A NEWMARK FMD SUB-CYCING ALGORITHM

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Abstract: This paper presents a Newmark method based sub-cycling algorithm, which is suitable for solving the condensed flexible multi-body dynamic (FMD) problems. Common-step update formulations and sub-step update formulations for quickly changing variables and slowly changing variables of the FMD are established. Stability of the procedure is checked by checking energy balance status. Examples indicate that the sub-cycling is able to enhance the computational efficiency without dropping results accuracy greatly.

1 INTRODUCTION

Flexible multi-body system (FMS) can be applied in various domains such as space flight, automobiles and robots. In these domains, accurate and efficient computation of the flexible bodies undergoing large overall motion is important for design and control of the system (Huang and Shao, 1996).

Conventional integration methods, such as the Newmark algorithm, the Runge-Kutta algorithm and the Gear algorithm and *etc* (Dan Negrut, et al, 2005), were widely applied to solve FMD equations.

Sub-cycling was proposed by Belytschko T. *et al* (Belytschko T.,1978). Mark *et al* (Mark, 1989) applied the method to simulate an impact problem and computational cost of the sub-cycling was only 15% of that without sub-cycling. Gao H *et al*. (Gao, 2005) used sub-cycling to simulate auto-body crashworthiness and declared that the cost is only 39.3% of no sub-cycling. Tamer *et al* (Tamer, 2003) pointed out that the FMD sub-cycling methods have not yet been presented in literatures.

2 SUB-CYCLING FOR FMD

A sub-cycling is constituted by two types of cycles, main-cycle and sub-cycle. The key for sub-cycling is to appropriately treat with interactions of nodes on the interface (Daniel, 1997).

2.1 Condensed Model of FMD

A flexible body is displayed in figure 1. Definition of variables can be referenced in the literature (Lu, 1996).

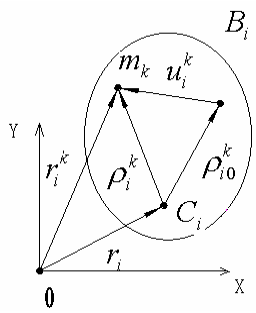

Figure 1: A spatial arbitrary flexible body.

The FMD equation can be established by means of the Lagrange formation.

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{C}_\mathbf{q}^T\boldsymbol{\lambda} = \mathbf{Q}_\mathbf{F} + \mathbf{Q}_\mathbf{v} \cdots (2-1)$$

$$\mathbf{C}(\mathbf{q}, t) = 0 \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2-2)$$

Thereinto, **M** is a general mass matrix, **K** is a general stiffness matrix, **C**(**q**,t) is the constrains, $\mathbf{Q_F}$ and $\mathbf{Q_V}$ are general external load and general centrifugal load. Λ is the Lagrange multiplier. By means of two times of differentials of the constrain equation, an augmented FMD equation can be obtained as below.

$$\begin{bmatrix} \mathbf{M}_\mathbf{r} & \mathbf{C}_\mathbf{q}^T \\ \mathbf{C}_\mathbf{q} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_\mathbf{F}^* \\ \mathbf{Q}_\mathbf{C} \end{bmatrix} \cdots\cdots\cdots (2-3)$$

Due to the constraints, variables in (2-3) are independent. By decomposition, a condensed FMD equation can be established as following (Haug, 1989).

$$\hat{\mathbf{M}}_i(\mathbf{q}_i,\mathbf{q}_d,t)\ddot{\mathbf{q}}_i=\hat{\mathbf{Q}}_i(\dot{\mathbf{q}}_i,\mathbf{q}_i,\dot{\mathbf{q}}_d,\mathbf{q}_d,t)\cdots(2-4)$$

Thereinto:

$$\hat{\mathbf{M}}_i=\mathbf{M}_{ii}-\mathbf{M}_{id}\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q}_i}-\mathbf{C}_{\mathbf{q}_d}^{T}\left(\mathbf{C}_{\mathbf{q}_d}^{-1}\right)^{T}\left[\mathbf{M}_{di}-\mathbf{M}_{dd}\mathbf{C}_{\mathbf{q}_d}^{-1}\mathbf{C}_{\mathbf{q}_i}\right]$$
$$\hat{\mathbf{Q}}_i=\mathbf{Q}_{\mathbf{F}i}^{*}-\mathbf{M}_{id}\mathbf{C}_{\mathbf{q}_d}^{-1}\gamma-\mathbf{C}_{\mathbf{q}_i}^{T}\left(\mathbf{C}_{\mathbf{q}_d}^{-1}\right)^{T}\left[\mathbf{Q}_{\mathbf{F}d}^{*}-\mathbf{M}_{dd}\mathbf{C}_{\mathbf{q}_d}^{-1}\gamma\right]$$

Equation (2-4) is a pure differential formation. It is suitable for sub-cycling (Lu, 1996).

2.2 Newmark Integration for FMD

The Newmark integration is as following. There, $\mathbf{q}_t$ denotes value of the general variable at time t. Δt is the step size. β and γ are adjustable parameters.

$$\mathbf{q}_{t+\Delta t}=\mathbf{q}_t+\dot{\mathbf{q}}_t\Delta t+\frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{\mathbf{q}}_t+2\beta\ddot{\mathbf{q}}_{t+\Delta t}\right]\cdots(2-5)$$
$$\dot{\mathbf{q}}_{t+\Delta t}=\dot{\mathbf{q}}_t+\Delta t\left[(1-\gamma)\ddot{\mathbf{q}}_t+\gamma\ddot{\mathbf{q}}_{t+\Delta t}\right]\cdots\cdots\cdots\cdots\cdots(2-6)$$

Define symbols below.

$$a_0=\frac{1}{\beta\Delta t^2}\quad a_1=\frac{\gamma}{\beta\Delta t}\quad a_2=\frac{1}{\beta\Delta t}\quad a_3=\frac{1}{2\beta}-1$$
$$a_4=\frac{\gamma}{\beta}-1\quad a_5=\frac{\Delta t}{2}\left(\frac{\gamma}{\beta}-2\right)\quad a_6=\Delta t(1-\gamma)\quad a_7=\gamma\Delta t$$

The dynamic equation at time t+Δ t can be established below.

$$\hat{\mathbf{M}}_i\left(\mathbf{q}_i^{t+\Delta t},\mathbf{q}_d^{t+\Delta t}\right)\left[a_0\left(\mathbf{q}_i^{t+\Delta t}-\mathbf{q}_i^{t}\right)-a_2\dot{\mathbf{q}}_i^{t}-a_3\ddot{\mathbf{q}}_i^{t}\right]$$
$$=\hat{\mathbf{Q}}_i\left({}^{(1)}\mathbf{q}_i^{t+\Delta t},{}^{(1)}\dot{\mathbf{q}}_i^{t+\Delta t},{}^{(1)}\mathbf{q}_d^{t+\Delta t},{}^{(1)}\dot{\mathbf{q}}_d^{t+\Delta t}\right)\cdots(2-7)$$

Finally, we can get the results.

$$\mathbf{q}_i^{t+\Delta t}=\left[a_0\hat{\mathbf{M}}_i^{t+\Delta t}\right]^{-1}\hat{\mathbf{Q}}_i^{t+\Delta t}+\mathbf{q}_i^{t}+\frac{a_2}{a_0}\dot{\mathbf{q}}_i^{t}+\frac{a_3}{a_0}\ddot{\mathbf{q}}_i^{t}\cdots(2-8)$$

Equation (2-8) need be solved iteratively. The iteration process is as following.

$${}^{(k)}\mathbf{q}_i^{t+\Delta t}=\left[a_0\,{}^{(k-1)}\hat{\mathbf{M}}_i^{t+\Delta t}\right]^{-1}{}^{(k-1)}\hat{\mathbf{Q}}_i^{t+\Delta t}$$
$$+\mathbf{q}_i^{t}+\frac{a_2}{a_0}\dot{\mathbf{q}}_i^{t}+\frac{a_3}{a_0}\ddot{\mathbf{q}}_i^{t}\cdots\qquad(2-9)$$

Thereinto, the top left corner marks represent the number of the iteration step. Start-up initialisation of iteration can be set-up below.

$${}^{(1)}\mathbf{q}_{t+\Delta t}=\mathbf{q}_t+\dot{\mathbf{q}}_t\Delta t+\frac{\Delta t^2}{2}\left[(1-2\beta)\ddot{\mathbf{q}}_t+2\beta^{(1)}\ddot{\mathbf{q}}_{t+\Delta t}\right]\cdots(2-10)$$

2.3 Newmark Sub-cycling for FMD

For simplification, all variables are separated into two categories. The smaller step size is set to be Δt. The larger step size is set to be $n\Delta t$. n is a positive integer. Thus, $\ddot{\mathbf{q}}$ is expressed as a decomposed format. And the condensed FMD formula can be decomposed as a block matrix format as following.

$$\begin{bmatrix}\hat{\mathbf{M}}_{LL}^{t} & \hat{\mathbf{M}}_{LS}^{t}\\ \hat{\mathbf{M}}_{SL}^{t} & \hat{\mathbf{M}}_{SS}^{t}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}_L\\ \ddot{\mathbf{q}}_S\end{bmatrix}=\begin{bmatrix}\hat{\mathbf{Q}}_L^{t}\\ \hat{\mathbf{Q}}_S^{t}\end{bmatrix}\cdots(2-12)$$

The subscript symbol, L and S, represent the larger step size and the smaller step size. **M** and **Q** are the general mass matrix and the general external force. According to balance status of the FMS at $t+\Delta t$ and $t+n\Delta t$, two groups of equations can be obtained as following.

$$\hat{\mathbf{M}}_{LLn}\ddot{\mathbf{q}}_{Ln}+\hat{\mathbf{M}}_{LSn}\ddot{\mathbf{q}}_{Sn}=\hat{\mathbf{Q}}_{Ln}\cdots(2-13)$$
$$\hat{\mathbf{M}}_{SL1}\ddot{\mathbf{q}}_{L1}+\hat{\mathbf{M}}_{SS1}\ddot{\mathbf{q}}_{S1}=\hat{\mathbf{Q}}_{S1}\cdots(2-14)$$

Thereinto, $\ddot{\mathbf{q}}_{Ln}$ and $\ddot{\mathbf{q}}_{Sn}$ are accelerations at $t+n\Delta t$. $\ddot{q}_{L1}$ and $\ddot{q}_{S1}$ are accelerations at $t+\Delta t$. The general mass matrix and the general external force are defined similarly. Define symbols below.

$$a_0^1=a_0,a_0^n=\frac{a_0^1}{n^2},a_2^1=a_2,a_2^n=\frac{a_2^1}{n},a_3^1=a_3,a_3^n=a_3^1$$

In order to compute interaction between the coupling variables at the common update, $\mathbf{q}_{Sn}$ and $\mathbf{q}_{L1}$ need be estimated simultaneously. Similar to the method (Daniel, 1997), $\mathbf{q}_{L1}$ can be linearly interpolated and $\mathbf{q}_{Sn}$ can be linearly extrapolated by means of the trapezoid rule. The formats of the interpolations are expressed below.

$$\mathbf{q}_{L1}=\frac{n-1}{n}\mathbf{q}_{L0}+\frac{1}{n}\mathbf{q}_{Ln}\cdots\cdots\cdots\cdots(2-15)$$
$$\mathbf{q}_{Sn}=n\left(\mathbf{q}_{Sn}-\mathbf{q}_{S0}\right)+\mathbf{q}_{S0}\cdots\cdots\cdots(2-16)$$

Hence action of the slowly changing variables to the rapidly changing variables is calculated below.

$$-\mathbf{F}_{SL1}=\hat{\mathbf{M}}_{SL1}\left[\frac{a_0^1}{n}\left(\mathbf{q}_{Ln}-\mathbf{q}_{L0}\right)-a_2^1\dot{\mathbf{q}}_{L0}-a_3^1\ddot{\mathbf{q}}_{L0}\right]\cdots(2-17)$$

The action of the rapidly changing variables to the slowly changing variables is calculated below.

$$-\mathbf{F}_{LSn}=\hat{\mathbf{M}}_{LSn}\left[na_0^n\left(q_{S_1}-\mathbf{q}_{S_0}\right)-a_2^n\dot{\mathbf{q}}_{S_0}-a_3^n\ddot{\mathbf{q}}_{S_0}\right]\cdots(2-18)$$

Imposing equations (2-17) and (2-18) into equations (2-13) and (2-14), and co-multiplying a number, n, to the left hand and the right hand of equation (2-13), the common update formula of the slowly changing variables and the rapidly changing variables can be obtained below.

$$a_0\begin{bmatrix}\hat{\mathbf{M}}_{SS1} & \hat{\mathbf{M}}_{SL1}\\ \hat{\mathbf{M}}_{LSn} & \hat{\mathbf{M}}_{LLn}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{S1}\\ \mathbf{q}_{Ln}/n\end{bmatrix}=\begin{bmatrix}\hat{\mathbf{Q}}_{S1}\\ n\hat{\mathbf{Q}}_{Ln}\end{bmatrix}+a_0\begin{bmatrix}\hat{\mathbf{M}}_{SS1} & \hat{\mathbf{M}}_{SL1}\\ \hat{\mathbf{M}}_{LSn} & \hat{\mathbf{M}}_{LLn}\end{bmatrix}\begin{bmatrix}\mathbf{q}_{S0}\\ \mathbf{q}_{L0}/n\end{bmatrix}$$
$$+a_2\begin{bmatrix}\hat{\mathbf{M}}_{SS1} & \hat{\mathbf{M}}_{SL1}\\ \hat{\mathbf{M}}_{LSn} & \hat{\mathbf{M}}_{LLn}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{q}}_{S0}\\ \dot{\mathbf{q}}_{L0}\end{bmatrix}+a_3\begin{bmatrix}\hat{\mathbf{M}}_{SS1} & \hat{\mathbf{M}}_{SL1}\\ n\hat{\mathbf{M}}_{LSn} & n\hat{\mathbf{M}}_{LLn}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{q}}_{S0}\\ \ddot{\mathbf{q}}_{L0}\end{bmatrix}\cdots(2-19)$$

In order to get the sub-step update formula, the following equation need be calculated at $t+(i+1)\Delta t$, $i=1,2\ldots(n-2)$.

$$\hat{\mathbf{M}}_{SL(i+1)}\ddot{\mathbf{q}}_{L(i+1)}+\hat{\mathbf{M}}_{SS(i+1)}\ddot{\mathbf{q}}_{S(i+1)}=\hat{\mathbf{Q}}_{S(i+1)}\cdots(2-20)$$

Also, according to equation (2-7), the following equation can be obtained.

$$\hat{\mathbf{M}}_{SL(i+1)}\left[a_0^1\left(\mathbf{q}_{L(i+1)}-\mathbf{q}_{Li}\right)-a_2^1\dot{\mathbf{q}}_{Li}-a_3^1\ddot{\mathbf{q}}_{Li}\right]+$$
$$\hat{\mathbf{M}}_{SS(i+1)}\left[a_0^1\left(\mathbf{q}_{S(i+1)}-\mathbf{q}_{Si}\right)-a_2^1\dot{\mathbf{q}}_{Si}-a_3^1\ddot{\mathbf{q}}_{Si}\right]=\hat{\mathbf{Q}}_{S(i+1)}\cdots(2-21)$$

In equation (2-21), the slowly changing variables, which are used to compute the interaction of the coupling variables, can be linearly interpolated in terms of the trapezoid rule.

$$\mathbf{q}_{Li}=\frac{n-i}{n}\mathbf{q}_{L0}+\frac{i}{n}\mathbf{q}_{Ln}\cdots\cdots\cdots\cdots(2-22)$$
$$\ddot{\mathbf{q}}_{L_i}=\frac{n-i}{n}\ddot{\mathbf{q}}_{L0}+\frac{i}{n}\ddot{\mathbf{q}}_{Ln}\cdots\cdots\cdots\cdots(2-23)$$

In terms of equations (2-22) and (2-23), the action of the slowly changing variables to the rapidly changing variables can be approximately assessed.

$$-\mathbf{F}_{SL(i+1)}=\frac{1}{n}\hat{\mathbf{M}}_{SL(i+1)}\left[\left(a_0\left(\mathbf{q}_{Ln}-\mathbf{q}_{L0}\right)-a_2\left((n-i)\dot{\mathbf{q}}_{L0}+i\dot{\mathbf{q}}_{Ln}\right)\right.\right.$$
$$\left.\left.-a_3^1\left((n-i)\ddot{\mathbf{q}}_{L0}+i\ddot{\mathbf{q}}_{Ln}\right)\right]\cdots\cdots\cdots\cdots\cdots\cdots(2-24)$$

Imposing equation (2-24) into equation (2-21), the sub-step update format can be got as following.

$$a_0\hat{\mathbf{M}}_{SS(i+1)}\mathbf{q}_{S(i+1)}=\hat{\mathbf{Q}}_{S(i+1)}+\hat{\mathbf{M}}_{SS(i+1)}\left(a_0\mathbf{q}_{Si}+a_2\dot{\mathbf{q}}_{Si}+a_3\ddot{\mathbf{q}}_{Si}\right)$$
$$-\frac{a_0}{n}\hat{\mathbf{M}}_{SL(i+1)}\left(\mathbf{q}_{Ln}-\mathbf{q}_{L0}\right)+\frac{a_2}{n}\hat{\mathbf{M}}_{SL(i+1)}\left((n-i)\dot{\mathbf{q}}_{L0}+i\dot{\mathbf{q}}_{Ln}\right)$$
$$+\frac{a_3^1}{n}\hat{\mathbf{M}}_{SL(i+1)}\left((n-i)\ddot{\mathbf{q}}_{L0}+i\ddot{\mathbf{q}}_{Ln}\right)\cdots\cdots\cdots\cdots\cdots\quad\cdots(2-25)$$

The energy balance status can be calculated by means of the equation below (Mark and Belytschko, 1989).

$$\left|W_n^{kin}+W_n^{\text{int}}-W_n^{ext}\right|\le\delta\|W\|\cdots\cdots(2-26)$$

Thereinto, W_n^{ext} is the work of the external forces at $n\Delta t$. W_n^{int} is the internal energy at $n\Delta t$. W_n^{kin} is the kinetic energy of the system at $n\Delta t.\delta$ is the available error coefficient.

3 NUMERICAL EXAMPLES

In this section, two numerical examples will be performed to validate availability and efficiency of the present sub-cycling algorithm.

3.1 A Bar-slider System

A bar-slider system is shown in figure 2. Mass of the rod is 2.0 kilograms and mass of the slider is 5.0 kilograms. The driving torque is 100 Nm/s. Stiffness of the spring is 1000 N/m. length of the rigid rod is 2 meters.

Results of rotational angle of the rod, the vibration amplification of the sliding block and the energy balance status computed by means of the sub-cycling and without sub-cycling are shown in figure 3 to figure 6. The scale values in brackets of the figure captions, such as (5:1), represent a sub-cycling with 5 sub-steps in one major step.

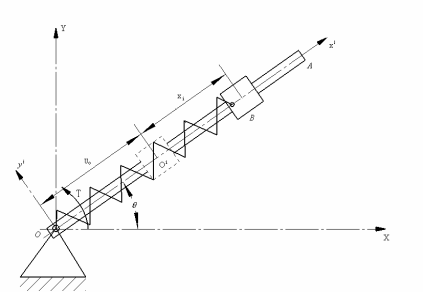

Figure 2: Rotational rod-spring-slider system.

Comparing the results in figure 3 to figure 6, we can see that no matter which scale of the sub-cycling is adopted, the results are very similar. The error of sub-cycling with scale 10:1 is a little larger than that of sub-cycling with scale 5:1. Yet all these two errors are very small if compare with the original results. The time cost of sub-cycling with scale 5:1 is 118 seconds and the time cost of the original algorithm without sub-cycling is 218 seconds. The proportion of time cost of these two algorithms is 54%. The time cost of sub cycling with scale 10:1 is 44 seconds and the proportion of time cost is only 20.2%. The results of balance checking illustrate that the sub-cycling is stable during the integral process.

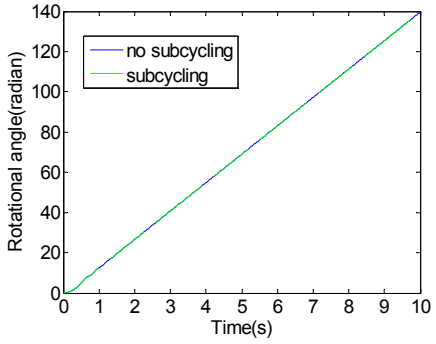

Figure 3: Rotational angle of the bar (5:1).

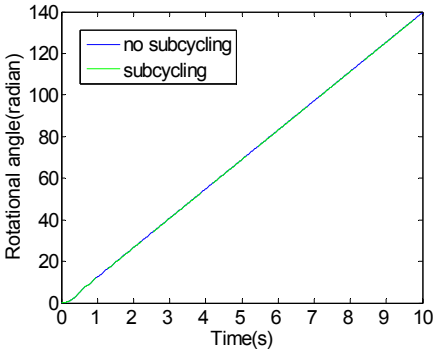

Figure 4: Rotational angle of the bar (10:1).

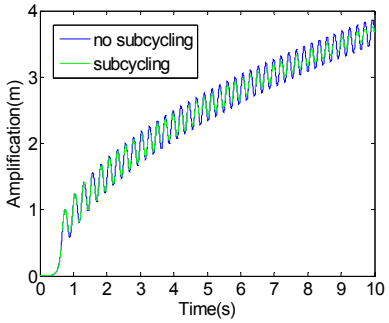

Figure 5: Vibration amplification of the slider (5:1).

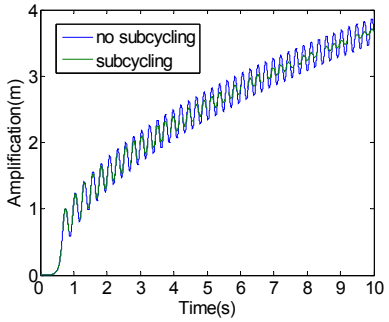

Figure 6: Vibration amplification of the slider (10:1).

3.2 Airscrew of a Jet Engine

FEM model of the airscrew of a jet engine are displayed in figure 7. Parameters of the model are as following (Units: *kg/N/mm/s*). The material of airscrew is aluminium alloy. EX=7e6, PR=3, DEN=2.78e-6. Diameter of the airscrew is D=900 *mm*, rotate speed is 8000 *rpm*.

We simulate the large range overall motion of the airscrew by means of the Newmark sub-cycling and the original Newmark respectively. The dynamic stress at the blade root is described in figure 8. The time cost of the Newmark sub-cycling is 1660 seconds and that of the Newmark is 2810 seconds. The computational efficiency is enhanced about 70%. The compared results show the good precision and stability of the Newmark sub-cycling.

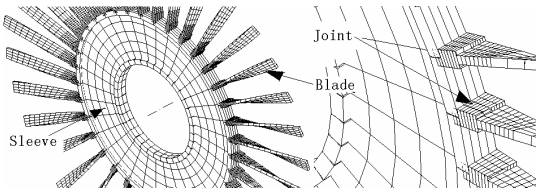

Figure 7: The FEM model and the local mesh.

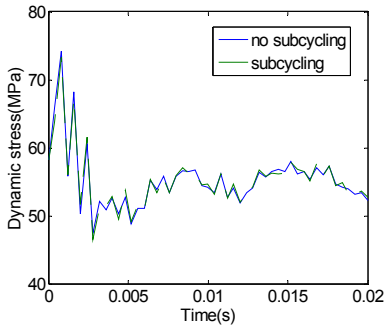

Figure 8: Dynamic stress of the blade root during rotation.

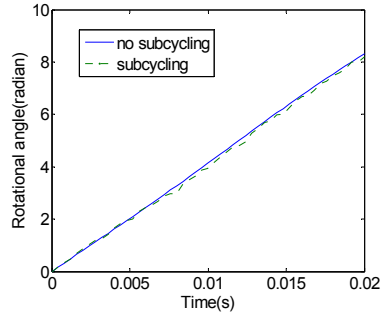

Figure 9: Rotational angle of the blade root.

4 CONCLUSIONS

This paper firstly presents a Newmark-based FMD sub-cycling algorithm. By modifying the Newmark integral formula to be fitted for sub-cycling of the FMD problems, not only the integral efficiency can be greatly improved, but also be more easy for convergence of the integral process.

Because of that different integral step sizes are adopted during the sub-cycling, the integral process can be more efficient and easier for convergence. At the same time, Unconditional stability of the original Newmark are still kept in the Newmark sub-cycling.

The number of the sub-step in one major cycle can be a little infection of the integral precision of a sub-cycling process. However, it is unobvious as the number is within a range. Generally speaking, the enhancement of the integral efficiency is more significant when the number is under a limitation.

By checking the energy balance status of the integral process real time and adjusting the step size when necessary, the sub-cycling procedure can keep a well convergence property and obtain the reasonable numerical computation results.

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