First-Order Typed Fuzzy Logics and their Categorical Semantics: Linear Completeness and Baaz Translation via Lawvere Hyperdoctrine Theory

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***Abstract*—It is known that some fuzzy predicate logics, such as Łukasiewicz predicate logic, are not complete with respect to the standard real-valued semantics. In the present paper we focus upon a typed version of first-order MTL (Monoidal T-norm Logic), which gives a unified framework for different fuzzy logics including, inter alia, Hajek's basic logic, Łukasiewicz logic, and Gödel logic. And we show that any extension of first-order typed MTL, including Łukasiewicz predicate logic, is sound and complete with respect to the corresponding categorical semantics in the style of Lawvere's hyperdoctrine, and that the so-called Baaz delta translation can be given in the first-order setting in terms of Lawvere's hyperdoctrine. A hyperdoctrine may be seen as a fibred algebra, and the first-order completeness, then, is a fibred extension of the algebraic completeness of propositional logic. While the standard real-valued semantics for Łukasiewicz predicate logic is not complete, the hyperdoctrine, or fibred algebraic, semantics is complete because it encompasses a broader class of models that is sufficient to prove completeness; in this context, incompleteness may be understood as telling that completeness does not hold when the class of models is restricted to the standard class of real-valued hyperdoctrine models. We expect that this finally leads to a unified categorical understanding of Takeuti-Titani's fuzzy models of set theory.**

***Index Terms*—first-order typed fuzzy logic, categorical semantics, completeness, Baaz translation, Lawvere hyperdoctrine**

I. Introduction

Monoidal T-norm Logic, denoted MTL, gives a unified framework for different fuzzy logics including, inter alia, Hajek's basic logic, Łukasiewicz logic, and Gödel logic (see, e.g., Esteva-Godo [4] and Hájek-Cintula [7]). MTL is sound and complete with respect to its algebraic semantics in terms of MTL algebras (see, e.g., Hájek [6] and Jenei-Montagna [10]). In this paper we give a first-order typed extension of MTL, and prove its (linear) completeness with respect to categorical semantics in terms of Lawvere hyperdoctrines (to be precise, its MTL extensions; see, e.g., Lawvere [13] and Pitts [20]); the (linear) completeness can be applied to any axiomatic extension of MTL. We also show that an analogue of the so-called Baaz delta translation can be given in the first-order setting in terms of Lawvere's hyperdoctrine. The methodology of the present paper builds upon our previous work [15], [16]; although the technical machinery we employ in this paper is basically the same as the one in there, we nonetheless emphasize fuzzy logical aspects here, such as linear completeness, which is a striking characteristic of fuzzy logic in general, and as the Baaz delta translation, which allows us to bridge between the fuzzy and the classical worlds. The general framework as developed in [**?**], [16], presumably, works for almost all kinds of logical systems; the present paper demonstrates this for fuzzy logical systems in particular, allowing for nuanced treatment of special characteristics of fuzzy systems as well.

Hyperdoctrines are category-theoretical concepts, and yet their essence is closer to the idea of algebraic logic. They basically give algebraic semantics for predicate logic, both first-order and higher-order. We call it fibred algebraic logic or fibred algebraic semantics; let us elaborate more on this idea. In general, hyperdoctrines are fibred algebras

$$(A_C)_{C\in\mathbf{C}}$$

where $\mathbf{C}$ is a category for the underlying type theory (or many-sorted structure); MTL hyperdoctrines are $(A_C)_{C\in\mathbf{C}}$ such that every fibre A_C is an MTL algebra. Hyperdoctrine semantics extends not only algebraic semantics, but also set-theoretical semantics, which corresponds to interpreting logic within a special class of set-theoretical hyperdoctrines; we shall explain more about this in subsequent sections.

It is known that some fuzzy predicate logics, such as Łukasiewicz predicate logic, are not complete with respect to the standard real-valued semantics. While the standard real-valued semantics for Łukasiewicz predicate logic is not complete, the hyperdoctrine (or fibred algebraic) semantics is complete because it encompasses a broader class of models that is sufficient to prove completeness; in this context, incompleteness may be understood as telling that completeness does not hold when the class of models is restricted to the standard class of real-valued hyperdoctrine models. The algebraic completeness of first-order fuzzy logics is usually shown via so-called safe valuations on algebras involved (see, e.g., Ono [18] and Ono [19]); yet safety is an ad hoc condition to guarantee the interpetability of quantifiers. In hyperdoctrinal or fibred algebraic semantics, there is no such ad hoc condition to ensure the interpretability of quantifiers. This gives us a particular rationale for the categorical approach to fuzzy predicate logic.

The rest of the present paper is organized as follows. In Section 2, we present the syntax of first-order typed MTL.

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In Section 3, we give categorical semantics for first-order typed MTL, which is shown to be sound and complete as desired. Note that linear completeness obtains as well as ordinary completeness. In Section 4, we prove a hyperdoctrinal analogue of the Baaz translation theorem in the first-order setting.

II. Typed Monoidal T-Norm Logic

Let us introduce a typed, or many-sorted, version $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ of quantified MTL on the basis of the unifying framework of substructural logics over full Lambek calculus as developed in Galatos et al. [5] and Ono [19]. The type theory of $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ follows the type theory of Pitts [20]. In categorical logic, typed or many-sorted logic is more popular than single-sorted one (Pitts [20], Lambek-Scott [12], Jacobs [9], and Johnstone [11]); this is because category theory itself is some sort of algebraic type theory. Yet at the same time, it is possible to reduce typed logic into single-sorted logic by considering one type or one sort only. From a different angle, typed logic is the integration of logic and type theory; a logic structure is coexistent with a type structure, both being able to be equipped with different additional structures such as different type constructors. The integrated nature of type logic is crucial in our construction of syntactic hyperdoctrines below; they are integrations of syntactic categories on the type theory side and Lindenbaum-Tarski algebras on the logic side. To put it succinctly, they are fibred Lindenbaum-Tarski algebras

$$(A_C)_{C\in\mathbf{C}},$$

each fibre A_C being a single algebra of propositions on a given type. Typed logic has different merits such as the resolution of the empty domain issue addressed below.

$\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ has the following logical connectives:

$$\otimes, \wedge, \vee, \rightarrow, 1, 0, \forall, \exists.$$

In $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$, any variable x has its type σ. Note that basic types are denoted by letters like σ and τ.

$$x : \sigma$$

is the formal statement that a variable x is of type σ. A type context is defined as a finite list of type declarations:

$$x_1 : \sigma_1, ..., x_n : \sigma_n.$$

We usually denote a context by Γ. In $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ we have typed predicate symbols (or predicates in context) and typed function symbols (or function symbols in context) as follows:

$$R(x_1, ..., x_n)\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$$

is the formal statement that R is a predicate with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ respectively;

$$f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$$

is the formal statement that f is a function symbol with n variables $x_1, ..., x_n$ of types $\sigma_1, ..., \sigma_n$ and with its values in τ. Formulae (or formulae-in-context) $\varphi\ [\Gamma]$ and terms (or terms-in-context) $t : \tau\ [\Gamma]$ are inductively defined in the usual manner. Sequents (or sequents-in-contexts) are defined as:

$$\Phi \vdash \varphi\ [\Gamma]$$

where Γ is a context, and Φ is a finitary list of formulae $\varphi_1, ..., \varphi_n$. Our notation and terminology basically follows those of Pitts [20], our system $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ being an adaptation of Pitts' typed intuitionistic logic to monoidal t-norm logic. There are several syntactic rules on the type theory of $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$, but they are the same as those of Pitts [20], and so we do not repeat them. Note that it is permitted to add a new variable $x : \sigma$ to a given context Γ: for instance, we can derive the following from $\Phi \vdash \varphi\ [\Gamma]$:

$$\Phi \vdash \varphi\ [\Gamma, x : \sigma].$$

Nevertheless, it is not allowed to delete existing variables, and we shall use this property below. Note also that we can change the ordering of contexts; for instance, it does not matter whether we write $[\Gamma, \Gamma']$ or $[\Gamma', \Gamma]$. Let us present the logical inference rules of $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ in the following. $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ has several structural rules. The identity and cut rules are as follows:

$$\frac{}{\varphi \vdash \varphi\ [\Gamma]}\ (id.) \qquad \frac{\Phi_1 \vdash \varphi\ [\Gamma] \quad \Phi_2, \varphi, \Phi_3 \vdash \psi\ [\Gamma]}{\Phi_2, \Phi_1, \Phi_3 \vdash \psi\ [\Gamma]}\ (cut)$$

where ψ may be empty; this applies to the following L (left) rules as well. $\mathrm{MTL}_{\mathrm{t}}^{\mathrm{q}}$ has the exchange and weakening rules:

$$\frac{\Phi_1, \psi, \varphi, \Phi_2 \vdash \psi\ [\Gamma]}{\Phi_1, \varphi, \psi, \Phi_2 \vdash \psi\ [\Gamma]}\ (ex.) \qquad \frac{\Phi_1, \varphi, \varphi, \Phi_2 \vdash \psi\ [\Gamma]}{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [\Gamma]}\ (weak.)$$

In the following, we list the rules of inference for the logical connectives. There are two kinds of conjunction:

$$\frac{\Phi, \varphi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \otimes \psi, \Psi \vdash \chi\ [\Gamma]}\ (\otimes L) \qquad \frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi \vdash \psi\ [\Gamma]}{\Phi, \Psi \vdash \varphi \otimes \psi\ [\Gamma]}\ (\otimes R)$$

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \wedge \psi, \Psi \vdash \chi\ [\Gamma]}\ (\wedge L) \qquad \frac{\Phi \vdash \varphi\ [\Gamma] \quad \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \wedge \psi\ [\Gamma]}\ (\wedge R)$$

There is only one disjunction with the following rules:

$$\frac{\Phi, \varphi, \Psi \vdash \chi\ [\Gamma] \quad \Phi, \psi, \Psi \vdash \chi\ [\Gamma]}{\Phi, \varphi \vee \psi, \Psi \vdash \chi\ [\Gamma]}\ (\vee L)$$

$$\frac{\Phi \vdash \varphi\ [\Gamma]}{\Phi \vdash \varphi \vee \psi\ [\Gamma]}\ (\vee R)$$

The rules for implication are as follows:

$$\frac{\Phi \vdash \varphi\ [\Gamma] \quad \Psi_1, \psi, \Psi_2 \vdash \chi\ [\Gamma]}{\Psi_1, \Phi, \varphi \rightarrow \psi, \Psi_2 \vdash \chi\ [\Gamma]}\ (\rightarrow L)$$

$$\frac{\varphi, \Phi \vdash \psi\ [\Gamma]}{\Phi \vdash \varphi \rightarrow \psi\ [\Gamma]}\ (\rightarrow R)$$

The rules for truth and falsity constants are as follows:

$$\frac{}{\Phi \vdash 1\ [\Gamma]}\ (1R) \qquad \frac{}{\Phi_1, 0, \Phi_2 \vdash \varphi\ [\Gamma]}\ (0L)$$

Finally, we have the following rules for quantifiers $\forall$ and $\exists$, in which type contexts explicitly change; notice that type contexts do not change in the rest of the rules presented above.

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}{\Phi_1, \forall x\varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}\ (\forall L) \qquad \frac{\Phi \vdash \varphi\ [x : \sigma, \Gamma]}{\Phi \vdash \forall x\varphi\ [\Gamma]}\ (\forall R)$$

$$\frac{\Phi_1, \varphi, \Phi_2 \vdash \psi\ [x : \sigma, \Gamma]}{\Phi_1, \exists x\varphi, \Phi_2 \vdash \psi\ [\Gamma]}\ (\exists L) \qquad \frac{\Phi \vdash \varphi\ [x : \sigma, \Gamma]}{\Phi \vdash \exists x\varphi\ [x : \sigma, \Gamma]}\ (\exists R)$$

Note that there are eigenvariable conditions on the rules above: x does not appear as a free variable in the bottom sequents of Rule $\forall R$ and of Rule $\exists L$. The deducibility of sequents in MTL_t^q is defined in the usual manner. As is well known, the following logics can be represented as axiomatic extensions of MTL_t^q: Hajek's basic logic, Łukasiewicz logic, and Gödel logic (see, e.g., Hájek [6]). Given a set of axioms (to be precise, axiom schemata), say X, we denote by $X\mathrm{MTL}_t^q$ the corresponding extension of MTL_t^q via X.

It is immediate to see the following:

Lemma 1. *The following are deducible in* MTL_t^q*:*

(i) $\varphi \otimes (\exists x\psi) \vdash \exists x(\varphi \otimes \psi)\ [\Gamma]$ *and its converse* $\exists x(\varphi \otimes \psi) \vdash \varphi \otimes (\exists x\psi)\ [\Gamma]$*;*

(ii) $(\exists x\psi) \otimes \varphi \vdash \exists x(\psi \otimes \varphi)\ [\Gamma]$ *and its converse* $\exists x(\psi \otimes \varphi) \vdash (\exists x\psi) \otimes \varphi\ [\Gamma]$,

where it is assumed that φ *does not contain* x *as a free variable, and that* Γ *contains type declarations on those free variables that appear in* φ *and* $\exists x\psi$*.*

In typed predicate logic, domains of discourse can be empty. Notice that they must be be non-empty in the standard Tarskian semantics. This is an interesting feature of typed logic, allowing us to remove the ad hoc, non-emptiness condition on domains of discourse (see, e.g., Marquis-Reyes [14]). This matters proof-theoretically as well as semantically. On one hand, the following sequent is not necessarily deducible in MTL_t^q:

$$\forall x\varphi \vdash \exists x\varphi\ [\ ].$$

On the other, the following sequent is deducible in MTL_t^q:

$$\forall x\varphi \vdash \exists x\varphi\ [x : \sigma, \Gamma].$$

The sequent above is thus provable if a type σ is inhabited. Note that deleting free variables is not allowed even if they do not appear in formulae concerned.

III. Hyperdoctrine: Fibred Algebraic Semantics

MTL algebras give complete semantics for propositional MTL (see, e.g., Hájek [6]). What we show here is the first-order extension of this: i.e., fibred MTL algebras, or MTL hyperdoctrines, give complete semantics for first-order typed MTL_t^q. Extending the logic to the first-order type setting amounts to extending a single algebra A to a fibred algebra $(A_C)_{C\in\mathbf{C}}$ indexed by a category $\mathbf{C}$. In the following we first define MTL algebras and then fibred MTL algebras or MTL hyperdoctrines.

Definition 2. $(A, \otimes, \wedge, \vee, \rightarrow, 1, 0, \top, \bot)$ *is called an* MTL *algebra iff*

- $(A, \otimes, 1)$ *is a monoid;* $(A, \wedge, \vee, 1, 0)$ *is a bounded lattice, which induces a partial order* $\leq$ *on* A*;*
- *for any* $a \in A$, $a \rightarrow (\text{-}) : A \rightarrow A$ *is a right adjoint of* $a \otimes (\text{-}) : A \rightarrow A$*: i.e.,* $a \otimes b \leq c$ *iff* $b \leq a \rightarrow c$ *for any* $a, b, c \in A$*; For any* $a, b \in A$, $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

A homomorphism of MTL *algebras is a map preserving the* MTL *operations* $(\otimes, \wedge, \vee, \rightarrow, 1, 0, \top, \bot)$. *Let* **MTL** *denote the category of* MTL *algebras and their homomorphisms.*

MTL is an algebraic category (or variety in universal algebra); an axiomatic extension XMTL of MTL corresponds to an algebraic full subcategory (or subvariety) of **MTL**, denoted X**MTL** (algebraicity follows from definability by axioms).

Definition 3 ([15], [16])**.** *An* MTL *hyperdoctrine is defined as an* **MTL***-valued functor (or equivalently, presheaf)*

$$P : \mathbf{C}^{\mathrm{op}} \rightarrow \mathbf{MTL}$$

such that $\mathbf{C}$ *is a category with finite products, and the following conditions hold (which come from Lawvere's idea of quantifiers as adjoints):*

- *For any projection* $\pi : X \times Y \rightarrow Y$ *in* $\mathbf{C}$, $P(\pi) : P(Y) \rightarrow P(X \times Y)$ *has a right adjoint*
$$\forall_\pi : P(X \times Y) \rightarrow P(Y).$$
Moreover, the Beck-Chevalley condition for $\forall$ *holds, i.e., the diagram below commutes for any arrow* $f : Z \rightarrow Y$ *in* $\mathbf{C}$ *(in the following* $\pi' : X \times Z \rightarrow Z$ *denotes a projection as usual):*
$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{\forall_\pi} & P(Y) \\ {\scriptstyle P(X\times f)}\downarrow & & \downarrow{\scriptstyle P(f)} \\ P(X \times Z) & \xrightarrow[\forall_{\pi'}]{} & P(Z) \end{array}$$
- *For any projection* $\pi : X \times Y \rightarrow Y$ *in* $\mathbf{C}$, $P(\pi) : P(Y) \rightarrow P(X \times Y)$ *has a left adjoint*
$$\exists_\pi : P(X \times Y) \rightarrow P(Y).$$
The Beck-Chevalley condition for $\exists$ *holds:*
$$\begin{array}{ccc} P(X \times Y) & \xrightarrow{\exists_\pi} & P(Y) \\ {\scriptstyle P(X\times f)}\downarrow & & \downarrow{\scriptstyle P(f)} \\ P(X \times Z) & \xrightarrow[\exists_{\pi'}]{} & P(Z) \end{array}$$
In addition, the following Frobenius Reciprocity conditions hold: for any projection $\pi : X \times Y \rightarrow Y$ *in* $\mathbf{C}$*, any* $a \in P(Y)$*, and any* $b \in P(X \times Y)$*,*
$$a \otimes (\exists_\pi b) = \exists_\pi(P(\pi)(a) \otimes b).$$

Given an axiomatic extension XMTL *of* MTL*, we define an* XMTL *hyperdoctrine to be such that the value category*

$\mathbf{MTL}$ *is replaced by* $X\mathbf{MTL}$*. We also call an* MTL *(resp.* XMTL*) hyperdoctrine a fibred* MTL *(resp.* XMTL*) algebra.*

We call the underlying category $\mathbf{C}$ the base category or type category, and P the predicate functor, since $P(C)$ is regarded as the algebra of predicates on a given type (or domain of discourse) C. We also call $P(C)$ a fibre of P (technically, an MTL hyperdoctrine is an indexed category, and the Grothendieck construction yields the corresponding fibration). We may intuitively regard an arrow f in $\mathbf{C}$ as a term, and $P(f)$ as a substitution operation. In this understanding, the Beck-Chevalley conditions and functoriality of P state that substitution commutes with the logical operations. In particular, the Beck-Chevalley conditions state that substitution after quantification is equivalent to quantification after substitution. All this is literally true in syntactic hyperdoctrines as we shall discuss below. Now, let us introduce the hyperdoctrine semantics for MTL_t^q.

Definition 4 ([15], [17])**.** *Given an* MTL *hyperdoctrine* $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MTL}$*, an interpretation* $[\![\text{-}]\!]$ *of* MTL_t^q *in* P *consists of the following data:*

- *assignment of an object* $[\![\sigma]\!]$ *in* $\mathbf{C}$ *to a basic type* σ *in* MTL_t^q*;*
- *assignment of an arrow* $[\![f : \tau\ [\Gamma]]\!] : [\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!] \to [\![\sigma]\!]$ *in* $\mathbf{C}$ *to a typed function symbol* $f : \tau\ [\Gamma]$ *in* MTL_t^q *where* Γ *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*;*
- *assignment of an element* $[\![R\ [\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*to each typed predicate symbol* $R\ [\Gamma]$ *in* MTL_t^q*; if the context* Γ *is* $x_1 : \sigma_1, ..., x_n : \sigma_n$*,* $[\![\Gamma]\!]$ *denotes* $[\![\sigma_1]\!] \times ... \times [\![\sigma_n]\!]$*.*

Terms are inductively interpreted as follows:

- $[\![x : \sigma\ [\Gamma_1, x : \sigma, \Gamma_2]]\!]$ *is defined as:*
$$\pi : [\![\Gamma_1]\!] \times [\![\sigma]\!] \times [\![\Gamma_2]\!] \to [\![\sigma]\!].$$
- $[\![f(t_1, ..., t_n) : \tau\ [\Gamma]]\!]$ *is defined as:*
$$[\![f]\!] \circ \langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$$
where $f : \tau\ [x_1 : \sigma_1, ..., x_n : \sigma_n]$*, and* $t_1 : \sigma_1\ [\Gamma], ..., t_n : \sigma_n\ [\Gamma]$*. In the above,* $\langle [\![t_1 : \sigma_1\ [\Gamma]]\!], ..., [\![t_n : \sigma_n\ [\Gamma]]\!] \rangle$ *is the product of arrows in* $\mathbf{C}$ *(which has finite products).*

Formulae are inductively interpreted as follows:

- $[\![R(t_1, ..., t_n)\ [\Gamma]]\!]$ *is defined as:*
$$P(\langle [\![t_1 : \sigma_1[\Gamma]]\!], ..., [\![t_n : \sigma_n[\Gamma]]\!] \rangle)([\![R\ [x : \sigma_1, ..., x_n : \sigma_n]]\!])$$
for a predicate symbol R *in context* $x_1 : \sigma_1, ..., x_n : \sigma_n$*.*
- $[\![\varphi \otimes \psi\ [\Gamma]]\!]$ *is defined as* $[\![\varphi\ [\Gamma]]\!] \otimes [\![\psi\ [\Gamma]]\!]$*. The rest of the binary connectives* $\wedge, \vee, \to$ *are interpreted in the same manner.* $[\![1\ [\Gamma]]\!]$ *is defined as the monoidal unit element of* $P([\![\Gamma]\!])$*. The rest of the constants* $0, \top, \bot$ *are interpreted in the same manner.*
- $[\![\forall x \varphi\ [\Gamma]]\!]$ *is defined as*
$$\forall_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!])$$
where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ *is a projection, and* φ *is a formula in context* $[x : \sigma, \Gamma]$*. Likewise,* $[\![\exists x \varphi\ [\Gamma]]\!]$ *is defined as:*
$$\exists_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!]).$$

Satisfaction of sequents is defined as follows:

- $\varphi_1, ..., \varphi_n \vdash \psi\ [\Gamma]$ *is satisfied in an interpretation* $[\![\text{-}]\!]$ *in* P *iff the following holds in* $P([\![\Gamma]\!])$*:*
$$[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq [\![\psi\ [\Gamma]]\!].$$
If the right-hand side of a sequent is empty, $\varphi_1, ..., \varphi_n \vdash [\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!] \leq 0$ *in* $P([\![\Gamma]\!])$*. If the left-hand side of a sequent is empty,* $\vdash \varphi\ [\Gamma]$ *is satisfied in* $[\![\text{-}]\!]$ *iff* $1 \leq [\![\varphi[\Gamma]]\!]$ *in* $P([\![\Gamma]\!])$*.*

An interpretation of $X\mathrm{MTL}_t^q$ *in an* XMTL *hyperdoctrine is defined by replacing* $\mathbf{MTL}$ *with* $X\mathbf{MTL}$ *and* MTL_t^q *with* $X\mathrm{MTL}_t^q$*.*

In the following we prove the soundness and completeness of this hyperdoctrine semantics. We prepare some notation: when Φ is $\varphi_1, ..., \varphi_n$, $[\![\Phi\ [\Gamma]]\!]$ denotes $[\![\varphi_1\ [\Gamma]]\!] \otimes ... \otimes [\![\varphi_n\ [\Gamma]]\!]$.

Proposition 5. *Assume that* $\Phi \vdash \psi\ [\Gamma]$ *is deducible in* MTL_t^q *(resp.* $X\mathrm{MTL}_t^q$*). Then,* $\Phi \vdash \psi\ [\Gamma]$ *is satisfied in any interpretation in any* MTL *(resp.* XMTL*) hyperdoctrine.*

Proof. Let us consider an MTL or XMTL hyperdoctrine P and an interpretation $[\![\text{-}]\!]$ in P. Note that initial sequents are necessarily satisfied, since $a \leq a$ in any fibre $P(C)$. Since $\otimes$ preserves $\leq$ and $\leq$ is transitive, the cut rule is valid (i.e., preserve satisfcation). It can be easily verified that all of the rules for the logical connectives are valid. So let us discuss the case of universal quantifier. We fist consider Rule $\forall R$. Let us assume that
$$[\![\Phi\ [x : \sigma, \Gamma]]\!] \leq [\![\varphi\ [x : \sigma, \Gamma]]\!]$$
in $P([\![\sigma]\!] \times [\![\Gamma]\!])$. We then have the following:
$$[\![\Phi\ [x : \sigma, \Gamma]]\!] = P(\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!])([\![\Phi\ [\Gamma]]\!])$$
where π is a projection. It then holds that:
$$P(\pi)([\![\Phi\ [\Gamma]]\!]) \leq [\![\varphi\ [x : \sigma, \Gamma]]\!].$$
By definition,
$$\forall_\pi : P([\![\sigma]\!] \times [\![\Gamma]\!]) \to P([\![\Gamma]\!])$$
is a right adjoint of $P(\pi)$, and so it holds that
$$[\![\Phi\ [\Gamma]]\!] \leq \forall_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!]) = [\![\forall x \varphi\ [\Gamma]]\!].$$
We then consider Rule $\forall L$. So let us assume that
$$[\![\Phi_1\ [x : \sigma, \Gamma]]\!] \otimes [\![\varphi\ [x : \sigma, \Gamma]]\!] \otimes [\![\Phi_2\ [x : \sigma, \Gamma]]\!] \leq [\![\psi\ [x : \sigma, \Gamma]]\!].$$
Since universal quantifier is an adjoint functor, it holds that
$$P(\pi)(\forall_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!])) \leq [\![\varphi\ [x : \sigma, \Gamma]]\!]$$
where $\pi : [\![\sigma]\!] \times [\![\Gamma]\!] \to [\![\Gamma]\!]$ is a projection. At the same time we have the following:
$$\begin{aligned} P(\pi)(\forall_\pi([\![\varphi\ [x : \sigma, \Gamma]]\!])) &= P(\pi)([\![\forall x \varphi\ [\Gamma]]\!]) \\ &= [\![\forall x \varphi\ [x : \sigma, \Gamma]]\!]. \end{aligned}$$
Note that $\otimes$ respects $\leq$. We have thus shown that
$$[\![\Phi_1\ [x : \sigma, \Gamma]]\!] \otimes [\![\forall x \varphi\ [x : \sigma, \Gamma]]\!] \otimes [\![\Phi_2\ [x : \sigma, \Gamma]]\!]$$

is less than or equal to

$$[\![\psi\ [x:\sigma,\Gamma]]\!].$$

Much the same argument allows us to prove the required properties for the case of existential quantifier $\exists$. $\square$

We define syntactic hyperdoctrines in the following. Note that they are type-fibred Lindenbaum-Tarski algebras.

Definition 6 ([15], [17])**.** *The syntactic hyperdoctrine of* MTL_t^q *is defined as follows. Note that the syntactic hyperdoctrine of* $X\mathrm{MTL}_t^q$ *is defined by replacing* $\mathbf{MTL}$ *with* $X\mathbf{MTL}$ *and* MTL_t^q *with* $X\mathrm{MTL}_t^q$*.*

The base category $\mathbf{C}$ *is defined as follows; it is analogous to syntactic categories as in categorical semantics of type theory. An object in* $\mathbf{C}$ *is a context* Γ *up to* α*-equivalence (i.e., modulo the renaming of variables). An arrow in* $\mathbf{C}$ *from an object* Γ *to another* Γ' *is a list of terms* $[t_1,...,t_n]$ *(up to equivalence) such that* $t_1:\sigma_1\ [\Gamma],...,t_n:\sigma_n\ [\Gamma]$ *where* Γ' *is* $x_1:\sigma_1,...,x_n:\sigma_n$*.*

We now define the syntactic hyperdoctrine

$$P:\mathbf{C}^{\mathrm{op}}\to\mathbf{MTL}$$

as follows. Given an object Γ *in* $\mathbf{C}$*, let*

$$\mathrm{Form}_\Gamma=\{\varphi\mid\varphi\textit{ is a formula in context }\Gamma\}.$$

And we define an equivalence relation $\sim$ *on* Form_Γ*: given* $\varphi,\psi\in\mathrm{Form}_\Gamma$*,* $\varphi\sim\psi$ *iff both* $\varphi\vdash\psi\ [\Gamma]$ *and* $\psi\vdash\varphi\ [\Gamma]$ *are deducible in* MTL_t^q*. We now define*

$$P(\Gamma)=\mathrm{Form}_\Gamma/\sim$$

where an MTL *algebra structure is given by the logical connectives.*

The arrow part of the functor P *is defined as follows. Given* $[t_1,...,t_n]:\Gamma\to\Gamma'$ *be an arrow in* $\mathbf{C}$ *where* Γ' *is* $x_1:\sigma_1,...,x_n:\sigma_n$*, we then define*

$$P([t_1,...,t_n]):P(\Gamma')\to P(\Gamma)$$

by

$$P([t_1,...,t_n])(\varphi)=\varphi[t_1/x_1,...,t_n/x_n]$$

where $t_1:\sigma_1\ [\Gamma],...,t_n:\sigma_n\ [\Gamma]$*, and* φ *is a formula in context* $x_1:\sigma_1,...,x_n:\sigma_n$*.*

Each fibre $P(\Gamma)$ is a Lindenbaum-Tarski algebra with respect to a context Γ, and the syntactic hyperdoctrine P, as a whole, is a bundle of them over all contexts Γ.

It is immediate to see that the operations of $P(\Gamma)$ are well defined on equivalence classes involved, and thus $P(\Gamma)$ is indeed an MTL algebra. It then remains to verify that P is well defined as a hyperdoctrine, which can be done in the following manner (note that the proof is mostly the same as the one given in [15]; yet we repeat it here, since it is the crucial part of our technical development).

Lemma 7. *The syntactic hyperdoctrine* $P:\mathbf{C}^{\mathrm{op}}\to\mathbf{MTL}$ *(resp.* $X\mathbf{MTL}$*) defined above is an* MTL *(resp.* XMTL*) hyperdoctrine; amongst other things, it has both universal and existential quantifiers as the adjoints that additionally satisfy the Beck-Chevalley and Frobenius Reciprocity conditions as specified above.*

Proof. Substitution and any of the logical connectives commute with each other; this tells us that $P([t_1,...,t_n])$ is actually a homomorphism of MTL algebras. This, in turn, tells that P is a contravariant functor. Let us show that the base category $\mathbf{C}$ of the hyperdoctrine P has finite products; it is enough to show that it has binary products. Given objects Γ,Γ' in $\mathbf{C}$, $\Gamma\times\Gamma'$ is defined in the following manner. Let us assume that Γ is $x_1:\sigma_1,...,x_n:\sigma_n$, and that Γ' is $y_1:\tau_1,...,y_m:\tau_m$. We can now define $\Gamma\times\Gamma'$ as follows:

$$x_1:\sigma_1,...,x_n:\sigma_n,y_1:\tau_1,...,y_m:\tau_m.$$

It comes equipped with a projection morphism $\pi:\Gamma\times\Gamma'\to\Gamma'$ defined as

$$[y_1,...,y_m]:\Gamma\times\Gamma'\to\Gamma'$$

where the context of each y_i is supposed to be $x_1:\sigma_1,...,x_n:\sigma_n,y_1:\tau_1,...,y_m:\tau_m$; it is not just $y_1:\tau_1,...,y_m:\tau_m$. We can define the other projection in the same manner. Thus, $\mathbf{C}$ has binary products.

What remains to show is that P has universal and existential quantifiers. We denote by $\pi:\Gamma\times\Gamma'\to\Gamma'$ the projection in $\mathbf{C}$ defined above. Let us prove that $P(\pi)$ has both right and left adjoints, which give the categorical structure of quantifiers. We can give those adjoint functors in the following manner. Firstly, note that Γ is $x:\sigma_1,...,x_n:\sigma_n$. Let then $\varphi\in P(\Gamma\times\Gamma')$. Note that we can identify φ with the equivalence class containing φ, and that the following argument certainly respects the equivalence. Now let us define

$$\forall_\pi:P(\Gamma\times\Gamma')\to P(\Gamma')$$

by

$$\forall_\pi(\varphi)=\forall x_1...\forall x_n\varphi.$$

Likewise we can define

$$\exists_\pi:P(\Gamma\times\Gamma')\to P(\Gamma')$$

by

$$\exists_\pi(\varphi)=\exists x_1...\exists x_n\varphi.$$

Let us verify that $\forall_\pi$ is indeed the right adjoint of $P(\pi)$. To this end, let us assume that

$$P(\pi)(\psi)\leq\varphi$$

in $P(\Gamma\times\Gamma')$ for $\psi\in P(\Gamma')$ and $\varphi\in P(\Gamma\times\Gamma')$. By the definition of P and π, it holds that

$$P(\pi)(\psi\ [\Gamma])=\psi\ [\Gamma,\Gamma'].$$

The ordering of $P(\Gamma\times\Gamma')$ is given through the lattice reduct of it, and thus it holds that $\varphi\wedge\psi=\psi$. By the definition of $P(\Gamma\times\Gamma')$, the following hold:

$$\varphi\wedge\psi\vdash\psi\ [\Gamma,\Gamma']$$

and

$$\psi\vdash\varphi\wedge\psi\ [\Gamma,\Gamma']$$

are deducible in MTL_t^q (resp. $X\mathrm{MTL}_t^q$). Thus, $\psi \vdash \varphi\ [\Gamma, \Gamma']$, too, is deducible. Applying Rule $\forall R$ several times, we can verify that

$$\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma']$$

is deducible. It thus holds that both

$$\psi \vdash \psi \wedge \forall x_1 ... \forall x_n \varphi\ [\Gamma']$$

and

$$\psi \wedge \forall x_1 ... \forall x_n \varphi \vdash \psi\ [\Gamma']$$

are deducible. We thus have the following:

$$\psi \leq \forall x_1 ... \forall x_n \varphi$$

in $P(\Gamma')$.

The converse can be shown in the following manner. Let us assume that

$$\psi \leq \forall x_1 ... \forall x_n \varphi$$

in $P(\Gamma')$. The same argument as above allows us to show that

$$\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma']$$

is deducible. Manipulating the context, we can further prove that

$$\psi \vdash \forall x_1 ... \forall x_n \varphi\ [\Gamma, \Gamma']$$

is deducible. Now,

$$\forall x_1 ... \forall x_n \varphi \vdash \varphi\ [\Gamma, \Gamma']$$

is deducible thanks to Rule $\forall L$, and the cut rule tells that $\psi \vdash \varphi\ [\Gamma, \Gamma']$ is deducible. We thus obtain the following:

$$P(\pi)(\psi) \leq \varphi$$

in $P(\Gamma \times \Gamma')$. We have now verified that $\forall_\pi$ is the right adjoint of $P(\pi)$. In a similar way, $\exists_\pi$ can be shown to be the left adjoint of $P(\pi)$.

We still have to verify the Beck-Chevalley conditions. Let us prove the Beck-Chevalley condition for $\forall$. Suppose that $\varphi \in P(\Gamma \times \Gamma')$, $\pi : \Gamma \times \Gamma' \to \Gamma'$ is a projection in $\mathbf{C}$, and $\pi' : \Gamma \times \Gamma'' \to \Gamma''$ is another projection in $\mathbf{C}$ for objects $\Gamma, \Gamma', \Gamma''$ in $\mathbf{C}$. The following then holds:

$$P([t_1, ..., t_n]) \circ \forall_\pi(\varphi) = (\forall x_1 ... \forall x_n \varphi)[t_1/y_1, ..., t_n/y_m]$$

where note that Γ is $x_1 : \sigma_1, ..., x_n : \sigma_n$, Γ' is $y_1 : \tau_1, ..., y_m : \tau_m$, and $t_1 : \tau_1\ [\Gamma''], ..., t_m : \tau_m\ [\Gamma'']$. We also have the following

$$\forall_{\pi'} \circ P([t_1, ..., t_n])(\varphi) = \forall x_1 ... \forall x_n (\varphi[t_1/y_1, ..., t_n/y_m]).$$

The Beck-Chevalley condition for $\forall$ thus follows. The Beck-Chevalley condition for $\exists$ can be verified in a similar way. The Frobenius Reciprocity condition for $\exists$ follows immediately from Lemma 1. □

As usual, we have the obvious, canonical interpretation of MTL_t^q (resp. $X\mathrm{MTL}_t^q$) within the syntactic hyperdoctrine of MTL_t^q (resp. $X\mathrm{MTL}_t^q$). And we can verify the following lemma by straightforward computation:

Lemma 8. *Assume that $\Phi \vdash \psi\ [\Gamma]$ is satisfied in the canonical interpretation in the syntactic hyperdoctrine of MTL_t^q (resp. $X\mathrm{MTL}_t^q$). Then $\Phi \vdash \psi\ [\Gamma]$ is deducible in MTL_t^q (resp. $X\mathrm{MTL}_t^q$).*

Combining the above lemmas, we obtain the completeness theorem. That is, if $\Phi \vdash \psi\ [\Gamma]$ is satisfied in any interpretation in any MTL (resp. XMTL) hyperdoctrine, then $\Phi \vdash \psi\ [\Gamma]$ is deducible in MTL_t^q (resp. $X\mathrm{MTL}_t^q$). So we finally have the following:

Theorem 9. *The following are equivalent:*

- *$\Phi \vdash \psi\ [\Gamma]$ is deducible in MTL_t^q (resp. $X\mathrm{MTL}_t^q$);*
- *$\Phi \vdash \psi\ [\Gamma]$ is satisfied in any interpretation in any* MTL *(resp. X*MTL*) hyperdoctrine.*

The completeness result above can be instantiated for a variety of fuzzy logics, including Łukasiewicz logic, Hájek's basic logic, and Gödel logic, by specifying axioms X in suitable ways; for example, Łukasiewicz logic is MTL extended with the following axiom:

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi).$$

Łukasiewicz predicate logic is known to be incomplete, and yet the hyperdosctrine semantics accommodate sufficiently many models to restore completeness.

We can refine the completeness result by restricting the class of models involved; it is actually sufficient to think of those MTL hyperdoctrines P that have all the fibres $P(C)$ linearly ordered. We call this strengthened property linear completeness, and call such MTL hyperdoctrines linear MTL hyperdoctrines. In order to show the linear completeness, we have to make the syntactic hyperdoctrine P defined above a linear MTL hyperdoctrine; this can be done by taking the quotient of each fibre $P(C)$ with respect to a prime filter that excludes the formula we want to refute (note that the so-called local deduction theorem tells that there is a formula φ such that $\Phi \vdash \psi\ [\Gamma]$ iff $\vdash \varphi \to \psi\ [\Gamma]$; for this, see, e.g., [7]). The quotient of an MTL algebra with respect to a prime filter is totally ordered. We thus obtain the following linear completeness theorem:

Theorem 10. *The following are equivalent:*

- *$\Phi \vdash \psi\ [\Gamma]$ is deducible in MTL_t^q (resp. $X\mathrm{MTL}_t^q$);*
- *$\Phi \vdash \psi\ [\Gamma]$ is satisfied in any interpretation in any linear* MTL *(resp. X*MTL*) hyperdoctrine $P : \mathbf{C}^{\mathrm{op}} \to \mathbf{MTL}$ (i.e., $P(C)$ is linearly ordered for any $C \in \mathbf{C}$).*

Set-theoretical semantics of logic corresponds to interpreting logic within set-theoretical hyperdoctrines as follows:

Proposition 11. *Let $\Omega \in \mathbf{MTL}$ with Ω complete. Then,*

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega) : \mathbf{Set}^{\mathrm{op}} \to \mathbf{MTL}$$

is an MTL *hyperdoctrine. The same holds for X***MTL** *as well.*

Proof. Let $\pi : X \times Y \to Y$ be a projection in $\mathbf{Set}$. We define $\forall_\pi$ and $\exists_\pi$ as follows: given $v \in \mathrm{Hom}(X \times Y, \Omega)$ and $y \in Y$, let

$$\forall_\pi(v)(y) := \bigwedge\{v(x, y) \mid x \in X\}$$

and

$$\exists_\pi(v)(y) := \bigvee\{v(x, y) \mid x \in X\}.$$

These yield the required quantifier structures satisfying the Beck-Chevalley and Frobenius Reciprocity conditions. □

Tarski semantics amounts to interpreting logic within the two-valued hyperdoctrine

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \mathbf{2}).$$

It is straightforward to check this by spelling out the definition of interpretation. Set-theoretical semantics of Ω-valued fuzzy logic amounts to interpreting logic within

$$\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega).$$

In this way, categorical semantics may be regarded as an extension of set-theoretical semantics.

IV. Hyperdoctrinal Baaz Translation

In the following we show the hyperdoctrinal version of Baaz' delta translation from MTL to classical logic CL. The Baaz delta Δ allows us to embed classical logic into fuzzy logic. If φ is true (i.e., its truth value is 1), $\Delta\varphi$ is true (i.e., its truth value is 1); otherwise, $\Delta\varphi$ is false (i.e., its truth value is 0). And thus $\Delta\varphi$ is bivalent; the Baaz delta operator makes the fuzzy world bivalent. Formally, the axioms for Δ are as follows:

- (i) $1 \to \Delta 1$; (ii) $\Delta\varphi \to \varphi$; (iii) $\Delta\varphi \to \Delta\Delta\varphi$;
- (iv) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$;
- (v) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$;
- (vi) $\Delta\varphi \to (\Delta\varphi \otimes \Delta\varphi)$; (vii) $\Delta\varphi \vee (\Delta\varphi \to 0)$.

$\Delta\mathrm{MTL}$ denotes MTL with Δ satisfying these axioms, and $\Delta\mathbf{MTL}$ the corresponding algebraic category. It is immediate to see that

$$\Delta\forall x \Delta\varphi \vdash \forall x \Delta\varphi.$$

The converse also holds as follows.

Lemma 12. *$\forall x \Delta\varphi \vdash \Delta\forall x \Delta\varphi$ $[\Gamma]$ holds in $\Delta\mathbf{MTL}$.*

Proof. Omitting the context, $\Delta\varphi \vdash \Delta\varphi$ allows us to derive

$$\Delta\varphi \vdash \forall x \Delta\varphi,$$

which in turn implies

$$\Delta\Delta\varphi \vdash \Delta\forall x \Delta\varphi.$$

Since Δ is idempotent, we have

$$\Delta\varphi \vdash \Delta\forall x \Delta\varphi,$$

from which we can derive

$$\forall x \Delta\varphi \vdash \Delta\forall x \Delta\varphi.$$

□

Baaz' Δ translation in fuzzy logic may be compared with Gödel's doubly negative translation $\neg\neg$ in intuitionistic logic. Algebraically, the fixpoints of $\neg\neg$ in a given Heyting algebra, i.e., those elements φ of the algebra that satisfy

$$\neg\neg\varphi = \varphi,$$

form a Boolean algebra, and this is the algebraic version of the Gödel translation. Likewise, the fixpoints of Δ form a Boolean algebra; this is logically clear because $\Delta\varphi$ satisfies both contraction and the excluded middle; MTL plus contraction and the excluded middle is classical logic. In the following we show that the quantifier structure is preserved in this process. Formally, we regard Δ as a functor Fix_Δ from $\Delta\mathbf{MTL}$ to $\mathbf{BA}$, which denotes the category Boolean algebras and their homomorphisms. We define

$$\mathrm{Fix}_\Delta(A) = \{a \in A \mid \Delta a = a\};$$

the arrow part is defined by restriction.

Replacing $\mathbf{MTL}$ by $\Delta\mathbf{MTL}$ and by $\mathbf{BA}$, we can define $\Delta\mathrm{MTL}$ hyperdoctrines and CL hyperdoctrines. Now we have the hyperdoctrinal Baaz translation theorem as follows.

Theorem 13. *Let*

$$P : \mathbf{C}^{\mathrm{op}} \to \Delta\mathbf{MTL}$$

be an $\Delta\mathrm{MTL}$ hyperdoctrine. Then, the following composed functor

$$\mathrm{Fix}_\Delta \circ P : \mathbf{C}^{\mathrm{op}} \to \mathbf{BA}$$

forms a CL hyperdoctrine.

Proof. When $\varphi \in \Delta \circ P(X \times Y)$ (i.e., $\Delta\varphi = \varphi$), the above lemma tells us

$$\Delta\forall_\pi\varphi = \forall_\pi\varphi$$

in $P(Y)$ where $\pi : X \times Y \to Y$. The quantifier structure of $\mathrm{Fix}_\Delta \circ P$, therefore, can be given by $\Delta\forall_\pi$, which satisfies the adjointness condition (because $\Delta\forall_\pi\varphi = \forall_\pi\varphi$ and $\forall_\pi$ satisfies the adjointness condition). The same applies to $\exists$ as well. □

We note that the hyperdoctrinal translation theorem above is more general than the syntactic translation theorem, in the sense that the latter corresponds to the case of syntactic hyperdoctrines in the former. Note also that proofs in the last section apply to $\Delta\mathrm{MTL}_t^q$, and (linear) completeness holds for $\Delta\mathrm{MTL}_t^q$.

V. Concluding Remarks

Categorical fuzzy logic (or categorical many-valued logic) has been limited in its applicability; in the present paper, we have taken first steps in developing a categorical approach to fuzzy predicate logic in typed form (rather than in single-sorted form, which is more limited). The striking feature of our theory lies in its modularity and broad applicability; it basically work for any first-order fuzzy logic extending the base system MTL.

The method developed in this paper is actually applicable for an even broader variety of fuzzy logics beyond MTL, including both first-order and higher-order (set-theoretical) versions, and we shall demonstrate this in future work. It is even applicable to fuzzy modal logic, which has been studied extensively in the fuzzy logic community in the last decade, yielding several applications in logic for artificial intelligence, including, inter alia, fuzzy description logic, fuzzy common knowledge logic, and fuzzy knowledge representation in general.

The so-called tripos-to-topos construction allows us to construct Heyting-valued models of set theory from Ω-valued hyperdoctrines $\mathrm{Hom}_{\mathbf{Set}}(\text{-}, \Omega)$ where Ω is a locale or complete Heyting algebra (see Hyland-Johnstone-Pitts [8]); in our future work we plan to do the same for Takeuti-Titani's fuzzy set theory by extending the tripos-to-topos construction as in [15], [16]. Note that higher-order hyperdoctrines are basically equivalent to what are called triposes (to be precise, there are different definitions of triposes, yet one of them is equivalent to the concept of higher-order hyperdoctrines for intuitionistic logic).

Acknowledgment

The author would like to thank his colleagues, both past and present, for the various discussions that have contributed to the developments of the materials presented in the paper in some way or other. The author hereby acknowledges that the present work was financially supported by JST PRESTO (grant code: JPMJPR17G9) and JSPS KAKENHI (grant code: 17K14231).

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