Universal Stone Duality via the Concept of Topological Dualizability and its Applications to Many-Valued Logic

Yoshihiro Maruyama
Research School of Computer Science
The Australian National University
yoshihiro.maruyama@anu.edu.au

***Abstract*—We propose the concept of topological dualizability as the condition of possibility of Stone duality, and thereby give a non-Hausdorff extension of the primal duality theorem in natural duality theory in universal algebra. The primal duality theorem is a vast generalization of the classic Stone duality for Boolean algebras, telling that any varieties generated by functionally complete algebras, such as the algebras of Emil Post's finite-valued logics, are categorically equivalent to zero-dimensional compact Hausdorff spaces. Here we show a non-Hausdorff extension of primal duality: any varieties generated by certain weakly functionally complete or topologically dualizable algebras are categorically dually equivalent to coherent spaces, a special class of compact sober spaces. This generalizes the Stone duality for distributive lattices and Heyting algebras (as a subclass of distributive lattices) in the spirit of primal duality theory. And we give applications of the general theorem to algebras of Łukasiewicz many-valued logics. The concept of topological dualizability is arguably the key to the universal algebraic unification of Stone-type dualities; in the present paper, we take the first steps in demonstrating this thesis.**

***Index Terms*—primal duality theory; non-Hausdorff duality; many-valued logic; Łukasiewicz logic; functional completeness**

I. Introduction

The Stone duality for Boolean algebras is one of the most important results in algebraic logic [33], and has been generalized in various directions (see, e.g., [2], [8], [11], [13], [16], [20]). Moreover, Stone-type dualities have been applied to diverse fields, including program semantics, non-classical logics and pointfree geometry (see, e.g., [1], [3], [4], [12], [20]). Stone-type dualities naturally connect logic, algebra and geometry, and therefore, for example, we can understand the geometric meanings of logics and their properties via Stone-type dualities. In the present paper, we discuss what is called primal duality in universal algebra [8].

Classical logic is known to be functionally complete:

- Logically speaking, any truth function is representable by a logical formula.
- Algebraically speaking, any function from $\mathbf{2}^n$ to $\mathbf{2}$ (where $\mathbf{2}$ denotes $\{0, 1\}$) is a term function of $\mathbf{2}$ (equipped with the Boolean operations).

And so we have the classic Stone duality for $\mathbb{ISP}(\mathbf{2})$, which is the variety of Boolean algebras. The so-called primal duality, arguably the most basic case of natural duality in universal algebra [8], generalizes this:

- If a finite algebra L is functionally complete, i.e., if any function from L^n to L is a term function of L, then we have the corresponding the Stone duality, i.e., $\mathbb{ISP}(L)$ is categorically dually equivalent to zero-dimensional compact Hausdorff spaces.

Such an algebra L has been called primal in universal algebra. The primal duality theorem applies to any $\mathbb{ISP}(L)$ generated by a primal algebra L. For example, the category of algebras of Emil Post's finite-valued logic is dually equivalent to the category of zero-dimensional compact Hausdorff spaces. In general, $\mathbb{ISP}(L)$ may be regarded as the algebras of L-valued logic; so the universal algebra of $\mathbb{ISP}(L)$ is directly connected with many-valued logic.

In this paper we show that a similar phenomenon actually exists for distributive lattices. We abstract properties of $\mathbf{2}$ as a distributive lattice, and it then turns out that any $\mathbb{ISP}(L)$ generated by a finite algebra L with those properties is categorically dually equivalent to coherent spaces, a special class of compact sober spaces (defined below). So what are the essential properties of $\mathbf{2}$ in light of Stone duality? Roughly, we consider that the coincidence of term functions and continuous maps yields a Stone-type duality; let us elaborate this idea in the following. Given a finite algebra L equipped with a topology, we consider a Stone-type duality for $\mathbb{ISP}(L)$. Let

$$\mathrm{TermFunc}_n(L)$$

denote the set of all n-ary term functions of L and

$$\mathrm{Cont}_n(L)$$

the set of all continuous maps from L^n to L. Then, L is said to be topologically dualizable with respect to the topology iff the following holds:

$$\forall n \in \omega \;\; \mathrm{Cont}_n(L) = \mathrm{TermFunc}_n(L).$$

Then our rough idea is that, if L is topologically dualizable with respect to the topology, a Stone-type duality holds for $\mathbb{ISP}(L)$. It may not always hold; we shall however show that it does hold for certain algebras L. This actually generalizes primality:

- L is primal iff L is topologically dualizable with respect to the discrete topology.

978-1-7281-6932-3/20/$31.00 ©2020 IEEE

Moreover, the following holds:

- **2** as a distributive lattice is topologically dualizable with respect to the Alexandrov topology
$$\{\emptyset, \{1\}, \{0,1\}\}.$$
Note that this follows from Proposition 47 below.

Thanks to this property, the variety of distributive lattices enjoys the Stone duality; this is the basic idea of the present paper. Note that the distributive lattices are generated by **2** (as a distributive lattice). Our main theorem may be summarized as follows.

- Let L be a finite algebra with a bounded join-semilattice reduct. We equip L with the Alexandrov topology with respect to the partial ordering induced by the join-semilattice reduct.
- We can then prove the following (Theorem 43): if L is topologically dualizable with respect to the Alexandrov topology, then the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms is dually equivalent to the category of coherent spaces and proper maps (defined below).

This is a universal algebraic generalization of the Stone duality for distributive lattices, just as the primal duality theorem is a universal algebraic generalization of the Stone duality for Boolean algebras.

The rest of the paper is organized as follows. In Section II, we review basics of general topology. In Section III, we introduce the concept of topological dualizability. In Section IV, we generalize the Stone duality for distributive lattices via the concept of topological dualizability; we also give applications to algebras with Łukasiewicz operations (except for negation, which makes topology Hausdorff). We finally conclude the paper with remarks on Stone-type dualities for Heyting algebras and their many-valued extensions.

II. Preliminaries from general topology

In this section, we review basic concepts and results from general topology.

Definition 1. *A Boolean space is defined as a zero-dimensional compact Hausdorff space.*

Definition 2. *For topological spaces S_1 and S_2, a map $f : S_1 \to S_2$ is proper iff $f^{-1}(O)$ is a compact open subset of S_2 for any compact open subset O of S_1.*

Note that a map f between Boolean spaces is continuous iff f is proper.

Definition 3. *A non-empty closed subset A of a topological space S is irreducible iff, for any closed subsets A_1 and A_2 of S, $A = A_1 \cup A_2$ implies either $A_1 = A$ or $A_2 = A$.*

Definition 4. *A topological space S is sober iff, for any irreducible closed subset A of S, there is a unique element x of S such that*
$$A = \overline{\{x\}},$$
where $\overline{\{x\}}$ denotes the closure of $\{x\}$.

In the following, we review basic facts on sober spaces (see [13], [20], [36]). A sober space is T_0. A Hausdorff space is sober.

Lemma 5. *Any product of sober spaces is also sober.*

Proof. See [17, Theorem 1.4] or [13, Exercise O-5.16]. □

Definition 6. *A coherent space S is defined as a compact sober space such that the set of compact open subsets of S forms an open basis of S.*

A proper map between coherent spaces is always continuous.

For example, the spectrum of a commutative ring is a coherent space (see [16]).

Definition 7. *Let S be a topological space and $\mathcal{B}$ the set of all compact open subsets of S. Then, the patch topology of S is defined as the topology generated by*
$$\mathcal{B} \cup \{S \setminus X \ ; \ X \in \mathcal{B}\}.$$
Let S^* denote the new space equipped with the patch topology.

Patch topology is useful for the study of sober and coherent spaces.

Lemma 8. *Let S be a coherent space. Then, S^* is a Boolean space.*

III. Topological dualizability

In this section, we introduce the notion of topological dualizability.

We mean by an algebra a set L equipped with a collection of finitary operations on L (for basic concepts from universal algebra, see [6], [8], [15]). Note that a constant of L is considered as a function from L^0 to L, where L^0 is a singleton. Throughout this paper, a lattice and a semilattice mean a bounded lattice and a bounded semilattice respectively.

For an algebra L,
$$\mathbb{ISP}(L)$$
denotes the class of all isomorphic copies of subalgebras of direct powers of L. $\mathbb{ISP}(L)$ may be seen as the algebras of L-valued logic. As usual, a homomorphism between algebras in $\mathbb{ISP}(L)$ is defined as a function which preserves all the operations of L. Note that a homomorphism preserves any term function.

Definition 9. *For an algebra L and $n \in \omega$,*
$$\mathrm{TermFunc}_n(L)$$
denotes the set of all n-ary term functions of L.

Any projection function from L^n to L is an element of $\mathrm{TermFunc}_n(L)$ by the definition of term functions.

Definition 10. *For a topological space S and $n \in \omega$,*
$$\mathrm{Cont}_n(S)$$

denotes the set of all continuous maps from S^n to S, where S^n is equipped with the product topology (S^0 is a singleton topological space).

Then, the notion of topological dualizability is defined as follows.

Definition 11. *Let L be a finite algebra equipped with a topology. Then, L is said to be topologically dualizable with respect to the topology iff*

$$\forall n \in \omega \ \ \mathrm{Cont}_n(L) = \mathrm{TermFunc}_n(L).$$

Any projection function from L^n to L is continuous by the definition of the product topology.

Let us review the notion of primal algebra.

Definition 12. *A finite algebra L is primal iff $\mathrm{TermFunc}_n(L)$ coincides with the set of all functions from L^n to L.*

Proposition 13. *Let L be a finite algebra equipped with the discrete topology. Then, L is primal iff L is topologically dualizable with respect to the discrete topology.*

Proof. This is immediate from the fact that, since L^n is a discrete space, $\mathrm{Cont}_n(L)$ coincides with the set of all functions from L^n to L. $\square$

Topological dualizability may thus be seen as a generalization of primality.

The following is the primal duality theorem:

Theorem 14. *Let L be a primal algebra. Then, the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms is dually equivalent to the category of Boolean spaces and continuous maps.*

Let $\mathbf{2}_b$ denote the two-element Boolean algebra. Then, $\mathbf{2}_b$ is a primal algebra. Notice that

$$\mathbb{ISP}(\mathbf{2}_b)$$

coincides with the class of all Boolean algebras, which follows from the ultrafilter theorem for Boolean algebras. Thus, the primal duality theorem is a generalization of the Stone duality for Boolean algebras.

IV. Topological dualizability entails Stone duality

In the remainder of paper, let L be a finite algebra such that

- L has a join-semilattice reduct;
- L has the greatest element 1 and the least element 0 with respect to a partial order $\leq$ defined by
$$x \leq y \Leftrightarrow x \vee y = y$$
for $x, y \in L$, where $\vee$ denotes the join operation of L.
- L is equipped with the Alexandrov topology with respect to $\leq$ above, i.e., the topology of L is generated by
$$\{\uparrow x \,;\, x \in L\},$$
where
$$\uparrow x = \{y \in L \,;\, x \leq y\}.$$

Note that the set of all open (resp. closed) subsets of L coincides with the set of all upward (resp. downward) closed subsets of L. For a set S,

$$L^S$$

denotes the set of all functions from S to L. We equip L^S with the product topology. In the remainder of the paper, we additionally assume:

- L is topologically dualizable with respect to the Alexandrov topology.

This is the last assumption. In the following, we prove a Stone-type duality for $\mathbb{ISP}(L)$.

Lemma 15. *Define a function $t_\wedge : L^2 \to L$ by*

$$t_\wedge(x,y) = \begin{cases} 1 & \text{if } x = y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $t_\wedge$ is a term function of L.

Proof. Since L is topologically dualizable with respect to the Alexandrov topology, it suffices to show that $t_\wedge$ is continuous, which is straightforward to verify. $\square$

In similar ways, we obtain the following lemmas.

Lemma 16. *Let $n \in \omega$. Define a function $t_\vee^n : L^n \to L$ by*

$$t_\vee^n(x_1, ..., x_n) = \begin{cases} 1 & \text{if } \ \exists i \in \{1, ..., n\} \ x_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, $t_\vee^n$ is a term function of L.

Lemma 17. *Let $r \in L$. Define a function $\tau_r : L \to L$ by*

$$\tau_r(x) = \begin{cases} 1 & \text{if } x \geq r \\ 0 & \text{otherwise.} \end{cases}$$

Then, τ_r is a term function of L.

τ_r is useful in many-valued logic [21], [22], [29], [35].

Lemma 18. *Let $r \in L$. Define a function $\theta_r : L \to L$ by*

$$\theta_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, θ_r is a term function of L.

Note that a homomorphism preserves the operations $t_\wedge$, $t_\vee^n$, τ_r and θ_r, since they are term functions.

A. The spectrum of an algebra in $\mathbb{ISP}(L)$

We define the spectrum $\mathrm{Spec}(A)$ of an algebra A in $\mathbb{ISP}(L)$ as follows.

Definition 19. *For $A \in \mathbb{ISP}(L)$, $\mathrm{Spec}(A)$ denotes the set of all homomorphisms from A to L. For $a \in A$, define*

$$\langle a \rangle = \{v \in \mathrm{Spec}(A) \,;\, v(a) = 1\}.$$

We equip $\mathrm{Spec}(A)$ with the topology generated by

$$\{\langle a \rangle \,;\, a \in A\}.$$

Note that, by Lemma 15,

$$\langle a \rangle \cap \langle b \rangle = \langle t_\wedge(a,b) \rangle$$

and that, by Lemma 16,

$$\langle a_1 \rangle \cup ... \cup \langle a_n \rangle = \langle t^n_\vee(a_1,...,a_n) \rangle.$$

Proposition 20. *Let $A \in \mathbb{ISP}(L)$. Then,*

$$\{\langle a \rangle \,;\, a \in A\}$$

forms an open basis of $\mathrm{Spec}(A)$.

Proof. It suffices to show that

$$\{\langle a \rangle \,;\, a \in A\}$$

is closed under $\cap$. Let $a, b \in A$. Then, we have

$$\langle t_\wedge(a,b) \rangle = \langle a \rangle \cap \langle b \rangle.$$

This completes the proof. □

Lemma 21. *Let $A \in \mathbb{ISP}(L)$. For $v, u \in \mathrm{Spec}(A)$, the following are equivalent:*

(i) $v = u$;
(ii) $v^{-1}(\{1\}) = u^{-1}(\{1\})$.

We omit the proof of this lemma; it is quite straightforward to see.

Definition 22. *Let $A \in \mathbb{ISP}(L)$ and $X \subset L^A$. For $a \in A$ and $r \in L$, define*

$$\langle a \rangle^r_X = \{f \in X \,;\, f(a) \geq r\}.$$

Define $X^\star$ as a topological space whose underlying set is X and whose topology is generated by

$$\{\langle a \rangle^1_X \,;\, a \in A\} \cup \{X \setminus \langle a \rangle^1_X \,;\, a \in A\}.$$

We then have the following lemmas (proofs are omitted; they shall be given in the fully extended journal version of the paper).

Lemma 23. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)$ ($\subset L^A$) *is a subspace of L^A, i.e., the topology of* $\mathrm{Spec}(A)$ *coincides with the relative topology induced by L^A on a set* $\mathrm{Spec}(A)$, *where L^A is equipped with the product topology.*

Let L_d denote the topological space whose underlying set is L and whose topology is the discrete topology.

Lemma 24. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)^\star$ *is a subspace of L^A_d, i.e., the topology of* $\mathrm{Spec}(A)^\star$ *coincides with the relative topology induced by L^A_d on a set* $\mathrm{Spec}(A)$, *where L^A_d is equipped with the product topology of L_d's.*

Lemma 25. *Let $A \in \mathbb{ISP}(L)$. Then, (i)* $\mathrm{Spec}(A)^\star$ *is compact; (ii) $\langle a \rangle$ is a compact subset of* $\mathrm{Spec}(A)^\star$ *for any $a \in A$.*

Proposition 26. *Let $A \in \mathbb{ISP}(L)$. Then, (i)* $\mathrm{Spec}(A)$ *is compact; (ii) $\langle a \rangle$ is a compact subset of* $\mathrm{Spec}(A)$ *for any $a \in A$.*

Proof. By Lemma 25, $\mathrm{Spec}(A)^\star$ is compact. Thus, since the topology of $\mathrm{Spec}(A)$ is weaker than or equal to that of $\mathrm{Spec}(A)^\star$, $\mathrm{Spec}(A)$ is also compact. It is verified in a similar way that $\langle a \rangle$ is a compact subset of $\mathrm{Spec}(A)$. □

Lemma 27. *Let $A \in \mathbb{ISP}(L)$. Then, L^A is a sober space.*

Recall the definition of patch topology (Definition 7).

Lemma 28. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)^\star$ *is equal to* $\mathrm{Spec}(A)^*$.

Proposition 29. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)$ *is a sober space.*

Proof. It is known that, for a sober space S and a subspace X of S, if X^* is a closed subspace of S^*, then X is sober (see [30, 1.1 and 1.5]). Thus, by Lemma 27, it suffices to show that $\mathrm{Spec}(A)^*$ is a closed subspace of $(L^A)^*$. By Lemma 23, $\mathrm{Spec}(A)$ is a subspace of L^A. As is shown in the proof of Lemma 25, $\mathrm{Spec}(A)^\star$ is a closed subspace of L^A_d. It is verified in a similar way to the proof of Lemma 24 that the topology of L^A_d is equal to the topology of $(L^A)^*$ (i.e., the patch topology of L^A). Hence, it follows from Lemma 28 that $\mathrm{Spec}(A)^*$ is a closed subspace of $(L^A)^*$. □

By the above facts, we obtain the following proposition.

Proposition 30. *Let $A \in \mathbb{ISP}(L)$. Then,* $\mathrm{Spec}(A)$ *is a coherent space.*

Proof. By Proposition 29 and Proposition 26, $\mathrm{Spec}(A)$ is a compact sober space. As is shown in the proof of Lemma 28, $\{\langle a \rangle \,;\, a \in A\}$ coincides with the set of all compact open subsets of $\mathrm{Spec}(A)$. By Proposition 20, $\{\langle a \rangle \,;\, a \in A\}$ forms an open basis of $\mathrm{Spec}(A)$. □

B. Categories and functors

In this subsection, we define categories $\mathrm{ISP}(L)$ and CohSp, and functors Spec and Prop between those categories.

Definition 31. $\mathrm{ISP}(L)$ *denotes the category of algebras in $\mathbb{ISP}(L)$ and homomorphisms.*

Definition 32. CohSp *denotes the category of coherent spaces and proper maps.*

Definition 33. *We define a contravariant functor*

$$\mathrm{Spec} : \mathrm{ISP}(L) \to \mathrm{CohSp}$$

as follows. For an object A in $\mathrm{ISP}(L)$, $\mathrm{Spec}(A)$ *has already been defined in Definition 19. For an arrow $f : A \to B$ in* $\mathrm{ISP}(L)$,

$$\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is defined by

$$\mathrm{Spec}(f)(v) = v \circ f$$

for $v \in \mathrm{Spec}(B)$.

The object part of the functor Spec is well-defined by Proposition 30. The arrow part of Spec is well-defined by the following lemma.

Lemma 34. *Let $f : A \to B$ be a homomorphism for $A, B \in \mathbb{ISP}(L)$. Then, $\mathrm{Spec}(f)$ is a proper map.*

Definition 35. *We define a contravariant functor*

$$\mathrm{Prop} : \mathrm{CohSp} \to \mathrm{ISP}(L)$$

as follows. For an object S in CohSp, define $\mathrm{Prop}(S)$ as the set of all proper maps from S to L endowed with the pointwise operations defined as follows: For each n-ary operation t of L and $f_1, ..., f_n \in \mathrm{Prop}(S)$, define

$$t(f_1, ..., f_n) : S \to L$$

by

$$(t(f_1, ..., f_n))(x) = t(f_1(x), ..., f_n(x)).$$

For an arrow $f : S \to S'$ in CohSp, define

$$\mathrm{Prop}(f) : \mathrm{Prop}(S') \to \mathrm{Prop}(S)$$

by

$$\mathrm{Prop}(f)(g) = g \circ f$$

for $g \in \mathrm{Prop}(S')$.

The functor Prop is well-defined by the following two lemmas.

Lemma 36. *Let S be a coherent space. Then, $\mathrm{Prop}(S)$ is in $\mathbb{ISP}(L)$.*

Lemma 37. *Let $f : S \to S'$ be a proper map between coherent spaces S and S'. Then, $\mathrm{Prop}(f)$ is a homomorphism.*

The lemma above follows immediately from the fact that the operations of $\mathrm{Prop}(S')$ are defined pointwise.

C. A Stone-type duality for $\mathbb{ISP}(L)$

In this subsection, we show a Stone-type duality theorem for $\mathbb{ISP}(L)$.

Theorem 38. *Let $A \in \mathbb{ISP}(L)$. Then, there is an isomorphism from A to $\mathrm{Prop} \circ \mathrm{Spec}(A)$.*

Proof. Define

$$\Phi : A \to \mathrm{Prop} \circ \mathrm{Spec}(A)$$

by

$$\Phi(a)(v) = v(a)$$

for $a \in A$ and $v \in \mathrm{Spec}(A)$. Let $r \in L$. By Lemma 17, we have

$$\Phi(a)^{-1}(\uparrow r) = \{v \in \mathrm{Spec}(A) \,;\, v(a) \geq r\} = \langle \tau_r(a) \rangle.$$

Thus, by Lemma 26, $\Phi(a) : \mathrm{Spec}(A) \to L$ is proper and so Φ is well-defined.

Let t be an n-ary operation of A for $n \in \omega$. For $a_1, ..., a_n \in A$ and $v \in \mathrm{Spec}(A)$, we have

$$\begin{array}{rcl} \Phi(t(a_1, ..., a_n))(v) & = & v(t(a_1, ..., a_n)) \\ & = & t(v(a_1), ..., v(a_n)) \\ & = & t(\Phi(a_1)(v), ..., \Phi(a_n)(v)) \\ & = & (t(\Phi(a_1), ..., \Phi(a_n)))(v). \end{array}$$

Therefore, Φ is a homomorphism.

We show that Φ is injective. Let $a, b \in A$ with $a \neq b$. By $A \in \mathbb{ISP}(L)$, A is isomorphic to a subalgebra A' of L^I for some I. Thus, we may identify A with A'. Then, a and b are functions from I to L. By $a \neq b$, there is $i \in I$ such that

$$a(i) \neq b(i).$$

Define $p_i : A \to L$ by

$$p_i(x) = x(i)$$

for $x \in A$. Note that $p_i(a) \neq p_i(b)$. Then, since the operations of L^I are defined pointwise, p_i is a homomorphism, which means $p_i \in \mathrm{Spec}(A)$. Moreover, we have

$$\Phi(a)(p_i) \neq \Phi(b)(p_i).$$

Thus, Φ is injective.

Finally, we show that Φ is surjective. Let $f \in \mathrm{Prop} \circ \mathrm{Spec}(A)$. Let $r \in L$. By Lemma 20 and the continuity of f, there are an index set K and $a_r^k \in A$ for $k \in K$ such that

$$f^{-1}(\uparrow r) = \bigcup_{k \in K} \langle a_r^k \rangle.$$

By Lemma 16 and the properness of f, there is $a_r \in A$ such that $f^{-1}(\uparrow r) = \langle a_r \rangle$. Then, we claim that

$$\Phi(\bigvee \{\theta_r(a_r) \,;\, r \in L\}) = f,$$

where θ_r is defined in Lemma 18. In order to show this, suppose that $v \in f^{-1}(\{s\})$ for $s \in L$. Then, we have: For each $r \in L$,

$$v(\theta_r(a_r)) = \begin{cases} r & \text{if } r \leq s \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\Phi(\bigvee \{\theta_r(a_r) \,;\, r \in L\})(v) = s = f(v).$$

Hence the above claim holds. □

We can then verify the following lemmas; full proofs shall be given in the fully expanded journal version of the paper.

Lemma 39. *Let S be a coherent space. Assume that P_i is a compact open subset or a closed subset of S for any $i \in I$. Then, if $\{P_i \,;\, i \in I\}$ has the finite intersection property, then $\bigcap \{P_i \,;\, i \in I\}$ is not empty.*

Lemma 40. *Let S be a coherent space and O a compact open subset of S. Define the indicator function $\mu_O : S \to L$ of O by $\mu_O(x) = 1$ for $x \in O$ and $\mu_O(x) = 0$ for $x \in S \setminus O$. Then, $\mu_O \in \mathrm{Prop}(S)$.*

Lemma 41. *Let S be a coherent space, $v \in \mathrm{Spec} \circ \mathrm{Prop}(S)$,*

$$G = \{\, f^{-1}(\{1\}) \,;\, v(f) = 1 \,\},$$

and

$$H = \{\, S \setminus f^{-1}(\{1\}) \,;\, v(f) \neq 1 \,\}.$$

Then, $G \cup H$ has the finite intersection property.

Theorem 42. *Let S be a coherent space. Then, there is a homeomorphism from S to $\mathrm{Spec} \circ \mathrm{Prop}(S)$.*

Proof. Define

$$\Psi : S \to \mathrm{Spec} \circ \mathrm{Prop}(S)$$

by

$$\Psi(x)(f) = f(x)$$

for $x \in S$ and $f \in \mathrm{Prop}(S)$. Since the operations of $\mathrm{Prop}(S)$ are defined pointwise, $\Psi(x)$ is a homomorphism and so Ψ is well-defined. We claim that Ψ is a homeomorphism. First, Ψ is proper, since we have the following for $f \in \mathrm{Prop}(S)$:

$$\Psi^{-1}(\langle f \rangle) = \{x \in S \,;\, \Psi(x) \in \langle f \rangle\} = f^{-1}(\{1\})$$

and since a compact open subset of $\mathrm{Spec} \circ \mathrm{Prop}(S)$ is of the form $\langle f \rangle$ for some $f \in \mathrm{Prop}(S)$ by Lemma 20 and Lemma 16.

Second, we show that Ψ is injective. Assume that $x, y \in S$ with $x \neq y$. Since S is a coherent space, S is T_0 and has an open basis consisting of compact open subsets of S. Thus, we may assume that there is a compact open subset O of S such that $x \in O$ and $y \notin O$. By Lemma 40, we have $\mu_O \in \mathrm{Prop}(S)$ and

$$\Psi(x)(\mu_O) = 1 \neq 0 = \Psi(y)(\mu_O).$$

Hence, we have $\Psi(x) \neq \Psi(y)$. Thus, Ψ is injective.

Third, we show that Ψ is surjective. Let $v \in \mathrm{Spec} \circ \mathrm{Prop}(S)$. Let

$$G = \{\, f^{-1}(\{1\}) \,;\, v(f) = 1 \,\}$$

and

$$H = \{\, S \setminus f^{-1}(\{1\}) \,;\, v(f) \neq 1 \,\}.$$

Since f is proper, $f^{-1}(\{1\})$ is compact open and $S \setminus f^{-1}(\{1\})$ is closed. By Lemma 41, $G \cup H$ enjoys finite intersection property. Therefore, by Lemma 39, there is $y \in S$ such that

$$y \in \bigcap (G \cup H) = (\bigcap G) \cap (\bigcap H).$$

Since $y \in \bigcap G$, if $v(f) = 1$ then

$$\Psi(y)(f) = f(y) = 1.$$

Since $y \in \bigcap H$, if $\Psi(y)(f) = f(y) = 1$ then $v(f) = 1$. Thus

$$v^{-1}(\{1\}) = \Psi(y)^{-1}(\{1\}).$$

By Lemma 21, we have $v = \Psi(y)$. Hence, Ψ is surjective.

Fourth, we show that Ψ is an open map. Let O be an open subset of S. Since S is coherent,

$$O = \bigcup_{i \in I} O_i$$

for some compact open subsets O_i of S. By Lemma 40, $\mu_{O_i} \in \mathrm{Prop}(S)$. We claim that

$$\Psi[O] = \bigcup \{\langle \mu_{O_i} \rangle \,;\, i \in I\}.$$

If $x \in O$, then $x \in O_i$ for some i, whence $\Psi(x) \in \langle \mu_{O_i} \rangle$. To show the converse, suppose $v \in \langle \mu_{O_i} \rangle$. Then $v = \Psi(y)$ for some $y \in S$, since Ψ is surjective. Since $\Psi(y) \in \langle \mu_{O_i} \rangle$, we have $\mu_{O_i}(y) = 1$. Thus, $y \in O_i$ by the definition of μ_{O_i}. Hence the claim holds.

Finally, since a continuous function maps a compact set to a compact set, Ψ^{-1} is a proper map. This completes the proof. $\square$

By the above results, we obtain the following duality theorem for $\mathbb{ISP}(L)$. Note that L is assumed to be topologically dualizable with respect to the Alexandrov topology.

Theorem 43. *The category $\mathrm{ISP}(L)$ is dually equivalent to the category CohSp via the functors Spec and Prop.*

Proof. Let $\mathrm{Id}_{\mathrm{alg}}$ denote the identity functor on $\mathrm{ISP}(L)$ and $\mathrm{Id}_{\mathrm{sp}}$ denote the identity functor on CohSp.

Define a natural transformation

$$\epsilon : \mathrm{Id}_{\mathrm{alg}} \to \mathrm{Prop} \circ \mathrm{Spec}$$

by

$$\epsilon_A = \Phi$$

for $\mathbb{ISP}(L)$, where Φ is defined in the proof of Theorem 38. It is verified by straightforward computation that ϵ is actually a natural transformation. By Theorem 38, ϵ is a natural isomorphism.

Define a natural transformation

$$\eta : \mathrm{Id}_{\mathrm{sp}} \to \mathrm{Spec} \circ \mathrm{Prop}$$

by

$$\eta_S = \Psi$$

for a coherent space S, where Ψ is defined in the proof of Theorem 42. It is verified by straightforward computation that η is actually a natural transformation. By Theorem 42, η is a natural isomorphism. $\square$

Let $\mathbf{2}_d$ denote the two-element distributive lattice. Since $\mathbb{ISP}(\mathbf{2}_d)$ coincides with the class of distributive lattices and since $\mathbf{2}_d$ is topologically dualizable with respect to the Alexandrov topology, Theorem 43 is a universal algebraic generalization of the Stone duality for distributive lattices.

D. Applications to Łukasiewicz algebras

In the following, we give sample applications to many-valued logics and their algebras (for basics of many-valued logics and algebras, see [5], [7], [14]).

Let $n \in \omega$ with $n > 1$ in this subsection.

Definition 44. *Let $\mathbf{n}$ denote*

$$\{0, 1/(n-1), 2/(n-1), ..., 1\}.$$

We equip $\mathbf{n}$ with all constants $(0, 1/(n-1), 2/(n-1), ..., 1)$ and with the operations $(\wedge, \vee, *, \wp)$ defined as follows:

$$\begin{aligned} x \wedge y &= \min(x, y) \\ x \vee y &= \max(x, y) \\ x * y &= \max(0, x + y - 1) \\ x \wp y &= \min(1, x + y). \end{aligned}$$

Note that $*$ and $\wp$ are defined as in Łukasiewicz n-valued logic. In the above definition, $\mathbf{n}$ is not equipped with $\neg$ or $\rightarrow$, which is because our aim here is to consider an n-valued version of distributive lattice.

The class of distributive lattices coincides with $\mathbb{ISP}(\mathbf{2})$, i.e., a distributive lattice can be defined as an isomorphic copy of a subalgebra of a powerset algebra $\mathbf{2}^X$ for a set X. Thus, it is natural to define an n-valued distributive lattice as an algebra in $\mathbb{ISP}(\mathbf{n})$, i.e., an n-valued distributive lattice is defined as an isomorphic copy of a subalgebra of an n-valued powerset algebra $\mathbf{n}^X$ for a set X:

Definition 45. *An n-valued distributive lattice is an algebra in $\mathbb{ISP}(\mathbf{n})$.*

A homomorphism of n-valued distributive lattices is a function which preserves the constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp)$.

DLat_n denotes the category of n-valued distributive lattices and homomorphisms of n-valued distributive lattices.

Note that 2-valued distributive lattices coincide with distributive lattices.

Applying Theorem 43, we can obtain a Stone-type duality for n-valued distributive lattices as follows.

Lemma 46. *Let $r \in \mathbf{n}$. Define*

$$\tau_r : \mathbf{n} \to \mathbf{n}$$

by letting $L = \mathbf{n}$ in Lemma 17. Then, τ_r is a term function of $\mathbf{n}$.

Proof. See [32, Section 1] (and also [35, Definition 3.7]). $\square$

We equip $\mathbf{n}$ with the Alexandrov topology.

Proposition 47. *In fact, $\mathbf{n}$ is topologically dualizable with respect to the Alexandrov topology, i.e.,*

$$\mathrm{Cont}_m(\mathbf{n}) = \mathrm{TermFunc}_m(\mathbf{n})$$

for any $m \in \omega$.

Proof. We first show that $\mathrm{Cont}_m(\mathbf{n}) \supset \mathrm{TermFunc}_m(\mathbf{n})$ for any $m \in \omega$, i.e., any term function of $\mathbf{n}$ is continuous. Since a composition of continuous functions is also continuous, it suffices to show that the constants $r \in \mathbf{n}$ and the operations $(\wedge, \vee, *, \wp)$ are continuous. Since a function on a singleton space is always continuous, the constants $r \in \mathbf{n}$ are continuous. We show that $* : \mathbf{n}^2 \to \mathbf{n}$ is continuous. This follows from the following fact:

$$*^{-1}\left(\uparrow \frac{k}{n-1}\right) = \bigcup_{i=0}^{n-1}\left(\left(\uparrow \frac{i}{n-1}\right) \times \left(\uparrow \frac{k-i+n-1}{n-1}\right)\right),$$

where we define $(\uparrow r) = \emptyset$ for $r > 1$. It is verified in similar ways that $(\wedge, \vee, \wp)$ are continuous.

Next we show that $\mathrm{Cont}_m(\mathbf{n}) \subset \mathrm{TermFunc}_m(\mathbf{n})$ for any $m \in \omega$. Let $f \in \mathrm{Cont}_m(\mathbf{n})$ for $m \in \omega$. For $i = 1, ..., m$, let $p_i : \mathbf{n}^m \to \mathbf{n}$ be the i-th projection function from $\mathbf{n}^m$ to $\mathbf{n}$. For $r \in \mathbf{n}$ and $i = 1, ..., m$, define $s_{i,r} \in \mathbf{n}$ as the least element of $p_i(f^{-1}(\uparrow r))$. Then we claim that

$$f(x_1, ..., x_m) = \bigvee_{r \in \mathbf{n}}\left(r \wedge \bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i)\right)$$

for any $(x_1, ..., x_m) \in \mathbf{n}^m$. To show this, suppose that $f(x_1, ..., x_m) = p$ for $p \in \mathbf{n}$. Since $f^{-1}(\uparrow r)$ is an open subset of $\mathbf{n}^m$, $p_i(f^{-1}(\uparrow r))$ is an open subset of $\mathbf{n}$ and so is upward closed. Thus, by Lemma 46, we have the following: For each $r \in \mathbf{n}$,

$$\bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i) = \begin{cases} 1 & \text{if } (x_1, ..., x_m) \in f^{-1}(\uparrow r) \\ 0 & \text{otherwise.} \end{cases}$$

Thus, since

$$(x_1, ..., x_m) \in f^{-1}(\uparrow q)$$

for any $q \in \mathbf{n}$ with $q \leq p$ and since

$$(x_1, ..., x_m) \notin f^{-1}(\uparrow q)$$

for any $q \in \mathbf{n}$ with $q > p$, we have

$$\bigvee_{r \in \mathbf{n}}\left(r \wedge \bigwedge_{i=1}^{m} \tau_{s_{i,r}}(x_i)\right) = p.$$

Hence the above claim holds. Therefore, it follows from Lemma 46 that f is a term function of $\mathbf{n}$. $\square$

By the above proposition and applying Theorem 43, we obtain the following Stone-type duality for n-valued distributive lattices.

Proposition 48. *DLat_n is dually equivalent to CohSp.*

In the last concluding section below, we also give brief remarks on Stone-type dualities for Heyting-type algebras, which obtain by restricting the dualities established thus far.

V. Concluding Remarks

In the present paper, we have proposed the concept of topological dualizability, and shown a non-Hausdorff extension of the primal duality theorem in universal algebra. The concept of topological dualizability is arguably the key to the universal algebraic unification of Stone-type dualities; here we have taken the first steps in demonstrating this thesis.

We conclude the paper with several remarks. The Stone duality for distributive lattices cuts down to the Stone duality for Heyting algebras. We can define a Heyting space S as a coherent space in which, for any Boolean combination B of compact open subsets of S, the interior of B is compact. The Heyting algebras, then, are categorically dual to the Heyting spaces. The notion of residuation (or relative pseudo-complement) plays an essential role in the concept of Heyting algebras. We can define residuation in a general context. Let A be an ordered algebra with a binary operation $*$.

- Then, A is called $*$-residuated iff, for all $x, y \in A$, the set of $z \in A$ such that $x * z \leq y$ has a greatest element, which is denoted by $x \rightarrow y$.

Assuming that L has a binary operation $*$ and L is $*$-residuated,

$$\mathbb{IRSP}(L)$$

is defined as the class of all isomorphic copies of $*$-residuated subalgebras of direct powers of L. Then,

$$\mathbb{IRSP}(\mathbf{2}_d)$$

coincides with the class of all Heyting algebras. Note that the class of all Heyting algebras cannot be equal to any $\mathbb{ISP}(M)$; logically rephrasing, intuitionistic logic cannot be many-valued logic (the same holds for modal logic; see [23], which proposes $\mathbb{ISP}_{\mathbb{M}}$). Now, our main theorem tells that:

- $\mathbb{IRSP}(L)$ is dual to the Heyting spaces (under the same conditions on L as per above); note that implication-preserving maps on the algebraic side correspond to open maps on the topological side.

This, in particular, yields the Stone-type duality for n-valued Heyting algebras, i.e.,

$$\mathbb{IRSP}(\mathbf{n})$$

where $\mathbf{n}$ is equipped with the Łukasiewicz operations as per above. We may call $\mathbb{IRSP}(\mathbf{n})$ the $*$-residuated variety generated by $\mathbf{n}$.

In the Hausdorff case, the primal duality theorem can be generalized to the quasi-primal duality theorem. In future work we shall explore whether the non-Hausdorff extension of the primal duality theorem can analogously be extended to the quasi-primal case; quasi-primality can be defined via topological dualizability. It is known that the extension is possible in a concrete example [29].

Stone-type dualities for propositional logics (see, e.g., [25], [27]) can actually provide models of the corresponding predicate logics (see [24], [26], [28]); the duality theory we have developed thus far, therefore, could be applied to categorical predicate logic. Categorical duality and categorical logic both build upon the methods of category theory, and yet they have been separated for some reason. So it would be significant to make a bridge between categorical logic and categorical duality theory. We shall take on this in our future work.

Acknowledgment

The author would like to thank his colleagues, both past and present, for the various discussions that have contributed to the developments of the materials presented in the paper in some way or other. Especially, the author is indebted to Kentaro Sato for his informative, helpful comments on an earlier draft of the paper. The author hereby acknowledges that the present work was financially supported by JST PRESTO (grant code: JPMJPR17G9) and JSPS KAKENHI (grant code: 17K14231).

References

[1] S. Abramsky, Domain theory in logical form, *Ann. Pure Appl. Logic* 51 (1991) 1-77.

[2] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia and A. Kurz, Bitopological duality for distributive lattices and Heyting algebras, *Mathematical Structures in Computer Science*, 20 (2010) 359-393.

[3] M. M. Bonsangue, Topological duality in semantics, *Electr. Notes Theor. Comput. Sci.* 8 (1998).

[4] C. Brink and I. M. Rewitzky, *A paradigm for program semantics: power structures and duality*, CSLI Publications, 2001.

[5] V. Boicescu , A. Filipoiu, G. Georgescu and S. Rudeanu, *Łukasiewicz-Moisil algebras*, North-Holland Publishing Co., 1991.

[6] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Springer-Verlag, 1981.

[7] R. L. O. Cignoli, I. M. L. D'Ottaviano and D. Mundici, *Algebraic foundations of many-valued reasoning*, Kluwer Academic Publishers, 1999.

[8] D. M. Clark and B. A. Davey, *Natural dualities for the working algebraist*, Cambridge University Press, 1998.

[9] L. Esakia, Topological Kripke models, *Soviet Mathematics Doklady* 15 (1974) 147-151.

[10] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Elsevier.

[11] M. Gehrke and J. Harding, Bounded lattice expansions, *J. Algebra* 239 (2001) 345-371.

[12] M. Gehrke, H. Nagahashi and Y. Venema, A Sahlqvist theorem for distributive modal logic, *Ann. Pure Appl. Logic* 131 (2005) 65-102.

[13] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove and D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, 2003.

[14] S. Gottwald, *A treatise on many-valued logics*, Research Studies Press, 2001.

[15] G. A. Grätzer, *Universal algebra*, Springer-Verlag, 1979.

[16] A. Grothendieck and J. Dieudonné, Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné) : I. Le langage des schémas, *Publications Mathématiques de l'IHÉS* 4 (1960) 5-228.

[17] R.-E. Hoffmann, On the sobrification remainder ${}^sX - X$, *Pacific J. Math.* 83 (1979) 145-156.

[18] T. K. Hu, Stone duality for primal algebra theory, *Math. Z.* 110 (1969) 180-198.

[19] T. K. Hu, On the topological duality for primal algebra theory, *Algebra Universalis* 1 (1971), 152-154.

[20] P. T. Johnstone, *Stone spaces*, Cambridge University Press, 1986.

[21] Y. Maruyama, Algebraic study of lattice-valued logic and lattice-valued modal logic, *Lecture Notes in Computer Science* 5378 (2009) 172-186.

[22] Y. Maruyama, A duality for algebras of lattice-valued modal logic, *Lecture Notes in Computer Science* 5514 (2009) 281-295.

[23] Y. Maruyama, Natural Duality, Modality, and Coalgebra, *Journal of Pure and Applied Algebra* 216 (2012) 565-580.

[24] Y. Maruyama, Full Lambek Hyperdoctrine: Categorical Semantics for First-Order Substructural Logics, *Lecture Notes in Computer Science* 8071 (2013) 211-225.

[25] Y. Maruyama, Categorical Duality Theory: Domains, Convexity, and the Distribution Monad, *Leibniz International Proceedings in Informatics* 23 (2013) 500-520.

[26] Y. Maruyama, Duality Theory and Categorical Universal Logic: With Emphasis on Quantum Structures, EPTCS 171 (2014) 100-112.

[27] Y. Maruyama, Topological Duality via Maximal Spectrum Functor, *Communications in Algebra* (in press; already published online).

[28] Y. Maruyama, Higher-Order Categorical Substructural Logic: Expanding the Horizon of Tripos Theory, *Lecture Notes in Computer Science* 12062 (2020) 187-203.

[29] Y. Maruyama and K. Sato, Intuitionistic many-valued logic and its geometric duality, *manuscript* (the same content was published in K. Sato's Ph.D. Thesis at Kobe University).

[30] I. D. Nel and R. G. Wilson, Epireflections in the category of T_0 spaces, *Fund. Math.* 75 (1972) 69-74.

[31] H. Ono, Substructural logics and residuated lattices - an introduction, *50 Years of Studia Logica*, Trends in Logic, Kluwer Academic Publishers, 21 (2003) 193-228.

[32] P. Ostermann, Many-valued modal propositional calculi, *Z. Math. Logik Grundlag. Math.* 34 (1988) 343-354.

[33] M. H. Stone, The representation of Boolean algebras, *Bull. Amer. Math. Soc.* 44 (1938) 807-816

[34] M. H. Stone, Topological representation of distributive lattices and Brouwerian logic, *Casopis pest. Mat. a Fys.* 67 (1937), 1-25.

[35] B. Teheux, A duality for the algebras of a Łukasiewicz $n+1$-valued modal system, *Studia Logica* 87 (2007) 13-36.

[36] S. Vickers, *Topology via logic*, Cambridge University Press, 1989