Then the operation I given by (7) is a fuzzy implication.*

Proof. Let $x_1 \le x_2$, $x_1, x_2 \in [0,1]$.
If $y \in |g(a_{k_0}), g(b_{k_0})|$ for some $k_0 \in A$, then we obtain the following three cases:
1. $b_{k_0} \le x_2$ and $x_2 \notin |a_{k_0}, b_{k_0}|$. Then $I(x_1, y) \ge 0 = I(x_2, y)$.
2. $x_1 \le a_{k_0}$ and $x_1 \notin |a_{k_0}, b_{k_0}|$. Then $I(x_1, y) = 1 \ge I(x_2, y)$.
3. $x_1, x_2 \in |a_{k_0}, b_{k_0}|$. Then using monotonicity of I_{k_0} we have

$$\begin{aligned} &I(x_1, y)\\ &\quad= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{x_1 - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{y - g(a_{k_0})}{g(b_{k_0}) - g(a_{k_0})}\right)\\ &\quad\ge c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{x_2 - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{y - g(a_{k_0})}{g(b_{k_0}) - g(a_{k_0})}\right)\\ &\quad= I(x_2, y). \end{aligned}$$

If $y \notin |g(a_k), g(b_k)|$ for all $k \in A$, then $g(x_1) \le g(x_2)$ and

$$\begin{aligned} I(x_1, y) &= \begin{cases} 0, & \text{if } y < g(x_1)\\ 1, & \text{otherwise} \end{cases}\\ &\le \begin{cases} 0, & \text{if } y < g(x_2)\\ 1, & \text{otherwise} \end{cases}\\ &= I(x_2, y). \end{aligned}$$

So, I satisfies (I1).

Next, let us consider the condition (I2). Let $x, y_1, y_2 \in [0,1]$, $y_1 \le y_2$. If $x \in |a_{k_0}, b_{k_0}|$ for some $k_0 \in A$, then we obtain the following cases.
1. $g(b_{k_0}) \le y_2$ and $y_2 \notin |g(a_{k_0}), g(b_{k_0})|$. Then $I(x, y_1) \le 1 = I(x, y_2)$.
2. $y_1 \le a_{k_0}$ and $y_1 \notin |g(a_{k_0}), g(b_{k_0})|$. Then $I(x, y_1) = 0 \le I(x, y_2)$.
3. $y_1, y_2 \in |g(a_{k_0}), g(b_{k_0})|$. Then using monotonicity of I_{k_0} we have

$$\begin{aligned} &I(x, y_1)\\ &\quad= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{x - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{y_1 - g(a_{k_0})}{g(b_{k_0}) - g(a_{k_0})}\right)\\ &\quad\le c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{x - a_{k_0}}{b_{k_0} - a_{k_0}}, \frac{y_2 - g(a_{k_0})}{g(b_{k_0}) - g(a_{k_0})}\right)\\ &\quad= I(x, y_2). \end{aligned}$$

If $x \notin |a_k, b_k|$ for all $k \in A$, then

$$\begin{aligned} I(x, y_1) &= \begin{cases} 0, & \text{if } y_1 < g(x)\\ 1, & \text{otherwise} \end{cases}\\ &\le \begin{cases} 0, & \text{if } y_2 < g(x)\\ 1, & \text{otherwise} \end{cases}\\ &= I(x, y_2). \end{aligned}$$

So, I satisfies (I2).

Now, let us notice that $I(0,0) = 1$, because of the assumption $0 \notin |a_k, b_k|$ and the simple fact that $0 \geqslant 0 = g(0)$, so I fulfils (I3).

Let us consider the value $I(1,1)$. If $1 \notin |a_k, b_k|$, then because of the fact that $1 \geqslant g(1)$ we have $I(1,1) = 1$. If for some $k_0 \in A$ $1 \in |a_{k_0}, b_{k_0}|$, then $b_{k_0} = 1$ and by assumption $d_{k_0} = 1$. So, in this case we have

$$\begin{aligned} I(1,1) &= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}\left(\frac{1 - a_{k_0}}{1 - a_{k_0}}, \frac{1 - a_{k_0}}{1 - a_{k_0}}\right)\\ &= c_{k_0} + (d_{k_0} - c_{k_0}) I_{k_0}(1,1)\\ &= c_{k_0} + (d_{k_0} - c_{k_0}) = d_{k_0} = 1. \end{aligned}$$

Thus, I fulfils (I4).

Finally $I(1,0) = 1$, because $0 \notin |a_k, b_k|$ and $0 < g(1) \neq 0$. So I fulfils (I5), which ends the proof. □

Example 2. Let

$$g(x) = \begin{cases} x^2, & \text{if } x \le 0.5\\ \sqrt{x}, & \text{otherwise,} \end{cases}$$

and let $\{[0.1, 0.3],\ (0.5, 0.64],\ (0.81, 0.9801)\}$, $\{[0.1, 0.7],\ [0.5, 1],\ [0.3, 0.4]\}$ be two families of intervals. Because we do not assume that the intervals $|a_i, b_i|$ and $|g(a_i), g(b_i)|$ are at the same time opened or closed, let us consider the family $\{[0.01, 0.09],\ [0.25, 0.8),\ [0.9, 0.99)\}$, and family of fuzzy implications $I_{k_1} = I_1$, $I_{k_2} = I_{LK}$, $I_{k_3} = I_0$. The implication obtained using formula (7) is of the form

$$I(x,y) = \begin{cases} 0.1, & \text{if } (x,y) = (0.3, 0.09),\\ 0.7, & \text{if } 0.1 \le x \le 0.3,\ 0.01 \le y \le 0.09\\ h(x,y), & \text{if } x \in (0.5, 0.64], y \in [0.25, 0.8)\\ 0.3, & \text{if } x \in (0.81, 0.9801), y \in [0.9, 0.99)\\ 0, & \text{if } y < g(x)\\ 1, & \text{otherwise,} \end{cases}$$

where $h(x,y) = \frac{1}{154}(197 - 275x + 70y)$.

Theorem 9. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and operation I be given by* (7)*.*

(i) *I satisfies* (SBC).
(ii) *I does not satisfy* (SCC0).
(iii) *I satisfies* (SCC1) *if and only if both $g(1) = 1$ and $cardA = 1$ with $|a_{k_1}, b_{k_1}| = (0,1|$ and I_{k_1} fulfilling* (SCC1).

Proof. (i) Let $x \in (0,1]$. Because $0 \notin |a_k, b_k|$ for all $k \in A$ and the function g is strictly increasing, so $g(x) > 0$ and $0 \notin |g(a_k), g(b_k)|$ for all $k \in A$. Consequently, by (7), $I(x,0) = 0$.

(ii) Directly by (i).

(iii) Obviously, for $x = 0$ or $y = 1$ we have $I(x,y) = 1$ as I is a fuzzy implication. Let us assume that $g(1) = 1$, and $cardA = 1$ with $|a_{k_1}, b_{k_1}| = (0,1|$ and I_{k_1} fulfilling (SCC1). Let $x, y \in [0,1]$ such that $x \neq 0$ and $y \neq 1$. If $(x,y) \in (0,1|^2$, then because I_{k_1} fulfils (SCC1), we have

$$\begin{aligned} I(x,y) &= c_{k_1} + (d_{k_1} - c_{k_1}) I_{k_1}(x,y) \\ &< c_{k_1} + (d_{k_1} - c_{k_1}) = d_{k_0} < 1. \end{aligned}$$

Conversely, let I satisfy (SCC1). First, let us suppose that $g(1) = v < 1$. In this case, by (7), we obtain $I(1,v) = 1$ which is a contradiction. So, we conclude that $g(1) = 1$. Now let us assume that there exists $k_0 \in A$ such that $|a_{k_0}, b_{k_0}| \neq (0,1|$. It means that there exist $x_0, y_0 \in (0,1)$ such that $y_0 \geqslant g(x_0)$. By (7) we have $I(x_0, y_0) = 1$, which is a contradiction. □

B. Construction based on the Gödel implication

Now, let us present a generalisation of the constructions of ordinal sum of fuzzy implications motivated by the ones with I_{GD} as a complement.

Definition 10. Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ (open, closed, or half-open) with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a finite or countably infinite index set. Moreover, let $\{[c_k, d_k]\}_{k\in A}$ be a family of subintervals of $[0,1]$, with $c_k \leq d_k$ for all $k \in A$ such that if $(1,1) \in |a_k, b_k| \times |g(a_k), g(b_k)|$, then $d_k = 1$, and $g : [0,1] \to [0,1]$ such that $g(0) = 0$, be a strictly increasing, continuous function. Let us define an operation $I : [0,1]^2 \to [0,1]$ given by the following formula

$$I(x,y) = \begin{cases} c_k + (d_k - c_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-g(a_k)}{g(b_k)-g(a_k)}\right), \\ \qquad\qquad \text{if } x \in |a_k, b_k|, y \in |g(a_k), g(b_k)| \\ 1, \qquad \text{if } y \geq g(x) \\ y, \qquad \text{otherwise.} \end{cases} \tag{8}$$

The general structure of an ordinal sum of fuzzy implications given by (8) are represented on Fig. 9.

Theorem 10. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications. The operation I given by* (8) *fulfils (I3), (I4) and (I5).*

Theorem 11. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications. If for all $k \in \mathcal{A}$ we have $c_k \geq g(b_k)$, then the operation I given by* (8) *is a fuzzy implication.*

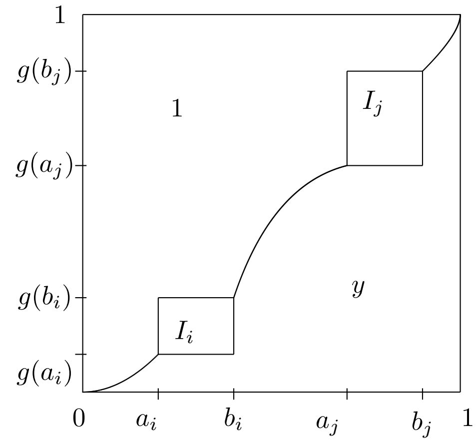

Fig. 9. The structure of an ordinal sum of fuzzy implications given by (8).

Theorem 12. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and the operation I given by* (8) *be a fuzzy implication.*

(i) *I satisfies* (SBC).
(ii) *I does not satisfy* (SCC0).
(iii) *I satisfies* (SCC1) *if and only if both $g(1) = 1$ and $cardA = 1$ with $|a_{k_1}, b_{k_1}| = (0,1|$ and I_{k_1} fulfilling* (SCC1).

Theorem 13. *Let $\{I_k\}_{k\in A}$ be a family of fuzzy implications and $\{|a_k, b_k|\}_{k\in A}$ be a family of pairwise disjoint subintervals of $[0,1]$ with $a_k < b_k$ and $0 \notin |a_k, b_k|$ for all $k \in A$, where A is a non-empty finite or countably infinite index set. Moreover, let $\{[c_k, d_k]\}_{k\in A}$ be a family of subintervals of $[0,1]$, with $c_k \leq d_k$ for all $k \in A$ such that if $(1,1) \in |a_k, b_k| \times |g(a_k), g(b_k)|$ then $d_k = 1$, and $g : [0,1] \to [0,1]$ such that $g(0) = 0$, be a strictly increasing, continuous function. Let us define an operation $I\colon [0,1]^2 \to [0,1]$ given by the following formula*

$$I(x,y) = \begin{cases} c_k + (d_k - c_k) I_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-g(a_k)}{g(b_k)-g(a_k)}\right), \\ \qquad\qquad \text{if } x \in |a_k, b_k|,\ y \in |g(a_k), g(b_k)| \\ I^*(x,y), \qquad \text{otherwise} \end{cases}$$

I is a fuzzy implication for arbitrary family of implications $\{I_k\}_{k\in A}$ and arbitrary family $\{|a_k, b_k|\}_{k\in A}$ if and only if

$$I^*(x,y) = \begin{cases} 1, & \text{if } g(x) < y \text{ or } (x,y) \in \{(0,0),(1,1)\} \\ 0, & \text{if } g(x) > y \\ \in [0,1], & \text{otherwise.} \end{cases}$$

IV. Conclusions

In this paper several methods of constructing ordinal some of fuzzy implications were presented. Two new methods of constructing ordinal sums of fuzzy implications were proposed. Sufficient properties of summands for obtaining a fuzzy implication as a result were examined. Some basic properties of such implications have been obtained.

Next step is to examine other properties of the component of introduced ordinal sums which can be preserved by the ordinal sums. It seems also interesting which of the fuzzy implication can we put as a complement instead of Rescher or Gödel implication in order to obtain a fuzzy implication with appropriate properties. It seems also desirable to study a newly published article on this topic by Zhou [18].

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