Coclustering algorithms are an alternative to classic one-sided clustering algorithms. Because of its ability to simultaneously cluster rows and columns of a dyadic data matrix, coclustering offers a higher value-added information: it offers column clusters besides row clusters, and the relationship between them in terms of co-clusters. Different structures of co-clusters are possible, and those that overlap in terms of rows or columns still represent an open question with room for improvements. In addition, while most related literature cites coclustering as a means of producing better results from one-side clustering, few initiatives study it as a tool capable of providing higher quality descriptive information about this clustering. In this paper, we present a new coclustering algorithm - OvNMTF, based on triple matrix factorization, which properly handle overlapped coclusters, by adding degrees of freedom for matrix factorization that enable the discovery of specialized column clusters for each row cluster. As a proof of concept, we modeled text analysis as a coclustering problem with column overlaps, assuming that given words (data matrix columns) are associated with over one document cluster (row cluster) because they can assume different semantic relationships in each association. Experiments on synthetic data sets show the OvNMTF algorithm reasonableness; experiments on real-world text data show its power for extracting high quality information.

Index Terms—coclustering, matrix factorization

I. INTRODUCTION

In clustering analysis, we use similarity between data to discover patterns that characterize them and their relationships [1]. This process organizes the data points into clusters to maximize the similarity between those in the same cluster and minimize similarity between those organized into distinct clusters [2]. One of the possible strategies to implement such a process is to partition the rows of a data matrix [3] where the rows represent the data under analysis and the columns represent the data descriptive attributes. In principle, the similarity analysis performed in clustering process considers all attributes, resulting in a holistic analysis [4].

Alternatively, in a co-clustering problems, we implement pattern discovery through similarity analysis applied simultaneously to data and attributes [5], i.e., data clustering is based on the distributions of attributes and attributes clustering is based on distribution of data [6]. Coclustering gives greater flexibility in defining clusters because it can perform partial similarity analysis and offer more precise data clustering. Besides, this process results in a cocluster structure that embed more detailed data clusters descriptions. This way of formulating descriptive data analysis has been promising for real problems characterized by subjective patterns interpretations, as in image and text data analysis [4], [5], [7]–[9].

In [7], the authors apply co-clustering in text data to explore the structure of coclusters and easily identify polysemic words and their context. Considering that coclusters represent relationships between row clusters (document clusters) and column clusters (word clusters) and recognizing that a word can take on different meanings depending on the context (document clusters), the usefulness of overlapping coclusters becomes noticeable. In this paper, we present a new algorithm OvNMTF (Overlapping Non-negative Matrix Tri-Factorization) capable of finding a coclustering solution that adequately addresses the coclusters overlap problem. This algorithm represents an evolution of our previous one, the BinOvNMTF (Overlapping Binary Non-negative Matrix Tri-Factorization) algorithm [10], which we proposed earlier and was restricted to making binary associations between row clusters and column clusters.

Matrix factorization methods have been widely applied in dyadic data analysis [5], [7], [11], [12], mainly for text data analysis. Non-Negative Matrix Factorization (NMF) is the basis of clustering and coclustering algorithms, such as: NMF for partial similarities-based data analysis [4], NMF for clustering [13], NMF for coclustering [14], Low Rank NMF [15], Semi-NMF (SNMF) [16], Orthogonal NNTF (ONMTF) [17], Graph regularized NMF (GNMF) [18], Fast NMTF (FNMTF) [8], BinOvNMTF [10], Word co-occurrence regularized NMTF (WC-NMTF) [19]. All of these algorithms were presented accompanied by experiments involving text data. The duality between rows and columns explored by coclustering algorithms has been shown to be effective in high dimensionality and sparse spaces [5], characteristic of the vector representation used for text data.

The research related to this class of algorithms shows that there is still room for improvement regarding the treatment of overlapping coclusters. Figure 1 illustrates this with three data matrices (I, II and III) with positive real values: the darker the

1In this case, “easily” means “with no further post-processing efforts”, since in text co-clustering results there is information about word clusters besides information about document clusters.
blue color the higher the value in the data matrix cell; and
the reconstruction of the original datasets by combining the
coclusters represented by submatrices in X denoted by $X_{K_pL_q}$,
being $k$ subsets $K \subseteq N$, $l$ subsets $L \subseteq M$, $p \in \{1, \ldots, k\}$
and $q \in \{1, \ldots, l\}$. In a solution to coclustering problems,
the cocluster $X_{K_pL_q}$ is a cluster of datapoints in $K_p$,
in view of the attributes in $L_q$. In this section, we present
the theoretical background concerning clustering and coclustering
problems. Such problems are detailed for one of the following
reasons: the problem is the basis for developing OvNMTF; the
algorithm that solves the problem was used in the experiments.

A. K-Means

The $k$-means clustering problem is one of the most studied
problems in the clustering field. This problem is classically
solved by applying the Lloyd algorithm, also called the $k$-
means algorithm [21]. The goal is to find $k$ prototype vectors
that quantize a dataset vector space, regarding a minimal
vector quantization error. Here, as in [22], we elaborate the
$k$-means clustering problem as the factorization of the data
matrix $X$ into two matrices, $U$ as a cluster indicator matrix and
$C$ as a prototype vector matrix, so that $X \approx UC$; $\|X - UC\|_F^2$
gives a reconstruction error of the original data matrix (see $F_1$).

$$F_1(U,C) = \min_{U,C} \sum_{i=1}^{n} \sum_{p=1}^{k} u_{ip} \|x_i - c_p\|^2$$
subject to $U \in \Psi^{n \times k}$, $C \in \mathbb{R}^{k \times m}$, $\sum_{p=1}^{k} u_{ip} = 1 \forall i$,
in which $\Psi = \{0,1\}$ and $\|\cdot\|_F$ is the Frobenius norm for
matrices.

B. NMF

Two reasons motivate the use of Non-Negative Matrix
Factorization (NMF) for clustering problems resolution: the
possibility of applying it as a data analysis method capable of
extracting knowledge about an object from the study of its parts [4],
therefore implementing partial similarity-based analysis; the adequacy of data representation used in various
clustering contexts to the factorization method requirements,
since such representations cover the relationship between pairs
of elements coming from two distinct finite sets (dyadic data)
[5]. For example, in text data clustering, two finite sets are
used to represent texts: documents and words. A positive
data matrix organizes information regarding the occurrence
or absence of a word in a document (a dyadic relationship),
allowing the use of the NMF method. NMF-based algorithms
have as input a data matrix $X \in \mathbb{R}^{n \times m}$, with $n$ rows
that constitutes a set of row vectors $N = \{x_1, \ldots, x_n\}$,
and $m$ columns that constitutes a set of columns vectors
$M = \{x_1, \ldots, x_m\}$. The relation between each row $x_i$
and each column $x_j$ is represented by $x_{ij}$, with $i \in \{1, \ldots, n\}$
and $j \in \{1, \ldots, m\}$.
and \( j \in \{1, \ldots, m\} \) \cite{4}. NMF can be seen as a double factor decomposition, in the form of the problem \( F_2 \):

\[
F_2(U, V) = \min_{U,V} \| X - UV^T \|_F^2,
\]

subj. to \( U \geq 0, V \geq 0, \)

in which \( U \in \mathbb{R}^{n \times k}_+, V \in \mathbb{R}^{m \times k}_+, \| \cdot \|_F \) is the Frobenius norm for matrices and \( \| X - UV^T \|_F^2 \) gives the reconstruction error. The Frobenius norm and the reconstruction error are also adopted in the other problems formulated in this and in the following sections.

According to \cite{7}, the columns in the factor matrix \( X \) correspond to basis vectors for the original data matrix reconstruction, while each row in the factor matrix \( U \) represents an encoding that gives the extent to which each basis vector will be used in the reconstruction process. Thus, the columns in \( V \) can be seen as the prototype vectors for row clusters extracted from the original data matrix.

C. BVD

Block Value Decomposition (BVD) searches for hidden block structures in a data matrix and can be used for analysis of dyadic data \cite{5}. It is suitable for implementing co-clustering solutions because BVD considers both data dimensions (rows and columns) simultaneously and explores their relationship by decomposing the data matrix \( X \in \mathbb{R}^{n \times m}_+ \) into three matrices (\( F_3 \)): \( U \) as a row coefficient matrix, \( S \) as a block-structured matrix and \( V \) as a column coefficient matrix.

\[
F_3(U, S, V) = \min_{U,S,V} \| X - USV^T \|_F^2,
\]

subj. to \( U \geq 0, V \geq 0, \)

in which \( U \in \mathbb{R}^{n \times k}_+, S \in \mathbb{R}^{k \times l} \) and \( V \in \mathbb{R}^{m \times l}_+ \).

The problem \( F_3 \) is an alternative to the problem \( F_2 \) because it uses triple factorization of matrices and can give us a co-clustering structure. The authors in \cite{5} provide the following interpretation: \( S \) is a compact representation of \( X \), the matrix \( US \) contains basis vectors for the columns in \( X \), the matrix \( SV^T \) contains basis vectors for rows in \( X \), and the factor matrices \( U \) and \( V \) denote the extent to which rows and columns are associated with their respective row/column clusters. Thus, prototype vectors can be extracted for both row and column clusters, and the notion of co-clusters can be explored by examining the information contained in the factor matrix \( S \), as in \cite{7}. The BVD problem restricted to positive data matrix, i.e., \( X \in \mathbb{R}^{n \times m}_+ \), results in the NBVD (Non-negative Block Value Decomposition) problem \cite{5}.

D. ONMTF

The problem \( F_4 \) was proposed in \cite{17}. In this problem, in addition to the nonnegativity constraints used in the \( F_2 \) and the triple factorization used in the \( F_3 \), two orthogonality constraints were added for row clusters and column clusters matrices, respectively: \( U^TU = I \) and \( V^TV = I \), in which \( I \) is an identity matrix. These constraints restrict the problem of factoring \( X \approx USV^T \) to a smaller number of possible solutions with more rigorous interpretation.

![Fig. 2. OvNMTF factorization process with five factor matrices](image)

\[
F_4(U, S, V) = \min_{U,S,V} \| X - USV^T \|_F^2,
\]

subj. to \( U \geq 0, S \geq 0, V \geq 0, U^TU = I, V^TV = I, \)

in which \( U \in \mathbb{R}^{n \times k}_+, S \in \mathbb{R}^{k \times l}_+ \) and \( V \in \mathbb{R}^{m \times l}_+ \).

III. OvNMTF

In this section, we formalize the Overlapping Non-negative Matrix Tri-Factorization (OvNMTF) problem and introduce an algorithm based on multiplicative update rules for solving it. The OvNMTF problem (Problem \( F_5 \)) is based on the assumptions established in NMF and BVD problems. We formulate (\( F_5 \)) as:

\[
F_5(U, S, V(1), \ldots, V(k)) = \min_{U,S,V(1),\ldots,V(k)} \| X - \sum_{p=1}^k I(p)SV^T(p) \|_F^2
\]

subj. to \( U \geq 0, S \geq 0, V(p) \geq 0, \forall p \)

in which \( U \in \mathbb{R}^{n \times k}_+, S \in \mathbb{R}^{k \times l}_+, V(p) \in \mathbb{R}^{m \times l}_+ \), \( p \in \{1, \ldots, k\} \) as the index for the set of matrices \( \{V(1), \ldots, V(k)\} \), \( I(p) \in \{0, 1\}^{k \times k} \) are constant selector matrices with zero in all cells except the unique cell \((i(p))_{pp}\) that assumes the value 1.

Each matrix \( SV^T(p) \) contains basis vectors for row clusters in \( X \). The set of selector matrices \( I(p) \) organizes the basis vectors by associating each one to a specific row cluster. Thus, in the minimization process, each row cluster is optimized with respect to one specific matrix \( SV^T(p) \). Similarly, the optimization of basis vectors for columns is oriented to specific column clusters. The set of matrices \( V(p) \) adds degrees of freedom in the factorization process. On the one hand, the association between columns and rows becomes more accurate, as illustrated in the experiments (Section IV); on the other hand, the time complexity of the algorithm used for factorizing \( X \) increases when compared, for example, to ONMTF Figure 2 shows a graphical visualization of the matrix factorization proposed in \( F_5 \).

The derivation of the multiplicative update rules to implement the minimization process for \( F_5 \) followed a gradient-based approach. The \( F_5 \) gradient calculation is as in \cite{7}, thus \( \nabla F_5 = [\nabla F_0]^+ - [\nabla F_0]^− \). Expanding \( F_5 \) using matrix trace properties [23]:

\[
\nabla F_5 = \frac{1}{2} \sum_{p=1}^k \left( [S^TV^T(p)S^TV^T(p)] - [S^TV^T(p)S^TV^T(p)]^− \right)
\]
The final gradients for \( U, S, V(\cdot), \forall \rho \in \{1, \ldots, k\} \) are:

\[
\begin{align*}
\nabla_U F_5 &= 2( -X \sum_{\rho=1}^{k} V(\rho) S^T I(\rho) + \sum_{\rho=1}^{k} U \sum_{\rho'=1}^{k} I(\rho') V(\rho') S^T I(\rho')) \\
\nabla_S F_5 &= 2( -\sum_{\rho=1}^{k} I(\rho) U^T X V(\rho) + \sum_{\rho=1}^{k} \sum_{\rho'=1}^{k} I(\rho') U^T U I(\rho') S^T V(\rho')) \\
\nabla_{V(\cdot)} F_5 &= 2( -X^T U I(\rho) S + \sum_{\rho=1}^{k} I(\rho') V(\rho') U^T U I(\rho') S) \\
\end{align*}
\]

Algorithm 1 implements the minimization process for \( F_5 \) by updating \( U, S \) and \( V \) using the multiplicative rule:

\[
X^{t+1} \leftarrow X^t \odot -\frac{\nabla_X F}{|\nabla_X F|}
\]

In this algorithm, \( t \) is an iteration counter, \( U^{(t)}, S^{(t)} \) and \( V^{(t)} \) are respectively the \( U, S \) and \( V(\cdot) \) matrices in the \( t \)th iteration, \( U(0,1) \in [0,1] \) is an uniformly distributed number generator, \( \odot \) is the Hadamard product and stop conditions as a maximum number of iterations \( t_{\text{max}} \) or the reconstruction error convergence according to the limit \( \epsilon \) (free parameter).

IV. EXPERIMENTS AND RESULTS ANALYSIS

We carried out two types of experiments:

- Experiment #1 (Section IV-A): We carried out this experiment on synthetic datasets to verify the reconstruction capacity provided by the OvNMTF algorithm in the presence of row or column overlapping cocluster structures. Here, the K-means algorithm is a reference for cluster capacity analysis; the ONMFT algorithm, based on multiplicative update rules [7], was chosen for illustration because of its similarity to OvNMTF algorithm.

- Experiment #2 (Section IV-B): We carried out this experiment on real-world text datasets to test the cluster discovery capacity of OvNMTF, and its power to produce information about topics (clusters of words) and how these topics describe clusters of documents. The cluster discovery evaluation was performed based on the Rand Index (RI) [24]. The evaluation of information production capacity was performed through the analysis of the words that made up each cocluster. Since the evaluation comprises cocluster analysis, only the results produced by ONMFT and by OvNMTF were analyzed.

A. Experiment #1

a) Datasets: Synthetic datasets were built on three of the eight data structure types [25]. These datasets are shown in the first column of Figure 1 and are related to: (I) coclusters with exclusive rows and columns; (II) coclusters with exclusive rows and overlapping columns; (III) coclusters with exclusive columns and overlapping rows. Structure I was chosen to show the effectiveness of the algorithm in solving the classic clustering problem. The structures II and III were chosen to see the effectiveness of the algorithm in solving the classic clustering problem.

\[\sum_{\rho=1}^{k} I(\rho') V(\rho') U^T U I(\rho') S = 2 \sum_{\rho=1}^{k} V(\rho') S^T I(\rho') U^T U I(\rho') S + \sum_{\rho' \notin \{1, \ldots, k\}} V(\rho') S^T I(\rho') U^T U I(\rho') S + \sum_{\rho \notin \{1, \ldots, k\}} V(\rho) S^T I(\rho) U^T U I(\rho) S = 2 \sum_{\rho=1}^{k} V(\rho) S^T I(\rho) U^T U I(\rho) S \]

In [25], the biclustering problem is covered. Biclustering and coclustering are similar problems. Although each has its own definitions, the latter can be seen as an extension to the former [26], [27].
Algorithm 1 OvNMTF algorithm

1: input: data $X$, number of rows clusters $k$, number of columns clusters $l$, max iterations $t_{max}$
2: initialize: $U^{(0)} \leftarrow \mathcal{U}(0,1), S^{(0)} \leftarrow \mathcal{U}(0,1), V^{(0)}_p \leftarrow \mathcal{U}(0,1), \forall p \in T \leftarrow 0$
3: while (no convergence) and ($t \leq t_{max}$) do
4:  $U^{(t+1)} \leftarrow U^{(t)} \odot \frac{\sum_{p=1}^{k} XV^{(t)}_{(p)}S^{(t)} X^{(t)} I(p)}{\sum_{p=1}^{k} \sum_{p'=1}^{k} U^{(t)} I(p) V^{(t)}_{(p')} S^{(t)} X^{(t)} I(p')}$
5:  for $p \leftarrow 1 : k$ do
6:      $V^{(t+1)}_{(p)} \leftarrow V^{(t)}_{(p)} \odot \frac{X^T U^{(t+1)} I(p) S^{(t)}}{\sum_{p'=1}^{k} V^{(t)}_{(p')} S^{(t)} I(p') U^T I(p) S}$
7:  end for
8:  $S^{(t+1)} \leftarrow S^{(t)} \odot \frac{\sum_{p=1}^{k} I(p) U^{(t+1)T} XV^{(t+1)}_{(p)}}{\sum_{p=1}^{k} \sum_{p'=1}^{k} I(p) U^{(t+1)T} U^{(t+1)} I(p') S^{(t)} X^{(t+1)} V^{(t+1)}_{(p')}}$
9:  $t \leftarrow t + 1$
10: end while
11: return $U^{(t)}, S^{(t)}, V^{(t)}_1, \ldots, V^{(t)}_k$

### Table I

<table>
<thead>
<tr>
<th>Reconstruction capacity:</th>
<th>ok - good reconstruction, correct information on overlapping rows/columns</th>
<th>x - good reconstruction, no information on overlapping columns</th>
<th>+ - poor reconstruction, partial information on overlapping rows/columns</th>
<th>o - beyond the scope of the algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>base (I)</strong></td>
<td>ok</td>
<td>ok</td>
<td>ok</td>
<td>ok</td>
</tr>
<tr>
<td><strong>base (II)</strong></td>
<td>ok, x</td>
<td>+</td>
<td>ok</td>
<td>ok</td>
</tr>
<tr>
<td><strong>base (III)</strong></td>
<td>o</td>
<td>+</td>
<td>ok</td>
<td>ok</td>
</tr>
</tbody>
</table>

*b) Parameters setup:* The following parameters setup was set: $k = 3$ for *k-means*, according to the actual number of clusters in the dataset, and $k = l = 3$ for ONMTF and OvNMTF, according to the actual number of row clusters and column clusters in the dataset; random initialization for $C$ in *k-means*, $U$, $S$, $V$ in ONMTF, and $U$, $S$ and $V_k$ in OvNMTF; stop conditions based on the maximum number of iterations (300 for *k-means*, 1000 for ONMTF and OvNMTF) and $\epsilon = 1e-04$ for reconstruction error convergence in ONMTF and OvNMTF; 10 runs of each algorithm in each dataset.

c) *Reconstruction capacity:* The reconstructions analyzed in this section concern the best result obtained for each algorithm in each dataset. Table I shows a summary of the results, and Figure 3 allows us a visual analysis.

In Figure 3, the first dataset (I) represents a classical clustering problem, without overlapping rows/columns. All algorithms tested could produce good reconstructions of the original dataset. The dataset II was properly reconstructed by *k-means* because if we analyze the problem represented in this dataset from the row clusters standpoint, it is equivalent to the classical clustering problem. In the datasets II and III, the ONMTF algorithm could not correctly associate rows/columns with over one row/column cluster with the chosen set of parameters. The OvNMTF algorithm properly reconstructed the datasets II and III, since it could properly organize the degrees of freedom conferred by the multiple matrices $V_k$.

The reconstruction errors for the dataset I are similar. For the dataset II, although the algorithm *k-means* offers a good reconstruction, with error similar to that produced by OvNMTF, it cannot produce information about column clusters, i.e. having a good reconstruction capacity does not guarantee good descriptions for data clusters, the reconstruction error...
produced by ONMTF are about seven times bigger than that produced by OvNMTF. In the dataset III, the reconstruction error produced by ONMTF are about ten times bigger than that produced by OvNMTF, and the $k$-means algorithm produced a very high reconstruction error. The ONMTF reconstruction errors can be improved if the parameter $l$ is set to higher values. However, in such a case, the discovered knowledge on column clusters will differ from the a priori knowledge.

B. Experiments #2

a) Datasets: The text data analysis has been chosen to illustrate the accuracy and added value of the information that OvNMTF can extract. This experiment was carried out on three text datasets, as in [10]; Table II lists quantitative information about these datasets:

1) Portuguese news items collection (PNC): A collection of Portuguese language news items. Each news item consists of an url, title, subtitle, body and topic in which the item was manually classified. The news items are distributed on 13 unevenly classes.

2) Portuguese news items collection (PNC toy): A subset of the PNC collection. It comprises 300 news items distributed in a balanced way in three topics (sports, games and activities for young people).

3) NIPS14-17 (NIPS): A dataset related to scientific papers published in the Neural Information Processing Systems Congress, 2001-2003 - volumes 14-17. The complete dataset comprises scientific papers published in 18 volumes, however, only the papers in the volumes 14 to 17 are labeled. Such documents are organized on topics that cover 13 technical areas and are unevenly distributed; documents from nine most voluminous areas were used.

b) Parameters setup: The following parameters setup was established: $k$ was set according to the actual number of classes/topics in each dataset and $l$ assumes a list of values, since there is no a priori knowledge about the actual number of word clusters, thus:

- PNC TOY: $k = 3 \ and \ l = \{2, 3, 4, 5, 6\}$;
- PNC: $k = 13 \ and \ l = \{7, 10, 13, 16, 19\}$;
- NIPS: $k = 9 \ and \ l = \{6, 9, 12, 15, 18\}$;

random initialization for factor matrices; stop conditions based on the maximum number of iterations (1,000 for PDN TOY, 10,000 for PNC and NIPS) and $\epsilon = 1e^{-04}$ for reconstruction error convergence; 10 runs of each combination; algorithm versus dataset versus $k, l$ combination values versus vector representations for text data.

c) Vector space model for text data: We chose a count-based distributional semantics model to build the vector space model for text data [28], [29]. It uses the text data in each document (news item) to produce a document-word matrix. In the preprocessing phase, stopwords were dropped, documents’ token were stemmed [30] and then, $tf$ and $tfidf$ scores [31] and their respective normalized versions ($tf_{norm}$, $tfidf_{norm}$) were computed for each document.

d) Clustering results: We evaluate clustering quality by using the Rand Index (RI). The results are presented in terms of: average RI for each combination of vector representations and algorithms; distribution of the RI values for all runs of each algorithm. Table III presents the best average RI values for each algorithm in the different vector representations; Figure 4 shows the distributions of the RI values.

OvNMTF presented the best average RI values in all cases shown on Table III. As the complexity of the problem in terms of the desired number of row clusters increased, all algorithm’s performance declined and the superiority of the algorithm OvNMTF over the others decreased. However, considering another aspect of the complexity of the problem — the sparsity of the data matrix (see Table II), OvNMTF has a positive highlight. These results reveal the good clustering capability presented by the algorithm introduced in this paper.

Considering all algorithms runs, OvNMTF stands out for its stability. The RI distributions graphs shown in Figure 4 illustrate that the variability in the results presented by the OvNMTF algorithm is smaller than the variability presented by the other algorithms. Specifically, for PNC and PNC toy datasets, the algorithm OvNMTF also achieves the best maximum RI values and concentrate the results on high RI values. For the NIPS dataset, the algorithm ONMTF has better maximum RI values, but concentrates most of its RI values around the lowest values presented by the algorithm OvNMTF.

The vector representation used meant a sensitivity issue for cluster quality in the case of the PNC toy dataset (the dataset with the most sparsity and least complexity in the number of row clusters). For this dataset, $k$-means and OvNMTF performed better with the vector representation purely based on

---

**TABLE II QUANTITATIVE INFORMATION ON TEXT REAL-WORLD DATASETS**

<table>
<thead>
<tr>
<th>DATASET</th>
<th>PNC</th>
<th>PNC toy</th>
<th>NIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td># unique terms</td>
<td>6,710</td>
<td>36,342</td>
<td>6,881</td>
</tr>
<tr>
<td># terms</td>
<td>69,301</td>
<td>1,187,334</td>
<td>746,826</td>
</tr>
<tr>
<td># documents</td>
<td>300</td>
<td>4,575</td>
<td>555</td>
</tr>
<tr>
<td># row clusters</td>
<td>3</td>
<td>13</td>
<td>9</td>
</tr>
<tr>
<td>% zeros in data matrix</td>
<td>0.997</td>
<td>0.993</td>
<td>0.804</td>
</tr>
</tbody>
</table>

**TABLE III AVERAGE RI FOR TEXT DATASETS, WITH $k = 3$ FOR PNC TOY, $k = 13$ FOR PNC AND $k = 9$ FOR NIPS AND THE BEST $l$ VALUES FOR EACH COMBINATION (DATASET $\times$ ALGORITHM $\times$ VECTOR REPRESENTATION)**

<table>
<thead>
<tr>
<th>COMBINATION</th>
<th>PNC TOY</th>
<th>PNC</th>
<th>NIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>k = 3</td>
<td>$tf_{norm}$</td>
<td>$tf_{norm}$</td>
<td>$tf_{norm}$</td>
</tr>
<tr>
<td>l = 5</td>
<td>$0.7086$</td>
<td>$0.6479$</td>
<td>$0.7466$</td>
</tr>
<tr>
<td>l = 3</td>
<td>$0.7862$</td>
<td>$0.7086$</td>
<td>$0.7466$</td>
</tr>
<tr>
<td>$tf$</td>
<td>$0.3869$</td>
<td>$0.1571$</td>
<td>$0.3455$</td>
</tr>
<tr>
<td>$tfidf$</td>
<td>$0.2784$</td>
<td>$0.1279$</td>
<td>$0.3534$</td>
</tr>
<tr>
<td>$tfidf_{norm}$</td>
<td>$0.2750$</td>
<td>$0.1184$</td>
<td>$0.3554$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NIPS</th>
<th>$tf$</th>
<th>$tf_{norm}$</th>
<th>$tfidf$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.1573$</td>
<td>$0.1579$</td>
<td>$0.1527$</td>
<td>$0.1519$</td>
</tr>
<tr>
<td>$0.1641$</td>
<td>$0.1611$</td>
<td>$0.1614$</td>
<td>$0.1519$</td>
</tr>
<tr>
<td>$0.1711$</td>
<td>$0.1712$</td>
<td>$0.1711$</td>
<td>$0.1519$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>COMBINATION</th>
<th>PNC TOY</th>
<th>PNC</th>
<th>NIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>k = 3</td>
<td>$tf_{norm}$</td>
<td>$tf_{norm}$</td>
<td>$tf_{norm}$</td>
</tr>
<tr>
<td>l = 6</td>
<td>$0.1571$</td>
<td>$0.1579$</td>
<td>$0.1527$</td>
</tr>
<tr>
<td>l = 15</td>
<td>$0.1641$</td>
<td>$0.1614$</td>
<td>$0.1519$</td>
</tr>
<tr>
<td>$tfidf$</td>
<td>$0.1368$</td>
<td>$0.1442$</td>
<td>$0.1519$</td>
</tr>
<tr>
<td>$tfidf_{norm}$</td>
<td>$0.1519$</td>
<td>$0.1318$</td>
<td>$0.1519$</td>
</tr>
</tbody>
</table>
e) Semantic features extraction: To extract the semantic features, we follow the ideas presented in [4] and [7]. In [4], the authors argue that it makes more sense for a document to be associated with a small set of topics than one or all possible topics; [7] argues that words have different meanings depending on the context in which they are used, and factor matrices allow us to identify word clusters with words in common but associated with different contexts.

Due to the addition of multiple matrices $V(p)$ in OvNMTF, and the use of each one as an independent basis for document clusters, each one specializes in generating topics for one document cluster. The association of this characteristic with the ideas mentioned above motivate our hypothesis that the feature extraction supported by OvNMTF will be more accurate than that offered by ONMTF and will trustingly express more reliable descriptions for document clusters. To test our hypothesis, we compared the best runs (in terms of RI) of the ONMTF and OvNMTF algorithms, for the PNC toy dataset.

Considering the RI values distribution, OvNMTF suggests greater stability, which puts it at an advantage in terms of vector representation independence.

f) Semantic features extraction: To extract semantic features to each cocluster of interest, we have the highest values. Thus, to extract semantic features to each cocluster of interest, we analyze them in terms of their 20 most relevant words. The relevance of the words is given by the values in the matrix $V$ that associate them with the word clusters. Figure 5 shows the words relevance through word clouds\(^4\). In the word cloud, the bigger the word, the bigger its relevance.

The word GAMES is the only one that appears in more than one cluster (WC#1 and WC#4). An informal interpretation of the words organization in the clusters allows us to infer that: WC#2 and WC#5 describe the news items in DC#1 as “soccer news”; WC#4 describes the news items in DC#2 as “e-sport news”; WC#1 describe the news items in DC#3 as “extreme sports news”; and WC#3 does not bring easily interpretable semantic information, imposing a degree of uncertainty about the definition of the topic referent to DC#3.

A normalized factor matrix $S$, resultant from the OvNMTF factorization, has each row associated with one matrix $S_{(p)}$, which determines the subset of word clusters optimized for one document cluster associated with that row in $S$. Thus, in the run under analysis (PCN toy dataset, $k = 3$, $l = 2$ and $t \_tfidf$), there are two word clusters for each one of the three document clusters. For DC#1, WC#1 = 0.38 and WC#2 = 0.62; for DC#2, WC#3 = 0.46 and WC#4 = 0.54; and for DC#3, WC#5 = 0.94 and WC#6 = 0.06. The semantic features extraction was carried out and the wordclouds are shown in Figure 6.

From wordclouds, we can give the following description for document clusters: WC#1 and WC#2 describe the news items in DC#1 as “e-sport news”; WC#3 and WC#4 describe the news items in DC#2 as “extreme sports news”; and WC#5 and WC#6 describe the news items in DC#3 as “soccer news”. As

\[^4\]Words used in word clouds were translated from English to Portuguese. GAME means JOGO; GAMES refers to the use of the English language word within the Portuguese language texts (commonly in the context of e-sports).
expected, the description of each document cluster is similar to that obtained from the ONMTF but, we declare two advantages arising from the OvNMTF coclustering framework:

- The arrangement of a subset of word clusters (for each document clusters) is completely independent of another subset, thus, OvNMTF can use the same word in different word clusters subsets more often. This makes it possible to better identify polysemic words.
- Each subset of word cluster concerns only one, document cluster. Thus, the word clusters analysis give us more accurate descriptions for document clusters. Moreover, the actual role of a polysemic word can be more easily identified in a context of more accurate interpretation.

V. CONCLUSION

In this paper, we formalize the overlapping non-negative matrix triple factorization problem (the OvNMTF problem), as an alternative to NMF-based problems. OvNMTF was designed to naturally deal with overlapping row and columns in coclustering analysis. To implement the OvNMTF problem minimization process, we derived an algorithm based on multiplicative update rules (the OvNMTF algorithm). The reasonableness of the algorithm was tested on synthetic data sets; its usefulness and its power to extract information were attested in real-world text datasets. OvNMTF produced better clustering results than $k$-means, and it was superior to ONMTF considering the experimentation model and the limits imposed for the variation of the parameters $k$ and $l$. In terms of semantic features extraction, the OvNMTF algorithm has also brought benefits as it allows us to infer more accurate descriptions for each document cluster. The results show that the proposed algorithm implements an optimization process that allows the use of degrees of freedom to efficiently compose coclusters. However, the OvNMTF algorithm is more complex in terms of runtime than ONMTF due to the larger number of matrices involved in the factorization process. Therefore, its use in real-world problems needs to consider this cost. This drawback is the point of attention for the next steps of this research.

REFERENCES

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